The principle of minimum rate of entropy production
An example

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The inevitable increase of entropy: one way ticket.
There is only one problem: how fast

The second principle of thermodynamics gives us a stone-solid truth: the entropy will ever grow. But, as Loschmidt has noticed to Boltzmann, only in the average. Probabilistic return to the initial state, with full restitution of the entropy is always possible because the equations of mechanics are invariant to time-reversal. But it has very low probability.

1 Preliminaries

Carnot 's principle (1824): No heat engine operating between the sametwo temperatures can be more efficient than the *reversible* one.

$$Q_1 \to W \text{ (work)} \to Q_2$$

William Thomson (Lord Kelvin): the efficiency must have an universal form

$$\eta = 1 - \frac{f\left(T_2\right)}{f\left(T_1\right)}$$

with $f\left(T\right) \equiv$ monotonic and increasing function of T.

$$\eta = 1 - \frac{T_2}{T_1}$$

$$
\begin{aligned}
1 - \frac{Q_2}{Q_1} &\leq 1 - \frac{T_2}{T_1} \\
\frac{Q_1}{T_1} + \frac{-|Q_2|}{T_2} &\leq 0 \\
\sum_i \frac{Q_i}{T_i} &\leq 0 \rightarrow \int \frac{\delta Q}{T} \leq 0
\end{aligned}
$$

This was then introduced by **Clausius**: a functional of the thermodynamic state of the system, *entropy***,** such that for two states we have

$$S_b - S_a = \int_a^b \frac{dQ}{T}$$

The *PATH* of integration only consists of *equilibrium states* (which means that the path is reversible) because the temperature T is only defined for equilibrium states, *NOT* for non-equilibrium states.

If however we take T to be the temperature of the thermal bath, the states can be of non-equilibrium and we have

$$S_{final} \geq S_{initial}$$

Notions: *Macrostate* characterized by macrovariables (intensive and extensive = scale with V or N).

Microstate. Boltzmann introduces the distribution function f and considers the collisions, with its conservations (momentum and energy)

$$f f_1 = f' f_1'$$

which suggested already that f must be an exponential function of the energy of the particle. He defines the H-function

$$H \equiv \int d\omega \; f \ln f$$

and the H-theorem

$$\frac{dH}{dt} \leq 0$$

This statement is true in terms of probability (*i.e.* on the average).

What is the number of microstates corresponding to a given macroscopic state.

$$\begin{aligned} N &= \sum_i n_i \\ E &= \sum_i \varepsilon_i \\ W &= \text{number of microstates} = \frac{N!}{\prod_i (n_i!)} \end{aligned}$$

and we find that the ratio

$$f_i \equiv \frac{n_i}{N}$$

is the *frequency* with which a particle is in a phase-space volume $d\omega_i$ with energy ε_i. Using Stirling approximation of the factorial $\ln N! \approx N \ln N - N$

$$\frac{1}{N} \ln W \approx -\sum_i f_i \ln f_i$$

Planck

$$S = -k \ln W$$

Einstein criticised the combinatorial calculation of W (as a multiplicity) and supported the probabilistic interpretation.

2 The example

A simple application

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \left(-\nabla \phi + \mathbf{v} \times \widehat{\mathbf{e}}_z \Omega\right) \cdot \frac{\partial f}{\partial \mathbf{v}} = C\left(f\right)$$

An expansion around a state of equilibrium

$$f = f_0 \left(1 + \widehat{f}\right)$$

and definition of the bilinear functional

$$K\left(f, g\right) = -\int d^3 v f \; C\left(g\right)$$

The functional K is the rate of irreversible entropy production. To see this one takes the definition of entropy

$$S = -\int d^3 v \; f \ln f$$

take the time derivative, replace the time variation of the entropy with *only* the explicit time dependence (no convection - it does not increase entropy), followed by an expansion if the small $\widehat{f}$,

$$\frac{\partial S}{\partial t} = K\left(f, f\right)$$

we obtain

$$\dot{S} = -\int d^3r \; \widehat{f} C\left(f\right)$$

and this is expressed as a sum over products of *currents* and *forces*.

$$\dot{S} = \int d^3v \; C\left(f\right) \left[vA_1 + \varepsilon v A_2 - \widehat{f} v A_3 \right]$$

where the drive is expressed in terms of gradients

$$A_k \equiv A_k \left(\nabla\rho, \nabla T, ...\right)$$

It results as

$$\dot{S} = -\left(A_1\Gamma + A_2 Q\right) - A_3 J$$

where Γ, Q and J are currents (fluxes) of mass, heat and some other quantity. The currents (I_n) are expressed as a linear combination of of the *forces* A_m,

$$I_n = \sum_m L_{nm} A_m$$

then

$$\dot{S} = -\sum_n I_n A_n = -\sum_n \sum_m A_n L_{nm} A_m$$

the extremum

$$\delta \dot{S} = 0$$

is then used to transform the relationships between the fluxes (currents) and the driving forces (gradients), L_{nm} , into an equation for the distribution function, $\widehat{f}$.

This effectively means the calculation of the diffusion coefficients.

Close to the equilibrium (remember we expanded f) the behavior is that of a minimum rate of entropy production.

Equal strength have the arguments for the maxximum rate of entropy production, in special circumstances.

Our basic idea is:

The system evolves according to the maximum rate of entropy production when it approaches the equilibrium. The system wants to reach equilibrium as fast as possible

When the equilibrium is reached the system reacts according to the principle of minimum entropy production. The system wants to preserve the equilibrium

The entropy gives information about the *irreversibility* and the *disorder*.

Regarding the *irreversibility*:

the system has an increase of the number of possible microscopic states that correspond to a given macroscopic state. For example the system is placed in contact with a thermal bath (let us say: a gas, or the molecules of a solid). The number of degrees of freedom now involved : *system + thermal bath* is very large. The measure of this irreversibility is the increase of the number of degrees of freedom involved, or, equivalently, the number of microscopic states that are now available behind the macroscopic state. This is the increase of entropy.

This example is actually the usual notion of dissipation by friction. This is an irreversible process and the entropy has increased.

Regarding the *disorder*:

the energy transfered from the system to the thermal bath is now located in many thermal fluctuations of the molecules. A representation of this fact would be the distribution of energy on elements of the spectral space (on spectral intervals). A certain amount of energy can be found on almost any spectral interval. equivalently, we say that many spectral elements are involved in the motion. This is disorder, as opposed to the situation where the energy is located in only few spectral elements. The entropy is a measure of spreading the energy on many degrees of freedom.

We conclude that

$$\delta S = \frac{dQ}{T}$$

should not be interpreted in the sense that the income of heat increases the entropy and the outflow of heat leads to decrease of the entropy:

the inflow of heat leads to increase of entropy only because more degrees of freedom are involved in the microscopic motion. They will almost never correlate such as to transfer back the energy they received.

and the fact that we take heat from a system leads to decrease of the entropy is actually due to the fact that the number of degrees of freedom is reduced.

For example:

suppose we have a system consisting of many particles that are bouncing between two perfectly reflecting walls. The particles are not colliding since their trajectories are simply perpendicular on the two walls. Suppose we give energy to the system of particles and in consequence they begin to move faster than before.

There is energy coming to the system but there is no increase in the disorder or in the number of degrees of freedom. Normally the entropy should not grow.

The morality: this is NOT a statistical system.

The expression of entropy for a gas of n moles

$$S = nC_v \ln \frac{T}{T_0} + nR \ln \frac{V}{V_0}$$

where C_v is the molar heat capacity at constant volume.

We have a recipient where there is a wall separating two volumes, V_a and V_b. In the first volume there are n_a moles of the gas and in the other there are n_b moles.

The experiment consisting of removing the wall between two cavities of volumes V_a and V_b at the same temperature leads to the increase of the entropy by

$$\Delta S = n_a R \ln \frac{V_{tot}}{V_a} + n_b R \ln \frac{V_{tot}}{V_b}$$

and for

$$x \equiv \frac{n_a}{n_a + n_b} = \frac{n_a}{n_{tot}} = \frac{V_a}{V_a + V_b} = \frac{V_a}{V_{tot}}$$

$$\Delta S = -n_{tot} R \left[x \ln x - (1 - x) \ln (1 - x) \right]$$

Then: what about the change of entropy when vapor of water are mixing in the dry air volume?

3 The contrast between low rate of entropy production and the sudden change of the pattern of flow.

The production of entropy is specific to thermodynamic systems that are in non-equilibrium state.

For non-equilibrium states the rate of production of entropy has the same role as the thermodynamic *potentials* for the equilibrium processes.

For systems that are not too far from equilibrium state, the rate of entropy production is lowest compatible with the constraints.

For a system that is far from equilibrium, there is the possibility of a sudden transition in which a large amount of entropy is released. But the final flow pattern of the system may be highly ordered, *i.e.* of lower entropy. The example is the first bifurcation of the Rayleigh-Benard system (from conduction to convection in regular cells). The new configuration produces entropy at a higher rate than before: it is a *dissipative structure*.

For oceans: the irreversible processes are *viscosity* and the *diffusion of salt and heat*.

4 Generalities

There is a difference bteween the two meaning of the entropy (or at least of the two utilisations of the notion of entropy)

1. the meaning associated with order. This is what is invoked to derive the *sinh*-Poisson equation and has also been used by **Schubert** et. al. The statistical aspect is emphasized and the number of microscopic states

is explicitely calculated and next is extremized (maximized) with constraints. It is invoked by **Chylek Lesins vertical distribution of entropy production** that the atmosphere is ordered: zonal flows. This - we note - is related to one of the meanings of the entropy

2. the second significance is associated with exchange of heat by radiation gain, radiation loss and by transport and by dissipative effects. Here the entropy is just another member of the family of thermodynamical variables. The deep statistical significance (and origin) of the concept entropy is not necessary here.

For example of mixing of the meanings, we note the statement from **Chylek Lesins** that it is necessary to identify the sources of *negative* entropy in the atmosphere: this is because we know that with the heating from the earth surface there is input (a stream) of positive entropy and - since we know that the atmosphere is organized - we need a flow of entropy outside or a flow of *negative* entropy inward to the atmosphere.

But: we underline: the order of the zonal flow and the order in the atmosphere are related to extremum of entropy arising from an evaluation of the number of internal microscopic states for given external parameters (macroscopic state); while, on the other hand, the exchange of positive or negative streams of entropy are simply related to the heat and dissipations and transport, thermodynamic processes that by themselves are not able to reduce the number of microscopic states corresponding to the macroscopic state and so to drive the system toward order.

The loss of energy by cooling does not lead necessarily to order: the energy is scaled but the number of states may not decrease. Then the system has globally lower energy but does not necessarily has more order.

5 Entropy in climate models

5.1 Entropy budget

The certitudes; the dissipative mechanisms.

The controversies: which are those mechanisms?

5.2 Vertical structure

(**Li, Chylek, Lesins**) ”The irreversible processes are accompanied by positive internal entropy production”.

The system absorbs energy in the form of heat, with a certain content of entropy. This increases the degree of internal disorder of the atmosphere.

On the other hand, the flows of the atmosphere show a high degree of organization (zonal flows, vortices). This would be impossible if the atmosphere have just accumulated entropy.

It is necessary to exist processes by which the atmosphere eliminates the entropy (processes with negative entropy, or, an outflow of entropy).

The atmosphere

- absorbs solar radiation: this is an input of low entropy
- radiates energy in the longwave spectrum and this is large outflow of entropy

It seems that this imbalance makes possible the organization of the flow.

The specific radiative entropy intensity of the blackbody radiation

$$dL_{\nu}\left(T\right)=\frac{dB_{\nu}\left(T\right)}{T}$$

where $B_\nu\left(T\right)$ is the Planck blackbody radiation function, T is the temperature of the blackbody, ν is the frequency of the emitted radiation

$$\begin{aligned} L_\nu\left(T\right) &= \frac{2h\nu^3}{c^2T}\left\{\frac{1}{\exp\left(\frac{h\nu}{kT}\right)-1}\right. \\ &\qquad \left. -\frac{kT}{h\nu}\ln\left[1-\exp\left(-\frac{h\nu}{kT}\right)\right]\right\} \end{aligned}$$

Integrating over the spectrum

$$\begin{aligned} L &= \int_0^\infty L_\nu d\nu \\ &= \frac{4}{3}\frac{\sigma}{\pi}T^3 \\ &= \frac{4}{3}\frac{U}{T} \end{aligned}$$

where σ is the Stefen Boltzmann constant and U is the radiative energy integrated over the spectrum.

Another expression of the spectral distribution of the specific radiative entropy intensity is obtained by first introducing $I_\nu \equiv$specific radiative energy intensity. Then the specific radiative entropy intensity as function of the specific energy radiative intensity, $J_\nu\left(I_\nu\right)$, is

$$\begin{aligned} J_\nu\left(I_\nu\right) \quad &= \quad \frac{2k\nu^2}{c}\left[\frac{c^2 I_\nu}{2h\nu^3}\ln\left(\frac{c^2 I_\nu}{2h\nu^3}\right)\right. \\ &\qquad \left. -\left(1+\frac{c^2 I_\nu}{2h\nu^3}\right)\ln\left(1+\frac{c^2 I_\nu}{2h\nu^3}\right)\right] \end{aligned}$$

The equation of transport of the *radiative energy intensity* I_ν is

$$\left(\widehat{\mathbf{n}}\cdot\nabla\right) I_\nu = -k_\nu I_\nu + i_\nu$$

The spatial decay is given by the *extinction* coefficient and i_ν is the source.

This was for radiated energy.

For the radiated entropy, we have

$$(\widehat{\mathbf{n}} \cdot \nabla) J_\nu = -k_\nu J_\nu + j_\nu$$

with the source term

$$j_\nu \equiv j_\nu (I_\nu, i_\nu)$$

having the expression

$$j_\nu = k_\nu \frac{2k\nu^2}{c^2} \left[\frac{c^2 i_\nu}{2h\nu^3 k_\nu} \ln \left(\frac{c^2 I_\nu}{2h\nu^3} \right) - \left(1 + \frac{c^2 i_\nu}{2h\nu^3 k_\nu} \right) \ln \left(1 + \frac{c^2 I_\nu}{2h\nu^3} \right) \right]$$

The *radiative entropy flux* is a vector resulted from the contribution of the radiative entropy intensity J_ν integrated over the solid angle and over the spectrum

$$\mathbf{H} = \int_{4\pi} d\Omega \int_0^\infty J_\nu \widehat{\mathbf{n}} d\nu$$

Returning to the general thermodynamical context, we identify for each system two sets of variables:

- extensive variables, like volume, magnetization, etc. The flux associated to an extensive variable is a *current*, $\mathbf{Y}_i$.

- intensive variables, like pressure, etc.They are conjugated to the extensive variables and are denoted a_i.

The total entropy flux is

$$\mathbf{Y} = \sum_i a_i \mathbf{Y}_i + \mathbf{H}$$

With the *flux of entropy* $\mathbf{Y}$ and the explicit time variation of the entropy content (S is the volume density of entropy) of the system we write the *entropy balance* equation

$$\frac{\partial S}{\partial t} + \nabla \cdot \mathbf{Y} = \mathbf{\Sigma}$$

where Σ is the source of entropy. The density of entropy is separated into a matter part and a radiation part. The convective variations contain divergences of the two fluxes, specific for each

$$\frac{\partial S_m}{\partial t} + \nabla \cdot \left(\frac{\mathbf{J}}{T} \right) + \frac{\partial S_{rad}}{\partial t} + \nabla \cdot \mathbf{H} = \Sigma$$

For the energy the balance is

$$\frac{\partial}{\partial t} \left(\rho c_p T \right) + \frac{\partial}{\partial z} \left(F_{sun} + F_{lw} + F_c \right) = 0$$

where

F_{sun}	=	solar radiative flux
F_{lw}	=	longwave (infrared) radiation
F_c	=	convective flux of energy

The current of energy density $\mathbf{J}$ has only z component in the $1D$ model and only the convection is present. This means the replacement

$$\nabla \cdot \left(\frac{\mathbf{J}}{T} \right) = \frac{\partial}{\partial z} \left(\frac{F_c}{T} \right)$$

and the entropy balance becomes

$$\begin{aligned} & \frac{\partial S_m}{\partial t} + F_c \frac{\partial}{\partial z}\left(\frac{1}{T}\right) + \frac{1}{T}\frac{\partial F_c}{\partial z} \\ & + \frac{\partial S_{rad}}{\partial t} + \frac{\partial H}{\partial z} \\ = \quad & \Sigma \end{aligned}$$

The *radiative energy transfer* equation is simplified to a $1D$ model, by considering a *diffusive* transfer and introducing a diffusion coefficient μ normalised to the density ρ.

For the infrared radiation

$$\mu \frac{\partial I_\nu}{\partial z} = -k_\nu I_\nu + k_\nu B_\nu \left(T\right)$$

where the source i_ν has been replaced with the Planck function $B_\nu \left(T\right)$ multiplied by the coefficient of spatial decay k_ν

$$i_\nu \to k_\nu B_\nu\left(T\right)$$

The equation that is obtained in this way can be solved and the solution is I_ν, the *radiative energy intensity*. This solution, I_ν, and the source $i_\nu = k_\nu B_\nu\left(T\right)$ are replaced in the expression of the *source* j_ν for the *ENTROPY radiative intensity* $\mathbf{J}_\nu$. We are interested in the regime where the z variation of the *radiative energy flux* I_ν is vanishing, which means that the divergence is zero. Then the solution simplifies to

$$I_\nu = k_\nu B_\nu\left(T\right)$$

and this will be used in the expression of the source j_ν of the *radiative entropy intensity* $\mathbf{J}_\nu$. Then in the $1D$ model the equation becomes

$$\mu \frac{\partial J_\nu}{\partial z} = -k_\nu J_\nu + k_\nu L_\nu\left(T\right)$$

This is the equation for the *current of radiative ENTROPY intensity* and can be solved

$$\begin{aligned} J_\nu^{\uparrow}\left(\tau_\nu,\mu\right) &= L_\nu\left[T\left(\tau_{0,\nu}\right)\right]\exp\left(-\frac{\tau_{0,\nu}-\tau_\nu}{\mu}\right) \\ &\quad +\int_{\tau_{0,\nu}}^{\tau_\nu}\frac{d\tau_\nu'}{\mu}L_\nu\left[T\left(\tau_\nu'\right)\right]\exp\left(-\frac{\tau_\nu'-\tau_\nu}{\mu}\right) \end{aligned}$$

for the upward flux of radiative entropy and

$$J_\nu^{\downarrow}\left(\tau_\nu,\mu\right)=\int_0^{\tau_\nu}\frac{d\tau_\nu'}{\mu}L_\nu\left[T\left(\tau_\nu'\right)\right]\exp\left(-\frac{\tau_\nu'-\tau_\nu}{\mu}\right)$$

for the downward flux.

The optical depth

$$\tau_\nu\left(z\right)=\int_z^\infty k_\nu\rho dz'$$

and the constant $\tau_{0,\nu}$ is defined as

$$\begin{aligned} \tau_{0,\nu} &\equiv \tau_\nu\left(z_0\right) \\ z_0 &\equiv \text{surface height} \end{aligned}$$

The transmission function is

$$\exp\left(-\frac{\tau_\nu}{\mu}\right)$$

Using the solutions for the current of radiative entropy intensity $J_\nu^{\uparrow\downarrow}\left(\tau_\nu, \mu\right)$ and replacing the diffusion coefficient by an effective value it is then possible to calculate the *fluxes of radiative entropy* by integrating the current J_ν over the solid angle.

$$H_\nu^{\uparrow}\left(\tau_\nu, \overline{\mu}\right) \quad \text{and} \quad H_\nu^{\downarrow}\left(\tau_\nu, \overline{\mu}\right)$$

Further the flux of radiative entropy is introduced in the equation of balance of entropy, taken at stationarity, *i.e.* with all explicit time variations vanishing

$$\Sigma = \frac{\partial H}{\partial z} + F_c \frac{\partial}{\partial z}\left(\frac{1}{T}\right) + \frac{1}{T}\frac{\partial F_c}{\partial z}$$

5.3 Horizontal structure

The study of the atmospheric entropy gives information about the *irreversibility* in the climate system.

The variation of the entropy must consider two components: *matter* and *radiation*.

The time change of the entropy is convective

$$\frac{\partial S_m}{\partial t} + \nabla\cdot\left(\frac{\mathbf{J}}{T}\right) + \frac{\partial S_{rad}}{\partial t} + \nabla \cdot \mathbf{H}$$

where

$\mathbf{J}$ ≡ heat flow

$\mathbf{H}$ ≡ entropy flux for the radiation

$$\nabla = \widehat{\mathbf{e}}_r \frac{\partial}{\partial r} + \widehat{\mathbf{e}}_\theta \frac{\partial}{r \partial \theta}$$

and

$$\nabla^\dagger = \widehat{\mathbf{e}}_r \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \cdot\right) + \widehat{\mathbf{e}}_\theta \frac{1}{r \cos\theta} \frac{\partial}{\partial \theta} \left(\cos\theta \cdot\right)$$

where θ is the latitude. Take

$$\sin\theta = x$$

$$\nabla = \widehat{\mathbf{e}}_r \frac{\partial}{\partial r} + \widehat{\mathbf{e}}_\theta \frac{1}{R_0} \left(1 - x^2\right)^{1/2} \frac{\partial}{\partial x}$$

$$\nabla^\dagger = \widehat{\mathbf{e}}_r \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \cdot\right) + \widehat{\mathbf{e}}_\theta \frac{1}{R_0} \frac{\partial}{\partial x} \left[\left(1 - x^2\right)^{1/2} \cdot\right]$$

One then considers the content of internal energy in the matter and in the radiation

$$\begin{aligned} & \frac{\partial U_m}{\partial t} + \nabla^\dagger \cdot \mathbf{J} \\ & + \frac{\partial U_{rad}}{\partial t} + \nabla^\dagger \cdot \mathbf{F} \\ = \quad & 0 \end{aligned}$$

where F is the total radiative energy flux.

There is the classical connection between the variation of the entropy and the variation of the internal energy (Gibbs)

$$\frac{\partial S_m}{\partial t} = \frac{1}{T}\frac{\partial U_m}{\partial t}$$

The *convective change* of the entropy is

$$\begin{aligned}\Sigma \quad &= \quad \frac{\partial S_{rad}}{\partial t} + \nabla^{\dagger} \cdot \mathbf{H} - \frac{1}{T}\frac{\partial U_{rad}}{\partial t} - \frac{1}{T}\nabla^{\dagger} \cdot \mathbf{F} \quad \text{(all terms are for radiation)} \\ &\quad + \mathbf{J} \cdot \nabla^{\dagger}\left(\frac{1}{T}\right) \quad \text{(change due to thermal conduction)}\end{aligned}$$

Integrating over the vertical (radius) coordinate in spherical geometry

$$\sigma \equiv \frac{1}{R_0^2} \int_{R_0}^{R_t} r^2 dr \ \Sigma$$

This represents the total change of the entropy of a volume of the atmosphere having a conical shape (due to sphericity) and having a unit surface at the base.

Other quantities resulting from radial integration

$$s_{rad} = \frac{1}{R_0^2} \int_{R_0}^{R_t} r^2 dr \ S_{rad}$$

$$u_r = \frac{1}{R_0^2} \int_{R_0}^{R_t} r^2 dr \; U_{rad}$$

$$h = \frac{1}{R_0^2} \int_{R_0}^{R_t} r^2 dr \; \nabla^\dagger \cdot \mathbf{H}$$

$$\mathbf{j} = \frac{1}{R_0^2} \int_{R_0}^{R_t} r^2 dr \; \mathbf{J} \quad \text{the total flux of thermal energy}$$

This is transferred from equator to the pole by conduction

$$\mathbf{j} = -D \nabla T \quad \text{(the Fick law)}$$

The radiative energy flux is

$$\begin{aligned} f &= \frac{1}{R_0^2} \int_{R_0}^{R_t} r^2 dr \; \nabla^\dagger \cdot \mathbf{F} \\ &= \frac{R_t^2}{R_0^2} F^t - F^0 \end{aligned}$$

where

$$R^t \equiv \text{radius at the top of the atmosphere}$$

and

$$F^t \equiv \text{net radiative energy flux at the top of the atmosphere}$$

$$F^0 \equiv \text{net radiative energy flux at the surface}$$

It then results that the meaning of f is

$$f \equiv \text{net trap of radiative energy flux by the column of atmosphere}$$

The follwoing expression is used

$$f = I\left(x, T\right) - QS\left(x\right) a\left(x_s, x\right)$$

where

$$I\left(x, T\right) \equiv \text{net outgoing infrared radiative energy flux}$$

and

$$QS\left(x\right) a\left(x_s, x\right) = \text{net incoming solar radiative flux}$$

where

$$\begin{aligned} Q &\equiv \frac{1}{4} \times \text{ solar constant} \\ S\,(x) &\equiv \text{mean annual meridional distribution of the solar radiation} \\ a\,(x_s, x) &\equiv \text{co-albedo} \\ x &\equiv \sin\,(\text{latitude}) \\ x_s &\equiv \text{line of ice} \end{aligned}$$

The radially integrated (over the atmosphere column, with sphericity included) divergence of the total radiative flux of entropy H is

$$h = \frac{4}{3}\frac{I}{T} - \frac{4}{3}\frac{QSa}{T}$$

Then the convective change σ of the entropy of matter + radiation, integrated over the radius, is

$$\begin{aligned}\sigma &= \frac{\partial s_{rad}}{\partial t} - \frac{1}{T}\frac{\partial u_{rad}}{\partial t} + \frac{1}{3}\frac{I}{T} - \frac{4}{3}\frac{QSa}{T_{equiv\_sun}} + \frac{QSa}{T} \\ &\quad + \mathbf{j} \cdot \nabla^{\dagger}\left(\frac{1}{T}\right)\end{aligned}$$

To this equation one must add the energy conservation

$$C\frac{\partial T}{\partial t} + f + \nabla^{\dagger} \cdot \mathbf{j} = 0$$

where C is the heat capacity of the atmosphere.

These equations are solved at stationarity.

The stationary production of entropy by thermal conduction (with diffusion coefficient D) from the higher temperature regions to lower temperature regions is maximum in the middle latitude region.

Vertical profile of production of entropy (Li, Chylek, Lesins)

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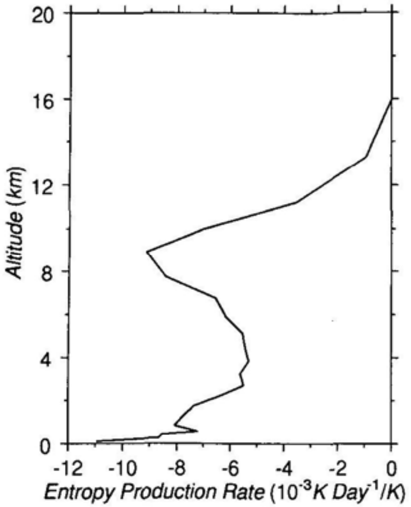

FIG. 3. The vertical distribution of the internal entropy production rate corresponding to the condensation term, Σ_c, for the Tropics.

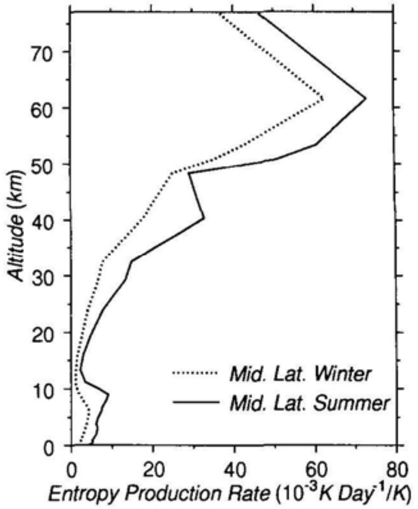

FIG. 5. The vertical distributions of the internal entropy production rates in the middle-latitude summer and in the middle-latitude winter.

Vertical 2

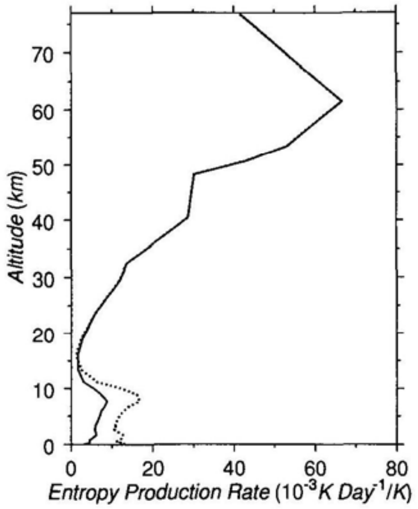

FIG. 4. The vertical distributions of the total internal entropy production rate (solid line), Σ, and the internal entropy production rate only by radiation (dotted line), Σ_r, for the tropical atmosphere.

Horizontal production of entropy (Li, Chylek, Lesins)

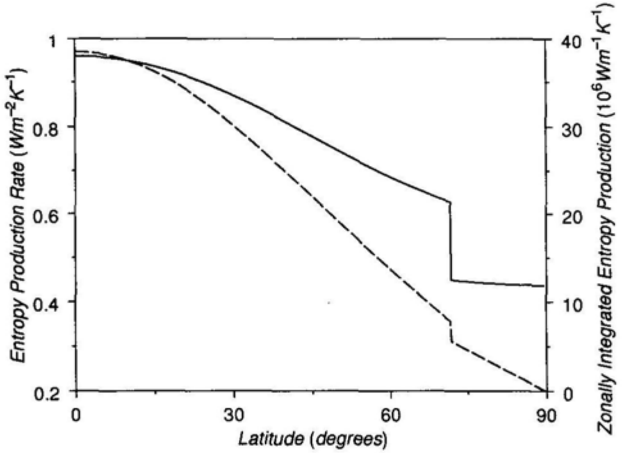

FIG. 1. The horizontal distributions of atmospheric internal entropy production rate (solid line), and the zonally integrated internal entropy production (dashed line).

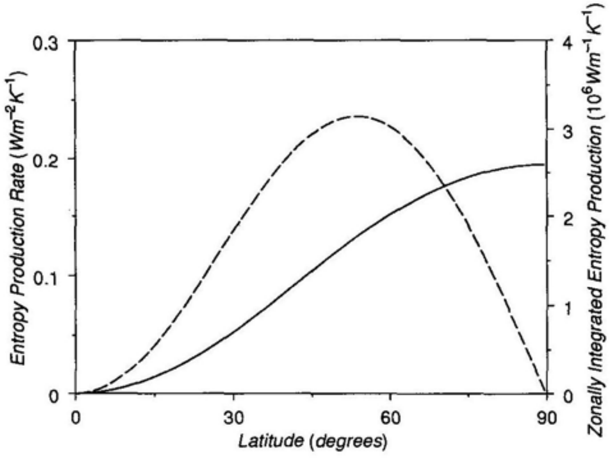

FIG. 2. The horizontal distributions of the internal entropy production rate due to thermal conduction (solid line), and the zonally integrated internal entropy production due to thermal conduction (dashed line).

Horizontal 2

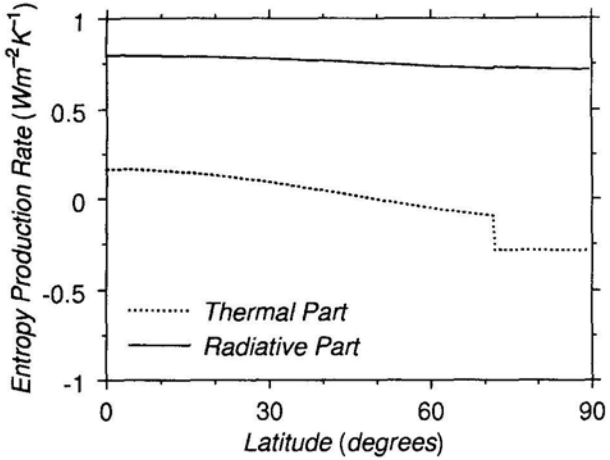

FIG. 3. The horizontal distributions of the internal entropy production rates corresponding to the horizontal thermal entropy flow and the vertical radiative entropy flow.

5.4 More than radiation - convection: dissipation due to moist phase changes

Frictional dissipation in a precipitating cloud. **Pauluis.** Difficult to use the entropy constraint, in this case.

Take the works of **Pauluis**. The budget of entropy of the atmosphere in *radiative-convective* equilibrium is a balance between the *entropy sink* due to the differential heating of the atmosphere and the **entropy production** due to dissipative processes.

The irreversible processes:

- frictional dissipation
- irreversible phase changes (evaporation of water vapors)
- diffusion of heat
- diffusion of water vapors

there is also entropy increase associated with the energy cascade from larger scales tosmaller scales (where the viscosity is able to suppress the motion converting it into heat). Pauluis finds that the frictional dissipation is substantially larger than the turbulent cascade.

This requires comment: the cascade generates entropy only if there is no reversed tendency to produce self-organization by inverse cascade: **this means we are in** $3D$ and NOT in $2D$.

Pauluis compares two processes:

- the radiation and convection equilibrium with production of entropy by dissipation and diffusion of heat
- the changes of states of aggregation (transition of phase) : the convection carries water vapors that are condensed and the precipitation falls. *The problem is to decide if the water (vapor, liquid, precipitation) is part of the system.*

The balance of energy involves

Q_{rad}	→	radiative cooling of the troposphere
Q_{sens}	→	flux of sensible heat at the surface
Q_{lat}	→	flux of latent heat at the surface

with the relation

$$Q_{rad} + Q_{sens} + Q_{lat} = 0 \quad \text{at equilibrium}$$

The mechanical work is done by the pressure force

$$W = \int_{vol} d^3r \; p\partial_k V_k$$

and exists because of non-incompressibility

$$\nabla \cdot \mathbf{V} \neq 0$$

or, air expansion.

The dissipation is

$$\begin{aligned} D &= D_p + D_k \\ D_p &\equiv \text{precipitation dissipation} \\ D_k &\equiv \text{spectral cascade} \end{aligned}$$

with the balance

$$W - D_p - D_k = 0$$

For moist air one introduces the quantities

q_t	$\equiv$	fraction of total liquid present in a volume
q_v	$\equiv$	fraction of water vapor
C_{pd}	$\equiv$	specific heat at constant pressure for DRY air
p_d	$\equiv$	pressure of the DRY air
C_l	$\equiv$	specific heat of liquid water
R_v and R_d	$\rightarrow$	gas constants for vapor and for dry air
$H = \frac{p}{p_{sat}}$	$\rightarrow$	relative humidity, ratio of water vapor pressure and the saturated vapor pressure

with these quantities one expresses the entropy density

$$
\begin{aligned}
s \quad = \quad & (1-q_t)\left(C_{pd}\ln T - R_d \ln p_d\right) \\
& + q_t C_l \ln T \\
& + \frac{q_v L_v}{T} - q_v R_v \ln H
\end{aligned}
$$

Several conclusions:

- molecular diffusion is negligible
- sensible flux loss due to detrainment is much smaller than the heat flux due to radiative cooling
- the sensible heat flux at the surface is smaller than the radiative cooling (for moist convection)

Irreversible entropy production by condensation of a mass of M water vapors is

$$\begin{aligned} \delta S_{pc} &= \int d^3r\ (C-E)R_v \ln\frac{p_v}{p_{v,sat}} && \substack{\text{irreversible entropy increase} \\ \text{due to condensation and re- evaporation}} \\ & \quad -\int d^3r\ J_{v,z} R_v \ln\frac{p_v}{p_{v,sat}} && \text{irreversible evaporation at the surface} \end{aligned}$$

Condensation is reversible and precipitation is NOT reversible.

Conclusion of **Pauluis**:

the kinetic energy of the convection is decreased due to frictional dissipation induced by precipitation, diffusion of water vapors and phase changes.

The irreversible production of entropy for moist air is due to dissipation and change of phase

$$
\begin{aligned}
\delta S &= \frac{D_k + D_p}{T_d} + \delta S_{pc} \ \ \text{which means:} \\
&= \frac{\text{diss.cascade} + \text{diss.precip.induced}}{\text{eff.temp. dissip}} + \text{irrev.entropy phase-changes}
\end{aligned}
$$

6 The connection between entropy and order for fluid systems

6.1 Theory of point-like vortices and the statistical approach

According to Joyce Montgomery (Journal of Plasma Physics, 1973)

The physical quantities describing the two-dimensional fluid dynamics are

$$\begin{aligned} \psi &\equiv \text{streamfunction} \\ \mathbf{v} &\equiv \text{velocity} \\ \omega\widehat{\mathbf{e}}_z &= \text{vorticity (perp. on the plane)} \end{aligned}$$

with the equations

$$\begin{aligned} \mathbf{v} &= -\nabla\psi \times \widehat{\mathbf{e}}_z \\ \omega &= \Delta\psi \end{aligned}$$

The formal solution of the last equation, connecting the vorticity and the streamfunction, can be obtained using the Green function for the Laplace operator

$$\Delta_{x,y} G\left(x, y; x', y'\right) = \delta(x - x')\delta\left(y - y'\right)$$

where (x', y') is a reference point in the plane. Then the Green function has the explicit expression

$$\begin{aligned} G\left(x, y; x', y'\right) &\equiv G\left(\mathbf{r}; \mathbf{r}'\right) \\ &= \frac{1}{2\pi} \ln\left(\left|\mathbf{r} - \mathbf{r}'\right|\right) \end{aligned}$$

after a normalization of the distances in plane by the length of the side L of the square domain.

Using the Green function for the laplacian in the plane we invert the equation relating ω and ψ :

$$\begin{aligned}\psi &= \iint dx' dy' G(\mathbf{r};\mathbf{r}')\,\omega(\mathbf{r}') \\ &= \iint dx' dy' \frac{1}{2\pi} \ln(|\mathbf{r}-\mathbf{r}'|)\,\omega(\mathbf{r}')\end{aligned}$$

Consider now the discretization of the vorticity field $\omega(x,y)$ in a discrete set of $2N$ point like vortices each carrying the elementary quantity ω_0 of vorticity which can be positive or negative

$$\omega_i = \pm\omega_0$$

N vortices with the vorticity $+\,\omega_0$ and
N vortices with the vorticity $-\,\omega_0$

The current position of a point-like vortex is (x, y) at the moment t. The total vorticity is

$$\omega\left(x,y\right)=\sum_{i=1}^{2N}\omega_i\delta\left(x-x_i\right)\delta\left(y-y_i\right)$$

from which we derive the streamfunction solution by inverting the Laplacian

$$\begin{aligned}
\Delta\psi\left(x,y\right) &= \sum_{i=1}^{2N}\omega_i\delta\left(x-x_i\right)\delta\left(y-y_i\right)\\
\psi &= \Delta^{-1}\omega\\
\psi &= \iint dx'dy'\frac{1}{2\pi}\ln\left(\left|\mathbf{r}-\mathbf{r}'\right|\right)\omega\left(\mathbf{r}'\right)\\
&= \iint dx'dy'\frac{1}{2\pi}\ln\left(\left|\mathbf{r}-\mathbf{r}'\right|\right)\sum_{i=1}^{2N}\omega_i\delta\left(x'-x_i\right)\delta\left(y'-y_i\right)
\end{aligned}$$

or

$$\psi\left(\mathbf{r}\right)=\sum_{i=1}^{2N}\omega_i\frac{1}{2\pi}\ln\left(\left|\mathbf{r}-\mathbf{r}_i\right|\right)$$

The velocity of the k-th point-vortex is

$$\begin{aligned}\mathbf{v}_k &= -\left.\nabla\psi\right|_{\mathbf{r}=\mathbf{r}_k}\times\widehat{\mathbf{e}}_z\\ &= -\sum_{i=1}^{2N}\omega_i\frac{1}{2\pi}\frac{\mathbf{r}_k-\mathbf{r}_i}{\left|\mathbf{r}_k-\mathbf{r}_i\right|^2}\times\widehat{\mathbf{e}}_z\end{aligned}$$

or

$$\begin{aligned}\frac{dx_k}{dt} &= v_x^{(k)}=-\sum_{i=1}^{2N}\omega_i\frac{1}{2\pi}\frac{y_k-y_i}{\left|\mathbf{r}_k-\mathbf{r}_i\right|^2}\\ \frac{dy_k}{dt} &= v_y^{(k)}=\sum_{i=1}^{2N}\omega_i\frac{1}{2\pi}\frac{x_k-x_i}{\left|\mathbf{r}_k-\mathbf{r}_i\right|^2}\end{aligned}$$

It is possible to define a Hamiltonian

$$H = \frac{1}{2\pi} \sum_{\substack{i=1 \\ i<j}}^{2N} \sum_{j=1}^{2N} \omega_i \omega_j \ln \left(|\mathbf{r}_i - \mathbf{r}_j| \right)$$

and the two conjugated variables are

$$\begin{pmatrix} p_j \\ q_j \end{pmatrix} = |\omega_j|^{1/2} \begin{pmatrix} x_j \\ y_j sign\left(\omega_j\right) \end{pmatrix}$$

We **NOTE** the difference in sign compared with the text of **Joyce Montgomery**. The reason is that the original paper was about *electric charges* and the relationship between the Laplacian of the electric potential ϕ and the electric charge density ρ is

$$\Delta\phi = -\frac{1}{\varepsilon_0}\rho$$

while in the case of the vorticity - streamfunction the relation is

$$\Delta\psi = \omega$$

END note.

6.2 Statistical approach

The ensemble of point-like vortices is treated with statistical methods.

Consider the explanation given by **Joyce Montgomery**.

The Hamiltonia system defined above has a *finite phase space*. This is because the two conjugated variables are coordinates of the point-like vortices and they evolve on a limited region in plane, a square with side L. The area of this square is the volume of the phase space for a single vortex.

It is defined the phase-space volume per unit interaction energy (so-called *structure function*) and its integration over the energy of interaction between the minimum $(-\infty)$ and a current value E is

$$\Phi\left(E\right) \equiv \int_{-\infty}^{E} \Omega\left(E'\right) dE'$$

is a function of E that increases monotonically from 0 to the maximum value

$$\prod_k \left(\left|\omega_k\right| L^2\right)$$

when the limit of integration E varies between $-\infty$ to ∞. Then it asaturates to a constant when $E \to \infty$. This means that the derivative of the function $\Phi\left(E\right)$ first has an increase and then a decrease, or

$$\frac{d\Phi\left(E\right)}{dE} \text{ has a maximum somewhere between } -\infty \text{ and } \infty$$

say at E_{max}. This means that the second derivative has a zero at E_{max}. Or, the second derivative is just the derivative of the *structure function* $\Omega\left(E\right)$.

For a statistical ensemble with fixed total energy E_0 the entropy and the temperature are defined as

$$\begin{aligned} S &= k \ln \Omega\left(E_0\right) + \text{const} \\ \frac{1}{kT} &= \frac{1}{\Omega\left(E_0\right)} \frac{d\Omega\left(E_0\right)}{dE_0} \end{aligned}$$

As said above, beyond the value of the maximum $E_{\max}$ the derivative of the structure function is negative. This means that

$$\text{for } E > E_{\max} \quad \text{we have } T < 0$$

6.3 Negative temperature in the statistical system of point-like vortices

General statistical treatment for a system for which we can specify the energy E_i of the discrete set of states it can have.

(Recommended **Isihara Statistics Book**).

$$Z = \sum_i \exp\left(-\beta E_i\right) \text{ partition function}$$

$$U = \frac{1}{Z} \sum_i E_i \exp\left(-\beta E_i\right) \text{ average energy}$$

$$F = -kT \ln Z \text{ free energy}$$

$$U = -\left(\frac{\partial \ln Z}{\partial \beta}\right)_V = \left(\frac{\partial \ln Z}{\partial T}\right) kT^2$$

$$F = U - TS \text{ free energy}$$

$$
\begin{aligned}
S &= \frac{U-F}{T} \text{ entropy} \\
&= \frac{1}{T}\left[\left(\frac{\partial \ln Z}{\partial T}\right) kT^2 + kT\ln Z\right] \\
&= k\left[\left(\frac{\partial \ln Z}{\partial T}\right) T + \ln Z\right] \\
&= \frac{\partial}{\partial T}\left(kT\ln Z\right)
\end{aligned}
$$

For the discrete set of point-like vortices (recommended paper **Edwards and Taylor**).

The distribution is *microcanonic* which means that it is specified **exactly** the energy E of the system

$$
\begin{aligned}
\text{distribution} &\equiv \rho\left(x_i, y_i\right) \\
\rho\left(x_i, y_i\right) &= \delta\left(E - H\left\{x_i, y_i\right\}\right)
\end{aligned}
$$

It results that the volume of the phase space where the energy is less than E

$$\phi\left(E\right)$$

and the statistical weight

$$\begin{aligned} \Omega\left(E,V,N\right) &= \frac{d\phi}{dE} \\ &= \int \delta\left(E-H\left(x_i,y_i\right)\right)d\Omega \end{aligned}$$

The *entropy* is

$$S\left(E,V,N\right)=\ln\Omega\left(E,V,N\right)$$

and it follows

$$\frac{1}{T}=\left(\frac{\partial S}{\partial E}\right)_{V,N}$$

$$p=-\left(\frac{\partial E}{\partial V}\right)_{S,N}=\frac{\left(\frac{\partial S}{\partial V}\right)}{\left(\frac{\partial S}{\partial E}\right)}$$

The statistical wight can be calculated after writting the explicit form of the Hamiltonian

$$\begin{aligned} H &= \sum_{i<j} \left(-\frac{2e_i e_j}{l} \right) \ln \left(|\mathbf{r}_i - \mathbf{r}_j| \right) \\ H\left\{x_i, y_i\right\} &= \sum_{i,j,i\neq j} e_i e_j \ln r_{ij} \end{aligned}$$

This two expressions, compared, explain the meaning of adopting the coefficient 2 in front of the plane Coulomb interaction (which is always possible since l is arbitrary). One can transform the sum over *distinct* pairs (i,j), $i<j$, into a sum over all pairs, *i.e.* counting two times a pair, (i,j) and (j,i) but dividing by 2. The second form has a sum which only excludes the selfinteractions $i=j$. This will be useful when one calculates sums like

$$\sum_i \sum_{j\neq i} e_i e_j = -Ne^2$$

where N is the number of positive charges and also the number of negative charges.

Then

$$\Omega = \int \prod dr_i \, \frac{d\lambda}{2\pi} \exp\left[i\lambda \left(E + \sum_{i,j,i\neq j} e_i e_j \ln r_{ij} \right) \right]$$

An important step is to extract the dimensional scale from the variables $\{x_i, y_i\}$ which represent the positions of the particles in plane, over which the integrations $\prod dr_i$ are to be calculated.

$$r \rightarrow r' \sqrt{V}$$

$$\Omega = \frac{V^{2N}}{2\pi} \int \prod dr'_i \, d\lambda \exp\left[i\lambda \left(E + \sum_{i,j,i\neq j} \frac{e_i e_j}{2} \ln V + \sum_{i,j,i\neq j} e_i e_j \ln r'_{ij} \right) \right]$$

In addition one takes into account that the number of positive and of negative charges are equal and that the magnitudes of the charges are the same. Then

$$\sum_{i,j,i\neq j} e_i e_j = -Ne^2$$

where N is the number of positive vortices and also the number of negative vortices.

It results

$$\Omega = \frac{V^{2N}}{2\pi} \int d\lambda \exp\left(i\lambda E - i\lambda \frac{Ne^2}{2} \ln V\right) \int \prod dr'_i \exp\left(i\lambda \sum_{i,j,i\neq j} e_i e_j \ln r'_{ij}\right)$$

At this point the equation of state can be derived using the definition of the thermodynamic quantities

$$PV = 2NT\left(1 - \frac{e^2}{2T}\right)$$

and shows that there is a *critical temperature*

$$T_c = \frac{1}{2} e^2$$

below which the pressure becomes *negative*.

The temperature is

$$T = \frac{\frac{d^2\phi}{dE^2}}{\frac{d\phi}{dE}}$$

As noted by **Montgomery** the treatment of **Taylor 1972** has introduced in the functional integral a constant term that represents the energy of interaction between all vortices if they are indefinitely close. This term is *infinite*. When this term is not included the result is even more explicit:

the statistical temperature of an ensemble of vortices is NEGATIVE as soon as the energy of the system is positive

$$E > 0 \quad \rightarrow \quad T < 0$$

This shows that for any configuration of point-like vortices in motion (*i.e.* with positive energy) the temperature is negative and the system evolves to self-organization.

6.4 Derivation of the equation for the asymptotic ordered states

It has been made the discretization taking an elementary $2D$ interval denoted σ. (In the notation of **Joyce Montgomery** it is used the notation Δ).

The *entropy* that is used in the theory of statistical mechanics of point-like vortices is the ln of

$$\mathcal{W} = \left\{ N! \prod_i \frac{\sigma N_i^+}{N_i^+!} \right\} \left\{ N! \prod_i \frac{\sigma N_i^-}{N_i^-!} \right\}$$

The quantity $\mathcal{W}$ represents (**Huang statistical**) the volume occupied in the space of all states of a system consisting of N particles distiguishable by the distribution function f corresponding to the occupation numbers $\{n_i\}$. It is equal to the number of ways to distribute N distinguishable molecules among K cells such that there are n_i of them in the cell i.

$$\Omega\left\{n_i\right\} = \frac{N!}{n_1!n_2!...n_K!}$$

The following formulation: *in every element of surface there are positive and negative vortices*

$$\begin{aligned} N_i^+ &\neq 0 \\ N_i^- &\neq 0 \end{aligned}$$

and there is no possibility that in an element of surface of the plane one of them to be absent.

Montgomery obtains

$$\ln N_i^+ + \alpha^+ - \beta \sum_j \Delta^{-1} \left(N_i^+ - N_i^-\right) = 0$$

and similar for $-$.

$$N_i^+ N_i^- = \text{const}$$

The two equations

$$\begin{aligned} \ln N_i^+ + \alpha^+ + \beta \sum_j \phi_{ij} \left(N_j^+ - N_j^-\right) &= 0 \\ \ln N_i^- + \alpha^- - \beta \sum_j \phi_{ij} \left(N_j^+ - N_j^-\right) &= 0 \end{aligned}$$

are obtained variationally under the constraints

$$\begin{aligned} \mathcal{E} &> 0\ ,\ \text{const} \\ \sum_i N_i^+ &= N = \text{const} \\ \sum_i N_i^- &= N = \text{const} \end{aligned}$$

and the proves that one obtains in SA the same equation *sinh*-Poisson even if one keeps non-zero total energy. The condition to obtain *sinh*-Poisson is the equality of the total number of positive and negative vortices.

We can find the multiplicity (but now there are $2N$ particles)

$$\ln W = \ln \left\{ N! \prod_i \frac{\sigma N_i^+}{N_i^+!} \right\} \left\{ N! \prod_i \frac{\sigma N_i^-}{N_i^-!} \right\}$$

$$\begin{aligned} &\ln W \approx 2\left(N \ln N - N\right) \\ &+ \sum_i \{ N_i^+ \ln \sigma - N_i^+ \ln N_i^+ + N_i^+ \quad \text{entropy} \\ &+ N_i^- \ln \sigma - N_i^- \ln N_i^- + N_i^- \} \end{aligned}$$

$$\begin{aligned} &\sum_i N_i^+ = N = \text{const} \\ &\sum_i N_i^- = N = \text{const} \end{aligned}$$

The two kinds of particles represent fluid rotation

N_+ clockwise

N_- counterclockwise

and

$$N_\pm = \exp\left[-\alpha_\pm \mp \beta \psi\left(x\right)\right]$$

In the statistical theory it is not necessary to have

$$N_+ N_- = 1$$

The equation for the density of point-like vortices is

$$\Delta\varphi = -\frac{1}{\varepsilon_0}\frac{e}{l}\left[n_+ \exp\left(-e\beta\psi\right) - n_- \exp\left(+e\beta\psi\right)\right]$$

This equation is derived under the condition

$$\beta < 0$$

There are constraints that the total number of positive and negative vortices are equal

$$n_\pm \int d^3r \ \exp\left(\mp e\beta\psi\right) = N \quad \text{(the same for both } N_\pm\text{)}$$

and

$$\lambda^2 \equiv -2\beta \frac{e^2}{\varepsilon_0 l} \sqrt{N_+ N_-}$$

the equation is

$$\Delta\psi + \lambda^2 \sinh(\psi) = 0$$

The negative temperature

$$T < 0$$

is introduced by

$$-\lambda^2 = \frac{1}{kT}$$

In the paper of **Book** it is also discussed the reason for the existence of so many solutions. This is found to be related with the large number of extrema of the *entropy*

$$\begin{aligned} S &= \ln W = 2\beta E \\ &= -\frac{\varepsilon_0 l}{e^2}\frac{\lambda^2}{n_0}E \end{aligned}$$

where W is the probability of the configuration. Then for fixed energy E

$$W = W_0 \exp\left(-\mathrm{const}\lambda^2\right)$$

where

$$\mathrm{const} > 0$$

The configurations with low λ^2 have higher probability, entropy, energy.

7 (My) Conclusions

- The principle of maximum rate of entropy production is TRUE

- The principle of minimum rate of entropy production is TRUE

We need to dissociate the classes of regimes where one of them is applicable.

Close to equilibrium (like in SOC) minimum rate.

When one offers to the system a large number of degrees of freedom (equivalently: many possibilities to get a microscopic configuration compatible with the macroscopic one) : maximum rate. Ex.: mixing of components, like mixing of gases; diffusion; direct spectral cascade of energy in $3D$.

Serious problem: coexistence of

- increase of entropy, the system is triumphally marching toward irrecuperable disorder
- instauration of high order, coherent flow.

Where does one end up and begins the other?

Can-we mix these tendencis (see recent proposals in oceanography).

8 Beyond conclusions: Connection with the Self-Organization at Criticality (SOC)

Application of the Bak-Snaeppen model and the Information Entropy

The BS model may be useful since it consists of sequential allocation of random numbers for a quantity in the point and in its surroundings where that quantity is minimal.

Regulation of the moist convection (Raymond 1995)

In the tropical oceanic atmosphere, only the *boundary-layer* is subject to instability. If we assume that there is *fast adjustment* such as to get quasi-equilibrium, we need to explain how effectively the instability developing in the boundary layer is suppressed by convection. A possibility: *reduce the equivalent potential temperature in the boundary layer*. (With the conservation of the vertically integrated enthalpy of the atmosphere).

Raymond: θ_e and r_t evolves through a succesion of *relaxation processes*, controlled by the surface fluxes, within the *convective layer*. The latter is the defined by the height at which a non-entraining surface parcel reaches the neutral buoyancy.

The *relaxation* is toward the saturated values, θ_{es} and r_{ts}.

The convection is controlled by the *difference in the buoyancy* *between the boundary layer parcel and the environmental air just above the boundary layer.* Therefore there is a threshold which is manifested as a value of the equivalent potential temperature θ_e^{thres}. The system makes oscillations such that the boundary layer is forced to remain close to the threshold value for convection.

The *budget of entropy* in the boundary layer

$$\begin{aligned} & \frac{\partial \theta_{eb}}{\partial t} + \left(\mathbf{u}_h \cdot \nabla_h\right) \theta_{eb} + w_e \frac{\partial \theta_{eb}}{\partial z} \\ = \quad & Q_{eb} \quad \text{mean radiative tendency of } \theta_{eb} \\ & + \frac{F_{es}}{b} \quad \text{surface flux/depth of the boundary layer} \end{aligned}$$

Here

$$w_e \equiv \text{entrainment volume flux}$$

which means the volume per area per time at which the boundary layer incorporates the immediately overlying air. This has the dimension of velocity

$$[w_e] = \frac{m}{s}$$

The derivative is approximated

$$w_e \frac{\partial \theta_{eb}}{\partial z} \rightarrow w_e \frac{\delta \theta_{eb}}{b}$$

where

$$\delta\theta_{eb} \equiv \text{decrease of } \theta_e \text{ across the top of the boundary layer}$$

The *mean radiative tendency* of the equivalent potential temperature is Q_{eb},

$$Q_{eb} = \theta_{eb} \frac{Q_b}{\theta_b}$$

where

$$\begin{aligned} Q_b &\equiv \text{radiative heating rate} \\ &= \left.\frac{d\theta}{dt}\right|_{rad} \end{aligned}$$

where θ is the potentail temperature, $\theta_b \equiv$ the potential temperature evaluated at the boundary layer.

The surface flux of θ_e is

$$F_{es} = C_d U_e \left(\delta\theta_e\right)$$

$$\begin{aligned} C_d &\approx 10^{-3} \\ \delta\theta_e &\equiv \theta_{es,s} - \theta_{eb} \\ U_e &\equiv \sqrt{\left|\mathbf{u}_h\right|^2 + w^2} \equiv \text{effective wind speed} \end{aligned}$$

$$\theta_{es,s} \equiv \text{saturated equiv. pot. temperature at the sea surface}$$

The definition of the threshold.

The threshold equivalent potential temperature for convection

$$\theta_e^{threshold}$$

is the *minimum* value of θ_{eb} for which an ascending parcel can *pass through the stable layer just above the cloud base.*

Soundings.

A boundary layer parcel experiences a small region of negative buoyancy between 880 and 950 mb, with *positive buoyancy* above. (See picture from **Raymond**).

Little or *no* buoyancy exists for parcels above the boundary layer. (Therefore if we want to trigger convection we can restrict to the boundary layer).

The dominant role of the boundary layer

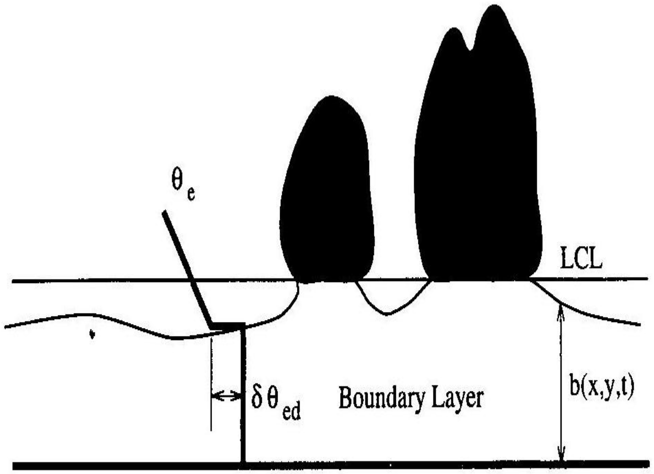

FIG. 1. Sketch of idealized boundary layer of depth $b(x, y, t)$. The lifting condensation level of boundary layer air is LCL and the difference between boundary–mixed-layer equivalent potential temperature and the air immediately above is $\delta\theta_{ed}$.

Boundary layer (Raymond 1995)

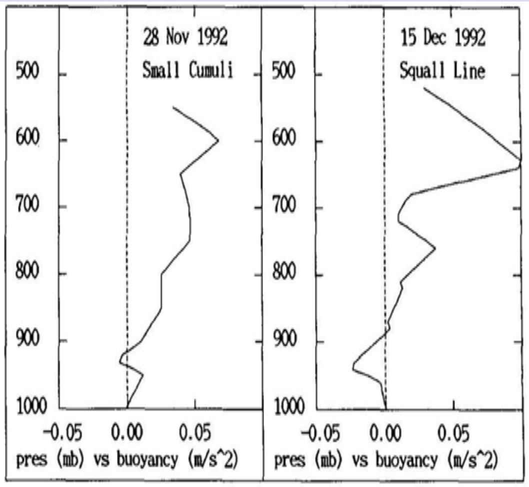

FIG. 5. Buoyancies of undilute 1000-mb parcels for the soundings shown in Figs. 3 and 4. The buoyancies include the virtual temperature effect and the weight of condensed water, which is assumed to be carried with the parcel. The negative area near 930 mb is roughly 0.7 J kg^{-1} for the 28 November case and 7 J kg^{-1} for the 15 December case.

Buoyancy (Raymond 1995)

The dominant role of the boundary layer (cont)

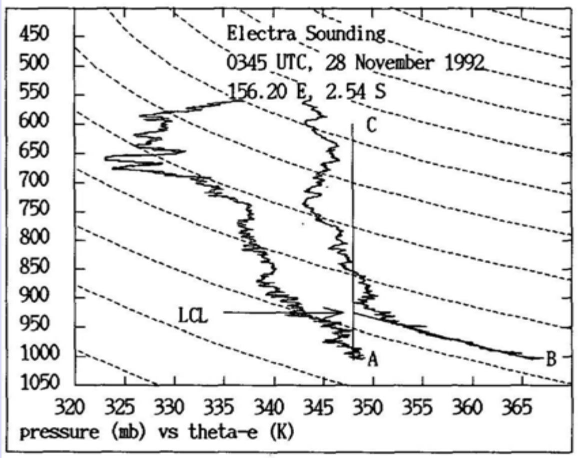

FIG. 3. Sounding by Electra aircraft, flight 4 of TOGA COARE, 28 November 1992. The left curve shows equivalent potential temperature as a function of pressure, while the right curve shows saturated equivalent potential temperature. The dashed lines are contours of potential temperature at 5 K intervals. The line B–LCL–C shows the saturated equivalent potential temperature of an ascending surface parcel, while the line A–C shows the equivalent potential temperature of this parcel.

The equivalent potential temperature

(1)

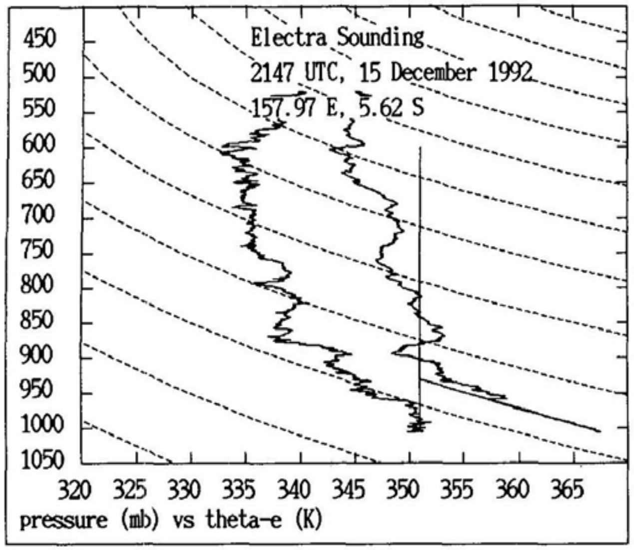

FIG. 4. Sounding by Electra aircraft, flight 15 of TOGA COARE, 15 December 1992. (See Fig. 3.)

The equivalent potential temperature

(2)

An example of threshold value of the equivalent potential temperature

$$\theta_e^{threshold} \sim 350\ K$$

The *energy barriers* were

$0.7\ J/kg$ or, in a different case,

$7\ J/kg$

The energy barrier is translated into *minimum vertical velocity* of the parcel. The minimum vertical velocities in the two cases are

$1.2\ m/s$ and, in the other case

$4\ m/s$

Corresponding to these values of the vertical velocity, the horizontal values were

$$2\ m/s \text{ and respectively}$$
$$8\ m/s$$

which shows that there were large differences in *eddies*.

The *convective deficit* defined as

$$I \equiv \theta_e^{threshold} - \theta_{eb}$$

The *threshold equivalent potential temperature for convection* can be approximated by the *saturated equivalent potential temperature averaged through a layer just above the cloud base.*

Sugestion: self-organized criticality may be the concept adequate for the state before major convective events

Conclusions