Notes on integrability

Florin Spineanu and Madalina Vlad
INFLPR, CEA-Cadarache, Univ. Aix-Marseille, Univ. Fukuoka

December 10, 2023

Abstract

General notes on integrability, on the geometrico-algebraic method of integration and application in plasma, fluid and laser physics. These are raw Notes.

1 Introduction

On the integrability.

Texts are in *plasma, selfmodulation.*

Some are in *Lasers, Notes NSEq, IST_periodic, Stability.*

The algebro-geometrical method for integrability.

In **Bogoyavlensky** there is reference to the situation where the system of linear equations whose compatibility is the nonlinear equation (the spectral matrix) is not 2×2 but higher. Paper **riemann sol bogoyavlensky** discusses the case 3×3.

Trigonal curve (instead of hyperelliptic curve), based on the characteristic polynomial of the Lax matrix.

In *integrability, biblio,* a special folder for *extension.*

The Self-Dual Yang Mills system, reduction and integrable equations.

The papers are in *biblio, physics, integrable.* And in *research, integrability, biblio.*

Papers by **Ward, Strachan, book Mason Woodhouse.**

One of the objectives is NSEq that may present the property "no blowup but cusp" caused by the existence of $n_+ n_- =$ const.

The Painleve property. It is defined for ordinary differential equations "*all movable singularities in the solutions are poles*"

It is extended to partial differential equations. **Ward integrability chiral system with torsion**.

We suggest to adopt the topological formulation.

Every singularity is a possible change of topology.

Poles are able to provide a particular form of change of topology.

The extension of the property to more complicated systems like partial differential equations must be made by looking to other changes of topology, as mediated by instantons. Or sphalerons ?

A text and folder on **Return_to_integrability**, or, **rate of generation (recuperation) of invariants at the return to integrability of an equation that has been structurally perturbed via the random scattering of the poles in the complex plane** is separate.

From **Chen Lee Pereira** for *Benjamin Ono* (see below), solution by **Olshanetsky Perelomov**

$$\begin{aligned} I_n &\equiv Tr\left(L^n\right) \\ &\quad \text{action variables} \end{aligned}$$

(**note** similarity to vorticity set of invariants $\int d^2x\ \omega^n$ see **Zabrodin Wiegmann** for mapping on complex plane, **Isichenko Gruzinov, Montgomery Kraichnan**)

$$\begin{aligned} J_n &\equiv Tr\left(BL^{n-1}\right) \\ &\quad \text{angle variables} \end{aligned}$$

(**question** if I_n $\equiv$action, are the vorticity invariants - then who are the angle variables B_n ?)

where

$$B_{ij} = a_i \delta_{ij}$$

is a diagonal matrix and

$$\frac{\partial B}{\partial t} = [A, B] + L$$

It is found that

$$\begin{aligned} \frac{dJ_n}{dt} &= I_n \\ \text{or } J_n(t) &= I_n(0)\ \ t + J_n(0) \end{aligned}$$

the angle is increasing linearly in time, as expected.

question is-this a generalization of the action-angle structure of tori (**Arnold**) in classical mechanics?

For the equation *Benjamin Ono* it has been found by **Olshanetsky and Perelomov** that the eigenvalues of the matricial operator

$$M(t, t_0) = B(t_0) + (t - t_0) L(t_0)$$

are the positions of the poles

$$a_i(t)$$

This suggests: take a matrix and place its eigenvalues then reconstitute the system from the positions of the poles.

After that, let the matrix $M(t, t_0)$ move in time, and find at t the eigenvalues and reconstruct the solution.

We should look also to **Blaizot**.

2 Selective decay of invariants

Montgomery Kraichnan

Zabrodin Weigmann for Hele Shaw, Schottky double.

3 Self-Dual Yang-Mills equations and integrability

Reduction from $\mathbf{R}^{2,2}$ to $(2+1)$.

From SDYM to chiral model.

From chiral model to NSEq.

From chiral model to Dirac equation in $1+1$.

Identify *zero-modes*, introduce N_R and N_L.

Find $N_R N_L = \text{const}$, at least for NSEq.

3.1 Definition of the Self-Dual Yang-Mills equations

The presentation of this subject is done very clearly by **Ward** in "Integrable and Solvable" in **Proc.Roy.Soc. 1985**.

A definition is offered in **9311119**.

There are papers on the derivation of ALL integrable equations from SD of YM. (**Sterling, Mason Woodhouse**) **Ward**.

Ablowitz self-dual integrable.

3.2 NSEq from SDYM Mason Sparling

The space

$$\mathbf{R}^{2,2}$$

coordinates

$$(x, y, u, t)$$

The metric

$$ds^2 = dx^2 - dy^2 + 2\ du\ dt$$

The gauge group is one of the REAL forms of

$$SL(2, \mathbf{C})$$

The solution must be symmetric to
- time translation
- a null translation that is transversal on the time direction

Covariant derivatives

$$\begin{aligned} D_a &= \frac{\partial}{\partial x^a} - A_a \\ &\text{where} \\ A_a &\in SL(2, \mathbf{C}) \end{aligned}$$

The condition of self-duality

$$\varepsilon_{ab}{}^{cd} [D_c \ , \ D_d] = [D_a \ , \ D_b]$$

or, in detail

$$[D_x + D_y \ , \ D_u] = 0$$

$$[D_x - D_y \ , \ D_x + D_y] + [D_u \ , \ D_t] = 0$$

$$[D_x - D_y \ , \ D_t] = 0$$

These equations are derived as *compatibility conditions* of a system. Let

$$\begin{aligned} \lambda \equiv\ & \text{spectral parameter} \\ & \text{"affine complex coordinate on the Riemann sphere"}\ CP^1 \end{aligned}$$

Let

$$s \equiv \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}$$

Now define

$$\begin{aligned} (D_x - D_y + \lambda D_u) \ s &= 0 \\ \text{notation } L_0 \ s &= 0 \end{aligned}$$

and

$$\begin{aligned} [\lambda (D_x + D_y) + D_t] \ s &= 0 \\ \text{notation } L_1 \ s &= 0 \end{aligned}$$

Now give explicit form to the covariant derivative operators

$$D_x = \frac{\partial}{\partial x} - A$$

$$D_u = \frac{\partial}{\partial u} - B$$

$$D_t = \frac{\partial}{\partial t} - C$$

$$D_y = \frac{\partial}{\partial y} - D$$

"two commuting symmetries which project to a pair of orthogonal spacetime translations one timelike and one null;"

"in our coordinates these are along"

$$\frac{\partial}{\partial y} \quad \text{and} \quad \frac{\partial}{\partial u}$$

The gauge functions A, B, C, D are independent of

$$y \quad \text{and} \quad u$$

In addition, impose

$$A + D = 0$$

"The gauge transformations are now restricted to $SL(2, C)$ valued functions of t alone under which A and B transform by conjugation"

With these simplifications (reductions) return to the equation defined by the commutators of the covariant derivative operators.

$$\frac{\partial B}{\partial x} = 0$$

$$\left[\frac{\partial}{\partial x} - 2A \ , \ \frac{\partial}{\partial t} - C\right] = 0$$

$$2\frac{\partial A}{\partial x} - [B, C] = \frac{\partial B}{\partial t}$$

Here A, B, C are 2×2 matrices with complex entries.

And the reduced versions of the operators $L_{0,1}$

$$L_0 s = \left(\frac{\partial}{\partial x} - 2A + \lambda \ B\right) \ s = 0$$

$$L_1 s = \left(\lambda \ \frac{\partial}{\partial x} - C + \frac{\partial}{\partial t}\right) \ s = 0$$

The matrix B can have the *normal* form
(α) form

$$B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

or
(β) form

$$B = \kappa\,(t) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

With B of the type β the solution of the equations are obtained after *adopting the ansatz*

$$2A = \begin{pmatrix} 0 & \psi \\ -\widetilde{\psi} & 0 \end{pmatrix}$$

$$\kappa\ 2C = \begin{pmatrix} \psi\widetilde{\psi} & \frac{\partial\psi}{\partial x} \\ \frac{\partial\widetilde{\psi}}{\partial x} & -\psi\widetilde{\psi} \end{pmatrix}$$

For this ansatz to provide solution the functions ψ and $\widetilde{\psi}$ must verify

$$2\kappa\ \frac{\partial\psi}{\partial t} = \frac{\partial^2\psi}{\partial x^2} + 2\ \psi^2\widetilde{\psi}$$

$$2\kappa\ \frac{\partial\widetilde{\psi}}{\partial t} = -\frac{\partial^2\widetilde{\psi}}{\partial x^2} - 2\ \widetilde{\psi}^2\,\psi$$

for

$$2\kappa = \begin{cases} 1 & \text{or} \\ -i & \end{cases}$$

When

$$A, B, C\ \in\ SU\,(2)$$

then

$$2\kappa = -i$$

and

$$\begin{aligned} \widetilde{\psi} &= \text{complex conjugate of } \psi \\ \widetilde{\psi} &= \overline{\psi} \end{aligned}$$

The equation becomes

$$i\frac{\partial\psi}{\partial t} + \frac{\partial^2\psi}{\partial x^2} + 2\,|\psi|^2\,\psi = 0$$

3.3 Anti-SD, Yang form, Belavin solution

From *chiral magnetic vortical* after **Mason Woodhouse**

I find in the **book Mason Woodhouse integrability** page 36 a reference to the solution of **Belavin et al. to Self Dual .**

First, define the coordinates and write the AntiSD equations.

Double null coordinates

The metric is $(2,2)$

$$ds^2 = 2\left(dz\ d\widetilde{z} - dw\ d\widetilde{w}\right)$$

$$vol\quad \nu = dw\ \wedge\ d\widetilde{w}\ \wedge\ dz\ \wedge\ d\widetilde{z}$$

Coordinate vectors are

$$\partial_w\ ,\ \partial_z\ ,\ \partial_{\widetilde{w}}\ ,\ \partial_{\widetilde{z}}$$

Different choices

Euclidean real slice is obtained by taking [here $\overline{z}$ is the complex conjugate of z]

$$\begin{aligned}\widetilde{w} &= -\overline{w}\\ \widetilde{z} &= \overline{z}\end{aligned}$$

or

$$\begin{pmatrix}\widetilde{z} & w\\ \widetilde{w} & z\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}x^0+ix^1 & -x^2+ix^3\\ x^2+ix^3 & x^0-ix^1\end{pmatrix}$$

The **Minkowski** real slice

$$\begin{aligned}z &= \text{real}\\ \widetilde{z} &= \text{real}\\ \overline{w} &= \widetilde{w}\end{aligned}$$

$$\begin{pmatrix}\widetilde{z} & w\\ \widetilde{w} & z\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}x^0+x^1 & x^2-ix^3\\ x^2+ix^3 & x^0-x^1\end{pmatrix}$$

The gauge potential function

$$\begin{array}{ll}\Phi_a & \text{gauge potential}\\ \text{non-Abelian} & \end{array}$$

then

$$F = F_{ab}\ dx^a \wedge\ dx^b$$

$$F_{ab} = \partial_a\Phi_b - \partial_b\Phi_a + [\Phi_a,\Phi_b]$$

where

$$a,b = w, z, \widetilde{w}\ ,\ \widetilde{z}$$

Write

$$D_w = \partial_w + \Phi_w$$

$$D_z = \partial_z + \Phi_z$$

$$D_{\widetilde{w}} = \partial_{\widetilde{w}} + \Phi_{\widetilde{w}}$$

$$D_{\widetilde{z}} = \partial_{\widetilde{z}} + \Phi_{\widetilde{z}}$$

The condition of ASD

$$\begin{aligned}
[D_z \ , \ D_w] &= 0 \\
[D_{\widetilde{z}} \ , \ D_{\widetilde{w}}] &= 0 \\
[D_{\widetilde{z}} \ , \ D_z] - [D_{\widetilde{w}} \ , \ D_w] &= 0
\end{aligned}$$

The Lax pair of operators

$$L = D_w - \zeta \ D_{\widetilde{z}}$$

$$M = D_z - \zeta \ D_{\widetilde{w}}$$

should commute for every value of the *spectral parameter* ζ.

For the group

$$SL\left(2, \mathbf{C}\right)$$

the **Belavin et al**. solution of the YANG form of the ASD equations

$$\Phi_w = \frac{f}{2} \begin{pmatrix} -\widetilde{w} & 0 \\ -2\widetilde{z} & \widetilde{w} \end{pmatrix}$$

$$\Phi_{\widetilde{w}} = \frac{f}{2} \begin{pmatrix} w & -2z \\ 0 & -w \end{pmatrix}$$

$$\Phi_z = \frac{f}{2} \begin{pmatrix} -\widetilde{z} & 2\widetilde{w} \\ 0 & \widetilde{z} \end{pmatrix}$$

$$\Phi_{\widetilde{z}} = \frac{f}{2} \begin{pmatrix} z & 0 \\ 2w & -z \end{pmatrix}$$

where

$$f = \frac{1}{1 - w\widetilde{w} + \widetilde{z}z}$$

In **McLerran Mottola Shaposhnikov** we have

$$A_\mu^a = \frac{1}{g} \frac{\eta_{\mu\nu}^a \ x^\nu}{\lambda^2 + \mathbf{x}^2 + \tau^2}$$

4 Systems

4.1 Calogero-Sutherland-Moser

For the Calogero system of N particles the Hamiltonian is

$$H = \sum_{i=1}^{N} \left(\frac{p_i^2}{2} + W(x_i) \right) + \sum_{j>i}^{N} V(x_j - x_i)$$

where W is an external field potential and V is the potential of the mutual interaction.

The system is completely integrable if one can find N involutory constants of motion (one of them being the Hamiltonian). The system is integrable for this choice of external and mutual potentials:

$$\begin{aligned} V(x) &= \frac{a}{x^2} \\ W(x) &= \gamma_1 x^4 + \gamma_2 x^2 + \gamma_3 x \end{aligned}$$

A connection has been found between the Calogero-Moser model given by this Hamiltonian *and* these potentials and the Burgers-Hopf equation.

See **Inozemtsev et al.**

The paper by **Chen Lee Pereira internal wave solitons** .
It is solved the equation *Benjamin Ono.*

$$2\frac{\partial q}{\partial t} + 2q\frac{\partial q}{\partial x} + H\left\{ \frac{\partial^2 q}{\partial x^2} \right\} = 0$$

where

$$\begin{aligned} H\{q(x)\} &= P\,\frac{1}{\pi}\int_{-\infty}^{\infty} d\xi \frac{q(\xi)}{\xi - x} \\ &\quad \text{Hilbert operator} \end{aligned}$$

the solution of the **benjamin Ono equation** is based on the motion of a set of *pairs* of poles complex conjugated.

The solution is obtained by connection with the Calogero-Moser problem.

There is a matrix, see above. We intended to take random this matrix to see the solution evolving.

4.2 Sigma model

The **Sigma model** in $1+1$ dimensions

There are NOTES on sigma models in Notes_ sci, Topological Theories, Notes.

Connection with $O(3)$ model, **Faddeev Hopf**.

This is in **9511002 construction Lax pairs**.
The model has the Lagrangian

$$\mathcal{L} = \int d^d x \, \mathrm{Tr}\left[\partial^\mu \left(g^{-1}\right) \partial_\mu g\right]$$

where g is a *group-valued* function defined on the Minkovsky space M^d

$$g \in G$$

The action has two global symmetries

$$\begin{aligned} g(x) &\rightarrow Lg(x) \\ g(x) &\rightarrow g(x) R \end{aligned}$$

where

$$L, R \in G$$

are *constant* (*i.e.* not dependent on the point x) group elements. From these symmetries there are two *Noether currents*

$$\begin{aligned} L_\mu &= g\partial_\mu \left(g^{-1}\right) \\ R_\mu &= g^{-1}\partial_\mu g \end{aligned}$$

These two *currents* are related by

$$R_\mu = -g^{-1} L_\mu g$$

They satisfy *zero-curvature* conditions

$$\begin{aligned} F_{\mu\nu}(L) &= \partial_\mu L_\nu - \partial_\nu L_\mu + [L_\mu, L_\nu] = 0 \\ F_{\mu\nu}(R) &= \partial_\mu R_\nu - \partial_\nu R_\mu + [R_\mu, R_\nu] = 0 \end{aligned}$$

The equation of motion for the σ model is

$$\Box g - \left[(\partial_\mu g)\left(\partial^\mu g^{-1}\right)\right] g = 0$$

The **LAX PAIR** for this equation is the system of *linear equations*

$$U^{-1}\partial_\mu U = \frac{1}{2}\left[(1-\cosh\lambda) L_\mu - \sinh\lambda \varepsilon_{\mu\nu} L^\nu\right]$$

where λ is the *spectral parameter*. The unknown function is $U \in G$ and the equations are written for each L fixed.

The *compatibility* condition is

$$[\partial_\mu, \partial_\nu] U = 0$$

and is satisfied for any L *pure gauge* with zero divergence (similar to the electromagnetic potential A_μ in the Coulomb gauge - no electrostatics $\mathbf{\nabla} \cdot \mathbf{A} = 0$)

The formal solution of the linear equations is

$$U = U_0 \mathbf{P} \exp \left\{ \int \frac{1}{2} \left[(1 - \cosh \lambda) L_\mu - \sinh \lambda \varepsilon_{\mu\nu} L^\nu \right] dx^\mu \right\}$$

(matricial solution).

Analogous *right* Lax pair or linear system can be constructed from the *right* current R_μ.

From the paper **stable and unstable solitary RLW**: the Lax definition that an equation has the property of *extended resolubility* if an initial perturbation is resolved into a set of solitons. This is possible if an initial function ψ has particular properties. If for a ψ there are

$$\begin{aligned} \text{wave speeds } c_i &> 1 \\ & \quad \text{phase translations } \ \theta_i(t) \\ i &< K \end{aligned}$$

such that if u is a solution of the equation corresponding to ψ, then

$$u(x,t) = \sum \phi_{c_i} \left(x - c_i t + \theta_i(t) \right) + r(x,t)$$

such that the remainder verifies

$$\lim_{t \to \infty} \sup_{x \in R} |r(x,t)| = 0$$

and the phase becomes constant

$$\lim_{t \to \infty} \theta_i(t) = \theta_i(\infty) = \text{constant}$$

NOTE
The stability of the soliton solutions is treated in this work.

From the paper **KPeqQuasiPeriodic**. Kadomtsev Petviashvili equation.
Two objects associated with integrability:

1. Riemann surface of genus g, with $2g$ cycles a_i and b_j;
2. Holomorphic differentials in number of g. The integrals of the g holomorphic differentials over the $2g$ *cycles* gives $2g^2$ numbers. The first g are used to normalize the differental forms and the other g form the *Riemann matrix*, whose real part is negative.

The Schottky problem in algebraic geometry: from all the Riemann matrices, which can be associated to a Riemann surface?

Krichever construction for KP

Riemann surface of genus g
$\rightarrow$Riemann matrix
$\rightarrow$*theta* function
$\rightarrow$ the parameters (k_j, l_j, w_j)

Krichever proves that the *theta* function θ is a solution of the KP equation.

Conclusion from Krichever: any Riemann surface generates a KP solution.

Conclusion from **Novikov/Shiota**: any solution of KP corresponds to a Riemann surface.

See **Calini Ivey** below.

4.3 Classical systems

The **Generalized Kuramoto Sivashinsky** equation is treated in **solitary and periodic nonintegrable**

$$u_t + 2u\ u_x + u_{xx} + 4u_{xxx} + u_{xxxx} = 0$$

using the Painleve property in terms of the singular manifold method.

It has the exact periodic solution expressed as

$$u(\xi) = \sum_{m=-\infty}^{\infty} u_s(\xi - 2m\delta)$$

where

$$\begin{aligned} u_s(\xi) &= 30a^2 \frac{(1+a)\exp(a\xi) + (1-a)\exp(2a\xi)}{[1+\exp(a\xi)]^3} \\ &= 30a^2 \left\{ \left[\sec h\left(\frac{a\xi}{2}\right)\right]^2 + \frac{d}{d\xi}\left(\left[\sec h\left(\frac{a\xi}{2}\right)\right]^2\right) \right\} \end{aligned}$$

The periodic solution becomes an expression of *cnoidal* terms

$$u = C_1 + C_2 [cn(b\xi)]^2 + C_3 \frac{d}{d\xi}\left([cn(b\xi)]^2\right)$$

The **Generalized Kuramoto Sivashinsky** equation is

$$\frac{\partial u}{\partial t} + u^r \frac{\partial u}{\partial x} + a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^3 u}{\partial x^3} + c\frac{\partial^4 u}{\partial x^4} + d\frac{\partial^5 u}{\partial x^5} = 0$$

The **KAWAHARA** equation describes the water waves with surface tension

$$\begin{aligned} a &= 0 \\ c &= 0 \\ r &= 1 \end{aligned}$$

Purely dispersive waves

$$\begin{aligned} & \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+b\frac{\partial^3 u}{\partial x^3} \quad \text{(until here it is KdV)} \\ & +d\frac{\partial^5 u}{\partial x^5} \qquad\qquad\qquad \text{(quintic term)} \\ = & \ 0 \end{aligned}$$

This is the KdV with quintic terms.

In the paper **9205110 tau func Krichever** it is mentioned the dispersionless KP equation, or the Khokhlov-Zabotinskaya equation

$$\frac{3}{4}\partial_{yy}u+\left(u_t-\frac{3}{2}u\ u_x\right)_x=0$$

and it is said that it is NOT a pure evolution equation.

4.4 Landau Lifshitz

The paper **bikbaev2014**

This is for ferromagnets

$$\begin{aligned} \frac{\partial \mathbf{S}}{\partial t} &= \mathbf{S}\times\frac{\partial^2 \mathbf{S}}{\partial x^2}+\mathbf{S}\times J\ \mathbf{S} \\ |\mathbf{S}|^2 &= S_1^2+S_2^2+S_3^2=1 \qquad \text{(like } O(3)\text{)} \end{aligned}$$

and

$$J\equiv diag\left(J_1,\ J_2\ ,J_3\right)$$

Take

$$\begin{aligned} J_1 &= J_2 \\ J &= diag\left(0,0,\varepsilon\right) \end{aligned}$$

There are two linear equations

$$\begin{aligned} \frac{\partial \Psi}{\partial x} &= U\ \Psi \\ \frac{\partial \Psi}{\partial t} &= V\ \Psi \end{aligned}$$

and the compatibility condition is

$$\frac{\partial U}{\partial t}-\frac{\partial V}{\partial x}+[U,V]=0$$

The explicit form of these operators U and V is based on definitions

$$\begin{aligned} w_1 &= w_2 = \sqrt{\lambda^2 - a^2} \\ w_3 &= \lambda \\ \text{for } \ a &= i\frac{\sqrt{\varepsilon}}{4} \end{aligned}$$

then

$$\begin{aligned} U &= -i\sum_{k=1}^{3} S_k \ w_k \ \sigma_k \qquad (\sigma_k \ \ \text{Pauli}) \\ V &= 2i\sum_{k=1}^{3} S_k \ w_1 \ w_2 \ w_3 \ w_k^{-1}\sigma_k - i\sum_{k=1}^{3}\left[\mathbf{S}\times\frac{\partial\mathbf{S}}{\partial x}\right]_k \ w_k \ \sigma_k \end{aligned}$$

5 Notes

The nonlinear differential equations can have, in principle, different classes of solutions. When we know a Lax pair of operators we can hope to generate a particular class of solutions, on periodic domains, based on the algebro-geometric procedure of integration. However, the two restrictions on this tentative are

- the systematic derivation of a Lax pair, as has been mentioned; even if we can find one by observation ingenuity, we should like to be able to derive them in general
- the *square eigenfunctions* f, g and h should have to be represented as finite polynomials of the spectral variable, p in the notation for sinh-Poisson equation. This is not sure in general and there is a condition which has been derived by Tracy, etc.

It must be interesting to examine the following problem: if the nonlinear equations derived from a condition of self-duality not only have a Lax pair but also have the property that the square eigenfunctions are polynomials in the spectral variable.

We **NOTE** the following similarity.
In applying the Hirota method for KdV, **Ablowitz Satsuma** write

$$\begin{aligned} u_t + 6u \ u_x + u_{xxx} &= 0 \quad \text{(the KdV eq.)} \\ u &= 2\left(\ln f_N\right)_{xx} \quad \text{(substitution, for } N \text{ soliton-solution)} \end{aligned}$$

On the other hand we get from our field-theoretical treatment of the second SD equation for the EULER case (the plasma case is very similar)

$$\begin{aligned} F_{+-} &= \left[\Psi^{\dagger}, \Psi\right] \\ \partial_+ a^* - \partial_- a &= \left[\Psi^{\dagger}, \Psi\right] \end{aligned}$$

and the right hand side can be written

$$F_{+-} = \Delta \ln \rho$$

We find that there is a possible connection between f_N and ρ:

$$f_N \sim \rho$$

However, what would be the meaning of

$$\rho = 1 + \exp(\xi_1) + \exp(\xi_2) + \exp(\xi_1 + \xi_2 + a_{12})$$

with $\xi_{1,2}$ linear combinations of the type

$$\xi = \alpha x + \beta y + \gamma t + \delta$$

The problem is where do we see something *similar* (we cannot expect identical) for the structure of ρ in FT?

It looks like a decomposition of ρ into objects that are similar with $\rho = \exp(\psi)$, with ψ a simple linear combination of terms. In FT we have

$$\rho = |\phi_1|^2$$

and has the nature of the *amplitude* of the process of creation of positive vorticity, since ϕ_1 is the coefficient of E_+. Then we have a sum over probabilities of increasing the vorticity along linear combinations of variables

$$\rho \sim \left|\phi_1^{(a)}\right|^2 + \left|\phi_1^{(b)}\right|^2 + p\left|\phi_1^{(a)}\right|^2 \left|\phi_1^{(b)}\right|^2$$

For example

$$\begin{aligned} & \left(\phi_1^{(a)} + \phi_1^{(b)}\right)^* \left(\phi_1^{(a)} + \phi_1^{(b)}\right) \\ = & \left|\phi_1^{(a)}\right|^2 + \left|\phi_1^{(b)}\right|^2 \\ & + \phi_1^{(b)*}\phi_1^{(a)} + \phi_1^{(a)*}\phi_1^{(b)} \end{aligned}$$

This suggests that there would be several scalar (matter) fields ϕ in the problem.

6 The method of scattering

From **solutions Camassa Holm Johnson 2003**.

Introduce a *potential*

$$Q(y;t)$$

and the Schrodniger equation

$$\frac{d^2\psi}{dy^2} - (Q-\mu)\,\psi = 0$$

for

$$\begin{aligned}
\psi &= \psi\,(y;t) \quad \text{and} \\
\mu &\equiv \text{spectral parameter}
\end{aligned}$$

Consider fixed t;
Find the scattering data.
Take a $Q\,(y; t=0)$.
Then find $Q\,(y,t)$ from the time evolution of the scattering data.

$$Q\,(y;t) = -2\frac{d}{dy}K\,(y,y;t)$$

where $K\,(y,x;t)$ is the solution of the Marchenko's equation

$$\begin{aligned}
& K\,(y,x,;t) + F\,(y+x;t) + \int_y^\infty dz\ K\,(y,z;t)\ \ F\,(z+z;t) \\
= \quad & 0
\end{aligned}$$

For reflectionless potentials

$$F\,(X;t) = \sum_{n=1}^{n} \exp\left\{-\frac{k_n}{\sqrt{\omega}}X - \frac{k_n}{\lambda_n}t + \alpha_n\right\}$$

Here

$$\begin{aligned}
\mu &= -\left(\lambda + \frac{1}{4\omega}\right) \\
\mu_n &= \frac{k_n^2}{\omega} \\
k_n &> 0
\end{aligned}$$

7 The Darboux transformation

From paper **Taimanov Tsarev two dimensional.**

Consider a Hamiltonian H for a one-dimensional Schrodinger problem

$$H = -\frac{d^2}{dx^2} + u\,(x)$$

and consider the zero-energy solution

$$H\omega = 0$$

The existence of a zero energy solution ω helps factorization of H as

$$H = A^T A$$

$$\begin{aligned} A &= -\frac{d}{dx} + v \\ A^T &= \frac{d}{dx} + v \end{aligned}$$

where

$$\begin{aligned} v &\equiv \frac{\omega_x}{\omega} \\ &= \frac{d}{dx} \ln \omega \end{aligned}$$

Verification

$$\begin{aligned} A^T A &= \left(\frac{d}{dx} + v\right)\left(-\frac{d}{dx} + v\right) \\ &= -\frac{d^2}{dx^2} + v^2 + \frac{dv}{dx} \end{aligned}$$

The equation

$$\frac{dv}{dx} + v^2 = u$$

is equivalent with

$$H\omega = 0$$

The DARBOUX transformation consists of the exchange of A^T and A,

$$H = A^T A \rightarrow \widetilde{H} = A A^T$$

or

$$u = v^2 + \frac{dv}{dx} \rightarrow \widetilde{u} = v^2 - \frac{dv}{dx}$$

If φ satisfies

$$H\varphi = E\varphi$$

with

$$E = \text{const}$$

then

$$\widetilde{\varphi} = A\varphi$$

is solution of

$$\widetilde{H}\widetilde{\varphi} = E\widetilde{\varphi}$$

8 Loop algebra

This part is also in *SURFACES FLOWS*.

A presentation is in the paper **schiff97camassaholm loopgroup**.

And in **Sultana** surfaces.

It is connected with **Dorfmeister Pedit Wu** method for the constant mean curvature surfaces. ("*The surface of the water*").

See **Magdalena Toda**.

The **book Pressley, Graeme Segal-Loop Groups**

8.1 The solution of the Camassa-Holm eq. using the loop group approach

This equation has *peakons* and *cuspons* and is useful for our study of **Extreme Events**, related to similarity between the extreme events and CUSPS.

There is a file *camassa-holm.tex* in *integrability, studies, camassa-holm.*

The aspect physics of CHM regime, meaning of terms in the $1D$ eq. is in *anomalies.tex*.

The aspect $n_+ n_-$ =const "no-blow-up but cusp" is in *FieldTh Model vortices*.

8.1.1 Introduction

The **Camassa Holm** equation with solutions: *peakons*.

$$\begin{aligned}
\frac{\partial u}{\partial t} &= 2\frac{\partial f}{\partial x}u + f\frac{\partial u}{\partial x} \\
\text{where} \quad u &= \frac{1}{2}\frac{\partial^2 f}{\partial x^2} - 2f
\end{aligned}$$

This is the physical system.

NOTE the second line seems a one-dimensional form of

$$\begin{aligned}
u &= \frac{1}{2}\frac{\partial^2 f}{\partial x^2} - 2f \\
&\rightarrow \Delta\psi - \frac{1}{\rho^2}\psi
\end{aligned}$$

which is the potential in geostrophic convection, Ertel's theorem, and occurs in the Charney Hasegawa Mima equation

$$\frac{\partial}{\partial t}\left(\Delta\psi - \frac{1}{\rho^2}\psi\right)$$

This is equal, in CHM equation with the advected vorticity

$$\left[\left(-\nabla\psi \times \widehat{\mathbf{n}}\right)\cdot\nabla\right]\Delta\psi$$

which in the present case is

$$[(-\boldsymbol{\nabla}\psi \times \widehat{\mathbf{n}}) \cdot \boldsymbol{\nabla}]\, \Delta\psi \rightarrow 2\frac{\partial f}{\partial x}u + f\frac{\partial u}{\partial x}$$

A possible association

$$\begin{array}{ccc} f & \rightarrow & \psi \\ u & \rightarrow & \Delta\psi - \frac{1}{\rho^2}\psi \\ 2\frac{\partial f}{\partial x}u + f\frac{\partial u}{\partial x} & \rightarrow & \text{(diamagnetic flow)} + [(-\boldsymbol{\nabla}\psi \times \widehat{\mathbf{n}}) \cdot \boldsymbol{\nabla}]\, \Delta\psi \end{array}$$

END NOTE.

It is Associated the system with "two times" t_0 and t_1,

$$\begin{aligned} \frac{\partial p}{\partial t_1} &= p^2 \frac{\partial f}{\partial t_0} \\ f &= \frac{p}{4}\frac{\partial}{\partial t_0}\left(\frac{1}{p}\frac{\partial p}{\partial t_1}\right) - \frac{p^2}{2} \end{aligned}$$

with

$$p > 0$$

(note

$$\begin{aligned} f &= \frac{1}{4}p\frac{\partial}{\partial t_0}\left(\frac{\partial \ln p}{\partial t_1}\right) - \frac{p^2}{2} \\ \frac{\partial \ln p}{\partial t_1} &= p\frac{\partial}{\partial t_0} f \end{aligned}$$

it exihibits a function $\frac{\partial \ln p}{\partial t_1}$ and an operator $p\frac{\partial}{\partial t_0}$. end)

This equation has a zero-curvature formulation

$$\frac{\partial Z_0}{\partial t_1} - \frac{\partial Z_1}{\partial t_0} = [Z_1, Z_0]$$

Where

$$Z_0 = \begin{pmatrix} 0 & \frac{1}{p} \\ \frac{p}{\lambda} + \frac{1}{p} & 0 \end{pmatrix}$$

$$Z_1 = \lambda \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} -\frac{1}{2p}\frac{\partial p}{\partial t_0} & 0 \\ -2f & \frac{1}{2p}\frac{\partial p}{\partial t_0} \end{pmatrix}$$

8.1.2 The Birkhoff factorization (or splitting)

Take

$$\varepsilon > 0$$

and define

$$\begin{aligned} \text{circle } C_0 &= \{|\lambda| = \varepsilon\} \\ &\quad \text{very small radius} \end{aligned}$$

and

$$\begin{aligned} \text{circle } C_\infty &= \left\{|\lambda| = \frac{1}{\varepsilon}\right\} \\ &\quad \text{very LARGE radius} \end{aligned}$$

Now define their reunion

$$C = C_0 \cup C_\infty$$

This set is composed of the circles with very small radius AND circles with very large radius.

This *reunion* does NOT mean there is a *pairing* of circles from the two families.

The set C simply has elements that consist of *circles* with extreme radius.

The *loop group* G is the group of smooth maps from the circles C to the group $SL(2, \mathbf{C})$

$$C \to SL(2, \mathbf{C})$$

Every element (map) of the *loop group* takes a circle $|\lambda|$ and obtains a 2×2 matrix with unit determinant and entries from $\mathbf{C}$.

(the matrix preserves its expression for all the points of the circle.

Here λ is a parameter and this radius of a circle $|\lambda|$ can vary between the two extreme values, given by ε and $1/\varepsilon$.

Take now a circle $|\lambda|$ =const, which is intermediate

$$\varepsilon < |\lambda| < \frac{1}{\varepsilon}$$

Consider the set of *analytic maps*

$$\left\{\varepsilon < |\lambda| < \frac{1}{\varepsilon}\right\} \to SL(2, \mathbf{C})$$

It is defined a subgroup from G,

$$G_+ \subset G$$

The maps from G_+ are boundary values of analytic maps defined above.

Now consider the set of analytic maps

$$\{|\lambda| < \varepsilon\} \cup \left\{ \frac{1}{\varepsilon} < |\lambda| \right\} \rightarrow SL(2, \mathbf{C})$$

These maps are denoted

$$S(\lambda)$$

Define the subgroup of G,

$$G_- \subset G$$

of maps $S(\lambda)$ which are boundary values of the analytic maps defined above. They should satisfy the following boundary conditions

$$S(\lambda = 0) = \begin{pmatrix} \frac{1}{\alpha} & 0 \\ \beta & \alpha \end{pmatrix}$$

$$S(\lambda = \infty) = \begin{pmatrix} \gamma & \sqrt{1+\gamma^2} \\ \sqrt{1+\gamma^2} & \gamma \end{pmatrix}$$

for some α, β, γ.

the group G has the property that, a map U from G can be factorized as

$$U = S^{-1} Y$$

where

$$S \in G_-$$

$$Y \in G_+$$

This is **Birkhoff factorization**.

8.1.3 The splitting of the algebra $\mathfrak{G}$ of G

Now, the *splitting of the Lie algebra* $\mathfrak{G}$ of G.

An element of the algebra

$$v \in \mathfrak{G}$$

has a Fourier decompositions on the two extreme circles

$$v = \sum_{n=-\infty}^{n=\infty} a_n \lambda^n \text{ on the circle } |\lambda| = \varepsilon$$

$$v = \sum_{n=-\infty}^{n=\infty} b_n \lambda^n \text{ on the circle } |\lambda| = \frac{1}{\varepsilon}$$

the coefficients of the expansion belong to the Lie algebra

$$a_n \quad , \quad b_n \in \mathfrak{sl}(2, \mathbf{C})$$

8.1.4 The mapping from the loop group G to solutions of the non-linear equation

This is done in the paper mentioned above for the Camassa Holm hierarchy.

There is a map from the elements of the loop group G to the solutions of the hierarchy of Camassa Holm.

How this mapping is constructed.

First define an affine manifold $\mathfrak{M}$ whose coordinates are the *time variables* of the hierarchy

$$...t_{-1}, t_0, t_1, ...$$

Define a differential one-form Ω on the manifold $\mathfrak{M}$ with values in the sub-algebra $\mathfrak{G}_+$

$$\Omega = \sum_{n=-\infty}^{n=\infty} \lambda^n \begin{pmatrix} 0 & 1 \\ 1+\frac{1}{\lambda} & 0 \end{pmatrix} dt_n$$

This one-form has the property

$$d\Omega = \Omega \wedge \Omega = 0$$

which is the condition of (Frobenius) integrability of a system like

$$dU(t) = \Omega U(t)$$

Here $U(t)$ is a mapping from the manifold $\mathfrak{M}$ to the group G. The solution is

$$U(t) = MU(0)$$

where

$$M = \exp\left[\sum_{n=-\infty}^{n=\infty} \lambda^n \begin{pmatrix} 0 & 1 \\ 1+\frac{1}{\lambda} & 0 \end{pmatrix} t_n\right]$$

which can be rewritten

$$\begin{aligned} M &= \cosh\left(z\sqrt{1+\frac{1}{\lambda}}\right)\mathbf{I} \\ &\quad + \sinh\left(z\sqrt{1+\frac{1}{\lambda}}\right)\begin{pmatrix} 0 & \frac{1}{\sqrt{1+\frac{1}{\lambda}}} \\ \sqrt{1+\frac{1}{\lambda}} & 0 \end{pmatrix} \end{aligned}$$

and

$$z \equiv \sum_{n=-\infty}^{n=\infty} \lambda^n t_n$$

Now, U, the solution of that matricial equation which depends on all times t_n of the hierarchy, is *splitted according to the Birkhoff factorization.*

$$U = S^{-1}Y$$

$$
\begin{aligned}
S &\in G_- \\
Y &\in G_+
\end{aligned}
$$

the assumed form of the solution, U, is replaced in the equation for u leading to

$$-dS\ S^{-1} + dY\ Y^{-1} = S\Omega S^{-1}$$

and projecting on the two components of the group G,

$$dS\ S^{-1} = -\left(S\Omega S^{-1}\right)_-$$

$$dY\ Y^{-1} = \left(S\Omega S^{-1}\right)_+$$

Introduce the new function

$$Z \equiv dY\ Y^{-1}$$

it results that it has the property

$$dZ = Z \wedge Z$$

and it can be written as an expansion

$$Z = \sum_{n=-\infty}^{n=\infty} Z_n dt_n$$

(since Z as it is defined, is a differential one-form defined on the manifold $\mathfrak{M}$). The equation above can be written in terms of these coefficients

$$\frac{\partial Z_n}{\partial t_m} - \frac{\partial Z_m}{\partial t_n} = [Z_m, Z_n]$$

these are zero-curvature equations.

From the equation

$$Z = dY\ Y^{-1} = \left(S\Omega S^{-1}\right)_+$$

one can obtain an explicit expression for the coefficients Z_n.

$$Z_n = \left[\lambda^n S \begin{pmatrix} 0 & 1 \\ 1+\frac{1}{\lambda} & 0 \end{pmatrix} S^{-1}\right]_+$$

Using the asymptotic forms of S for the two extreme circles, one can obtain the expressions for Z_n.

It is shown that the Z_n obtained this way are those from the zero-curvature formulation of the Associated Camassa Holm system.

References

[1] A paper on exact solutions for few particles.

[2] Inozemtsev and Meshcheryakov, the connection between Burgers-Hopf and Calogero-Moser Eqs.

[3] Fractional statistics of solitons in Calogero continuum.

[4] Magdalena Toda 0307270, math.

9 Notes on geometric-algebraic theory of integrability

NSEq

In *lasers, NSEq, (1) IST periodic, (2) stability*

This can be found in *plasma, models, self-modulation*.

It is copied here and then expanded.

Other sources

- **Grinevich** on the connection Vorticity Filament Equation and the NSEq, (**Hasimoto**). Isoperiodic deformations and un-pinching of eigenvalues on the spectral plane with increase of genus of the Riemann surface of the NSEq.
- **Calini Ivey**, on VFE.

9.1 Exact solutions of the Nonlinear Schrodinger Equation and the nonlinear stability problem

9.1.1 Introduction

The Nonlinear Schrodinger Equation can be solved *exactly* on an infinite spatial domain using the Inverse Scattering Transform. The first step is to represent the nonlinear equation as a condition of compatibility of two *linear* equations. This is achieved by finding a pair of linear operators (Lax operators) and noticing that for the first of them the eigenfunction problem is a Schrodinger equation leading to a quantum scattering problem. The unknown function of the NSE is the potential of the Schrodinger equation. Using the initial condition of the NSE the scattering data are determined at $t = 0$. The scattering data have simple time dependence and so they can be evloved in time to the desired moment. Returning from this new set of scattering data (to the potential that has produced it) is achieved by solving an integral equation (Gelfand-Levitan-Marchenko) and provides the solution of the NSE. In general this solution consists of solitons and "radiation".

The Inverse Scattering Transform on periodic domain [**?**], [**?**], [**?**] is a more powerful procedure since it reveals and takes advantage of the deep geometric and topological nature of the problem. The admissible solutions of the Lax eigenvalue problem can now only be periodic functions. Since after one period the change of any solution can only be linear (via the *monodromy* matrix) the periodic or antiperiodic functions must be found as eigenfunctions of this monodromy operator (matrix), under the constrain that the eigenvalues are complex of unit absolute value. This singles out a set of complex values of the spectral parameter λ (the formal eigenvalue in the Lax equation), the *main spectrum*. From this set, the particular values that make the two eigenfunctions to coincide are called *non-degenerate*. Here the squared Wronskian (which is a space and time invariant) has zeros on the complex λ plane. Then the non-degenerate λ's are singular points for the Wronskian. Removing the square-root indeterminancy one has to connect pairs of non-degenerate eigenvalues by cuts in the λ plane. The Wronskian becomes a hyperelliptic Riemann surface. The evolution of the unknown NSE solutions (called hyperelliptic functions $\mu_j(x,t)$) on this surface is as complicated as the original NSE equation. However, via the *Abel map* the Riemann surface is mapped onto a torus and on this torus the motion magically becomes *linear*. After obtaining the space-time dependence on the torus we need to come back to the original framework. This is called the Jacobi inversion and can be done exactly in terms of Riemann *theta* functions, giving at the end the exact solution of the NSE.

Not only the geometric approach is more clear but it also allows a treatment of the stability problem for the solutions of the NSE since it allows to trace the changes of the main spectrum after a perturbation of the initial condition. The following subsections provide a more detailed discussion of the IST of a periodic domain. The next section will discuss the stability. This information is available from the abundant literature on the IST and is only mentioned here in order to understand the mechanism governing the stability of the envelope of the ion wave turbulence. For this reason we will focus on the determination of the *main spectrum* and its role in the construction of the solution. The effective steps to be undertaken to obtain the solution will only be briefly described. We strongly recommend the lecture of Ref.([**?**]) on which this presentation is based.

9.1.2 The Lax operators and the main spectrum

The IST method starts by introducing a pair of linear operators, called the **Lax pair** (see [**?**]) which allow to express the nonlinear equation as a compatibility condition for a system of two linear equations. For the cubic Schrödinger equation

$$i\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial x^2}+2\left|u\right|^2 u=0 \tag{1}$$

the **Lax pair** of operators is

$$L\equiv\begin{pmatrix} i\partial_x & u(x,t) \\ -u^*(x,t) & -i\partial_x \end{pmatrix} \tag{2}$$

$$A \equiv \begin{pmatrix} i\left|u\right|^2 - 2i\lambda^2 & -u_x + 2i\lambda u \\ u_x^* + 2i\lambda u^* & -i\left|u\right|^2 + 2i\lambda^2 \end{pmatrix} \tag{3}$$

The action of the operators on a two-component (column) wave function $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ is given by the equations

$$L\phi = \lambda\phi \tag{4}$$

$$\frac{\partial \phi}{\partial t} = A\phi \tag{5}$$

and the condition of compatibility of these two equations $\phi_{xt} = \phi_{tx}$ is precisely the cubic Nonlinear Schrödinger Equation.

9.1.3 Solving the eigenvalue equation for the operator L

As in the "infinite-domain" IST, we have to solve the eigenvalue equation Eq.(4) using the initial condition for the function $u(x,t)$, $u(x,0)$. However, for the particular case of **periodic** solutions of the NSE, we must have $u(x+d,0) = u(x,0)$ where d is the length of the spatial period (*i.e.* actually L in the previous notation; however we use d in this and the next section, keeping L for the *Lax* operator). Fixing an arbitrary base point $x = x_0$ one considers the two independent solutions of the equation (4) which take the following "initial" values at $x = x_0$:

$$\phi(x_0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \widetilde{\phi}(x_0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{6}$$

The matrix $\Phi(x, x_0; \lambda)$ of solutions is given by

$$\Phi(x, x_0; \lambda) \equiv \begin{pmatrix} \phi_1(x, x_0; \lambda) & \widetilde{\phi}_1(x, x_0; \lambda) \\ \phi_2(x, x_0; \lambda) & \widetilde{\phi}_2(x, x_0; \lambda) \end{pmatrix} \tag{7}$$

i.e. this solution satisfies the equation

$$L\Phi = \lambda\Phi \quad \text{and} \quad \Phi(x_0, x_0; \lambda) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{8}$$

It can be seen that the Wronskian of $\left(\phi, \widetilde{\phi}\right)$ is the *determinant* of the matrix of fundamental solutions

$$W\left(\phi, \widetilde{\phi}\right) = \det(\Phi)$$

It can be shown that $\frac{\partial W}{\partial x} = 0$ *i.e.* the Wronskian is constant, which gives

$$\det(\Phi(x)) = \det(\Phi(x_0)) = 1$$

Using the initial condition for Φ one can solve the equation (8) and obtain $\Phi(x, x_0; \lambda)$; in particular we can obtain it at $x_0 + d$, *i.e.* after a period :

$$\Phi(x_0 + d, x_0; \lambda) = \begin{pmatrix} \phi_1(x_0 + d, x_0; \lambda) & \widetilde{\phi}_1(x_0 + d, x_0; \lambda) \\ \phi_2(x_0 + d, x_0; \lambda) & \widetilde{\phi}_2(x_0 + d, x_0; \lambda) \end{pmatrix} \tag{9}$$

This is called **the monodromy matrix** and is noted

$$M(x_0;\lambda) \equiv \Phi(x_0 + d, x_0;\lambda) \tag{10}$$

The monodromy matrix is the matrix of the fundamental solutions calculated for one spatial period. In general the *monodromy matrix* is the element of the monodromy group associated to a loop and to its homotopy class, for a marked point on a manifold (here the circle $S^1 \equiv [0,d)$. The linear monodromy group acts by replacing the elements of the vector column by *linear* combinations of the initial ones. For changes of the marked point x_0 the matrix has two invariants : **the trace** and **the determinant**:

$$[Tr\, M](x_0;\lambda) = [Tr\, M](\lambda) \equiv \Delta(\lambda)$$

$$\det M = 1$$

The function $\Delta(\lambda)$ is called **the discriminant** and is independent of x_0.

Then the discriminant

$$\Delta(\lambda)$$

is the *trace* of the Wronskian calculated on an initial x_0 and final-one-period later, $x_0 + d$

$$M(x_0;\lambda) \equiv \Phi(x_0 + d, x_0;\lambda)$$

which is a 2×2 matrix.

The discriminant $\Delta(\lambda)$ only depends on the spectral variable λ.

In other words, the *monodromy* itself only depends on the spectral variable λ.

Looking for Bloch (Floquet) solutions The values the discriminant $\Delta(\lambda)$ takes on the complex λ-plane control the monodromy and by consequence select the values of λ for which admissible eigenfunction of the Lax problem exist. In other words Δ governs the *spectral properties of the operator L*. To find under what conditions periodic solutions are possible, we construct the Bloch (or Floquet) solutions of the equation $L\phi = \lambda\phi$. The Bloch function has the property

$$\psi(x + d;\lambda) = e^{ip(\lambda)}\psi(x;\lambda) \tag{11}$$

where $p(\lambda)$ is the **Floquet exponent**. Like any other solution of the Lax eigenproblem ψ can be expressed as a linear combination of the two fundamental solutions ϕ and $\widetilde{\phi}$

$$\psi(x;\lambda) = A\phi(x;\lambda) + B\widetilde{\phi}(x;\lambda)$$

For the particular point x_0 we have, taking into account the "initial" conditions (6)

$$\psi(x_0;\lambda) = A\phi(x_0;\lambda) + B\widetilde{\phi}(x_0;\lambda) = \begin{pmatrix} A \\ B \end{pmatrix}$$

After one period d the function ψ is linearly modified by the monodromy matrix and, according to Eq.(11) is multiplied by a complex number of unit absolute value, *i.e.* by a phase factor $\exp(ip)$. We write $m(\lambda)$ for this number and look for those λ's where this is complex of modulus one

$$\begin{aligned} m(\lambda) \quad \rightarrow \quad & \exp[ip(\lambda)] \\ & \text{pure phase, } p \text{ is the Floquet} \end{aligned}$$

The regions of the λ plane where $m(\lambda)$ has modulus different of one correspond to unstable functions ψ. We write

$$\psi(x_0 + d; \lambda) = m(\lambda)\ \psi(x_0; \lambda) \tag{12}$$

$$M \begin{pmatrix} A \\ B \end{pmatrix} = m(\lambda) \begin{pmatrix} A \\ B \end{pmatrix} \tag{13}$$

The Bloch functions are invariant directions for the *monodromy operator* M acting in the base formed by the fundamental solutions of the Lax eigenproblem. The monodromy operator preserves these directions and multiplies the Bloch vector with $m(\lambda) = \exp(ip(\lambda))$.

$$\det(M - m) = m^2 - (Tr\, M)\, m + \det M = m^2 - \Delta(\lambda)\, m + 1 = 0$$

This gives

$$m^{\pm}(\lambda) = \frac{\Delta(\lambda) \pm \left(\Delta^2(\lambda) - 4\right)^{1/2}}{2}$$

In general $\Delta(\lambda)$ is an analytic function on the plane of the complex variable λ. The equation $\operatorname{Im}[\Delta(\lambda)] = 0$ is a single relation with two unknowns, $(\operatorname{Re}\lambda, \operatorname{Im}\lambda)$ and calculating one of them leaves the dependence on the other as a free parameter. This gives a curve in the plane and the real-λ axis is such a curve: for $\operatorname{Im}\lambda = 0$ we have $\operatorname{Im}\Delta(\lambda) = 0$.

We look at the variation of $\Delta(\lambda)$ on the complex plane to find the effect on $m^{\pm}(\lambda)$.

This means to see for whic λ's it is a pure phase factor $\exp[ip(\lambda)]$ and when it is not (where ψ is unstable, after a period it increases or decreases).

We **note** from the expression of $m^{\pm}(\lambda)$ that for

$$\Delta(\lambda) = \pm 2$$

the factor $m^{\pm}(\lambda)$ is 1 in absolute magntitude.

This means that the values of λ for which $\Delta(\lambda) = \pm 2$ ensure the periodicity with just multiplication of the Bloch function by the pure phase factor.

These values of λ are called *main spectrum.*

- Where the Floquet multiplier is a complex number of magnitude unity $|m^{\pm}(\lambda)| = 1$, the functions are only effected by a phase factor after one period. The Bloch functions (eigenfunctions of L) are stable under translations with d. The Bloch functions do not increase or decrease. The *real* λ axis is a region of stability. The region where Δ is real and

$$\Delta(\lambda) < 4$$

is the **band of stability**, the modulus of m being 1.

- When λ is such that the discriminant ($\Delta(\lambda) \equiv Tr(M)$) is equal to 4, the eigenvalues of the monodromy matrix, $m(\lambda)$, are ± 1, and the Bloch functions are *periodic* or *antiperiodic*. **This is the Main Spectrum** in λ and is noted $\{\lambda_j\}$. The main spectrum consists of a set of discrete values λ_j.
- for those λ which gives $|m(\lambda)| \neq 1$ the Bloch eigenfunctions are unstable. They grow of decrease after a period.

Consider the case where $\Delta(\lambda)$ is **complex**. For all values of the parameter λ on the complex plane, [with the exception of the Main Spectrum where $\Delta(\lambda) = \pm 2$], the eigenvalues of the monodromy matrix are distinct which means that the eigenfunctions Bloch are distinct and **Independent** (The **Wronskian** is different of zero).

$$\begin{aligned} \psi^+(x+d) &= m^+\psi^+(x) \\ \psi^-(x+d) &= m^-\psi^-(x) \end{aligned}$$

However we can only be intersted in points of the *main spectrum*, as they can provide admissible (periodic) eigenfunctions of the Lax operator. (1) There are points λ_j in the main spectrum where the two eigenfunctions Bloch are distinct and independent: these points are called *degenerate* (with the meaning that for two independent eigenfunctions there is only one eigenvalue). (2) There are points λ_j where the two eigenfunctions Bloch are NOT independent: these points are called *nondegenerate*. For such values of λ the Wronskian is zero.

The important class of initial conditions $u(x, t=0)$ with the property that there is only a *finite number of non-degenerate eigenvalues* λ_j is called **finite-band potential**.

9.1.4 The "squared" eigenfunctions

Starting from the initial condition $u(x,0)$ and leting it evolve in time according to the NSE, $u(x,0) \overset{NSE}{\rightarrow} u(x,t)$, the **discriminant** remains invariant. All the spectral structure obtained from the discriminant $\Delta(\lambda)$, *i.e.* the main spectrum, the stability bands, are invariant. The eigenvalues of the monodromy matrix λ_j are invariant.

9.1.5 The two Bloch functions for the operator L

Consider the two Bloch eigenfunctions of the operator L:

$$\psi^+ = \begin{pmatrix} \psi_1^+ \\ \psi_2^+ \end{pmatrix} \quad \text{and} \quad \psi^- = \begin{pmatrix} \psi_1^- \\ \psi_2^- \end{pmatrix}$$

and define the following functions

$$f(x,t;\lambda) \equiv -\frac{i}{2}\left(\psi_1^+\psi_2^- + \psi_2^+\psi_1^-\right) \tag{14}$$

$$g(x,t;\lambda) \equiv \psi_1^+\psi_1^- \tag{15}$$

$$h(x,t;\lambda) \equiv -\psi_2^+\psi_2^- \tag{16}$$

These functions have the property that they are periodic, $f(x+d,t;\lambda) = f(x,t;\lambda)$for all values of λ. Since f is composed of ψ^+ and ψ^- and these are Bloch solutions for the eigenvalue equation $L\varphi = \lambda\varphi$ we have

$$\frac{\partial f}{\partial x} = u^* g - uh \tag{17}$$

$$\frac{\partial g}{\partial x} = -2i\lambda g - 2uf \tag{18}$$

$$\frac{\partial h}{\partial x} = 2i\lambda h + 2u^* f \tag{19}$$

and similarly we write the time derivatives taking account of the compatibility condition which involves the operator A.

$$\frac{\partial f}{\partial t} = -\left(u_x^* + 2i\lambda u^*\right) g + i\left(-u_x + 2i\lambda u\right) h \tag{20}$$

$$\frac{\partial g}{\partial t} = 2i\left(|u|^2 - 2\lambda^2\right) g + 2i\left(-u_x + 2i\lambda u\right) f \tag{21}$$

$$\frac{\partial h}{\partial t} = -2i\left(|u|^2 - 2\lambda^2\right) h - 2i\left(-u_x^* + 2i\lambda u^*\right) f \tag{22}$$

It is shown in Ref.([**?**]) that the condition on the initial function $u(x,0)$ for there to be only a finite number of nondegenerate points in the main spectrum is equivalent to the requirement that f, g and h be finite-order polynomials in the parameter λ, which we take of degree $N+1$: $f(x,t;\lambda) = \sum_{j=0}^{N+1} f_j(x,t)\lambda^j$, $g(x,t;\lambda) = \sum_{j=0}^{N+1} g_j(x,t)\lambda^j$, $h(x,t;\lambda) = \sum_{j=0}^{N+1} h_j(x,t)\lambda^j$. From the Eqs.(17 - 19) it results however that g and h have degree N in λ.

One can check that the following combination is invariant in time and space and we note that it is actually the square of the Wronskian:

$$\frac{\partial}{\partial x}\left(f^2 - gh\right) = \frac{\partial}{\partial t}\left(f^2 - gh\right) = 0 \tag{23}$$

$$f^2 - gh = -\frac{1}{4}\left[W\left(\psi^+, \psi^-\right)\right]^2$$

The Wronskian only depends on the spactral parameter λ. We also know that for a subset of the *main spectrum*, the nondegenerate eigenvalues, the Wronskian is zero. Then the number of the nondegenerate eigenvalues is $2N+2$ (the degree of $f^2 - gh$ as a polynomial in λ) and we express the function $f^2 - gh$ as a polinomial with constant coefficients (not depending on x and t).

$$-\frac{1}{4}\left[W\left(\psi^+, \psi^-\right)\right]^2 = f^2 - gh \equiv P(\lambda) = \sum_{k=1}^{2N+2} P_k \lambda^k = \prod_{j=0}^{2N+2} (\lambda - \lambda_j) \qquad (24)$$

9.1.6 Product expansion of g and its zeros $\mu_j(x,t)$. Introduction of the *hyperelliptic* functions μ_j

The functions g and h are both of order N in λ. For g we will note the N roots by $\mu_j(x,t)$. The coefficient of λ^{2N+2} in $f^2 - gh$ is f_{N+1}^2. Due to Eq.(23) it is a constant which can be taken 1. Now since $f_{N+1} = 1$ the coefficient of λ^N in g is $iu(x,t)$. Written as a product

$$g = iu(x,t) \prod_{j=1}^{N} \left[\lambda - \mu_j(x,t)\right] \qquad (25)$$

By similar arguments we find that the coefficient of λ^N in h is $iu^*(x,t)$ and the zeros of h are $\mu_j^*(x,t)$. The functions $\mu_j(x,t)$ are called the *hyperelliptic functions*. It will be proved below that finding the *hyperelliptic* functions leads immediately to the solution $u(x,t)$.

Now we calculate the expression in Eq.(24) for $\lambda =$ a zero of the function g, *i.e.* λ is equal to *hyperelliptic function* $\mu_j(x,t)$:

$$f^2(x,t;\lambda = \mu_m) - gh = f^2(x,t;\mu_m) = P(\lambda = \mu_m)$$

or

$$f(x,t;\mu_m) = \sigma_m \sqrt{P(\mu_m)} \qquad (26)$$

Here the factor σ_m is a sheet index that indicates which sheet of the Riemann surface associated with $\sqrt{P(\lambda)}$ the complex μ_m lies on.

Let us calculate from Eq.(18) and Eq.(25) the derivative at x of μ_m:

$$\begin{aligned}\frac{\partial g(x,t;\lambda)}{\partial x} &= i\frac{\partial u(x,t;\lambda)}{\partial x} \prod_{j=1}^{N} \left(\lambda - \mu_j(x,t)\right) \\ &\quad -iu(x,t;\lambda) \sum_{j=1}^{N} \frac{\partial \mu_j(x,t)}{\partial x} \prod_{k=1, k\neq j}^{N} (\lambda - \mu_k(x,t))\end{aligned}$$

Replace here a zero of g: $\lambda = \mu_m$

$$\frac{\partial g(x,t;\mu_m)}{\partial x} = -iu(x,t;\mu_m) \frac{\partial \mu_m(x,t)}{\partial x} \prod_{k=1, k\neq m}^{N} (\mu_m - \mu_k(x,t))$$

On the other hand we have, from Eq.(18)

$$\begin{aligned}\frac{\partial g}{\partial x} &= -2i\mu_m g\left(x,t;\lambda=\mu_m\right)-2u\left(x,t;\lambda=\mu_m\right)f\left(x,t;\lambda=\mu_m\right)\\ &= -2uf\end{aligned}$$

then, since we have calculated $f\left(x,t;\lambda=\mu_m\right)$, Eq(26), we obtain

$$\frac{\partial\mu_m\left(x,t\right)}{\partial x}=\frac{-2i\sigma_m\sqrt{P\left(\mu_m\right)}}{\prod_{k=1,k\neq m}^{N}\left(\mu_m-\mu_k\left(x,t\right)\right)} \tag{27}$$

for $m=1,2,...,N$.

The coefficients of the term with λ^N in the equation (18) are matched and we obtain $iu_x=-2ig_{N-1}-2uf_N$ .Using Eq.(25) we find

$$\partial_x\ln u=2i\left(\sum_{j=1}^{N}\mu_j+f_N\right)$$

From the Eq.(24) we obtain the coefficient of the term λ^N, *i.e.* f_N

$$f_N=-\frac{1}{2}\sum_{k=1}^{2N+2}\lambda_k$$

Then from the preceding equation it results

$$\partial_x\ln u=2i\left(\sum_{j=1}^{N}\mu_j-\frac{1}{2}\sum_{k=1}^{2N+2}\lambda_k\right) \tag{28}$$

The same procedure is applied to the equation for g_t, (21) and we obtain

$$\frac{\partial\mu_j\left(x,t\right)}{\partial t}=-2\left(\sum_{m\neq j}\mu_m-\frac{1}{2}\sum_{k=1}^{2N+2}\lambda_k\right)\frac{\partial\mu_j\left(x,t\right)}{\partial x} \tag{29}$$

$$\begin{aligned}\partial_t\ln u &= 2i\left[\sum_{j>k}\lambda_j\lambda_k-\frac{3}{4}\left(\sum_{k=1}^{2N+2}\lambda_k\right)^2\right]\\ &\quad -4i\left[\left(-\frac{1}{2}\sum_{k=1}^{2N+2}\lambda_k\right)\left(\sum_{j=1}^{N}\mu_j\right)+\sum_{j>k}\mu_j\mu_k\right]\end{aligned} \tag{30}$$

We can see that the problem has been reformulated: from the equation for the function $u\left(x,t\right)$ the Lax-operators formulation leads to the problem for Bloch functions, ψ^+ and ψ^- and their Wronskian; then, using the “squared” eigenfunctions f, g and h we arrive at a formulation for the *hyperelliptic* functions $\mu_j(x,t)$. Finding $\mu_j(x,t)$ leads immediately to $u(x,t)$.

These operations have a significative geometric counterpart: every point of the complex plane of the spectral parameter λ is mapped, via Eq.(24) on the two-sheeted Riemann surface $\sqrt{-\frac{1}{4}W(x,t;\lambda)} = \sqrt{P(\lambda)}$ with singularities at the *nondegenerate* points of the main spectrum, λ_j. Removing the indeterminancy (by cutting and glueing the two sheets) we obtain a **hyperelliptic Riemann surface** of genus $g = N$. It results that we have to consider the variables $\mu_j(x,t)$, $j = 1, N$, as points on this surface but the motion of $\mu_j(x,t)$ is not simpler than the equation itself.

Now the next steps would be: (1) using the initial condition $\mu_j(0,0)$ on the function $\mu_j(x,t)$ we can solve the two equations (27) and (29) and find $\mu_j(x,t)$; (2) then using the initial condition $|u(0,0)|$ for the function $u(x,t)$ we can solve the equations (28) and (30). The procedure will be:

- Take the parameters $\{\lambda_j | \, j = 1, 2, ..., 2N+2\}$ as known; these parameters are **non-degenerate eigenvalues** from the main spectrum, they can be found from the equation $\Delta(\lambda) = \pm 2$ and Δ is determined from $u(x,0)$, the initial condition.
- Choose initial conditions $\mu_j(0,0)$ and $|u(0,0)|$ (see below)
- solve the equations for $\mu_j(x,t)$; this means to find the *hyperelliptic functions*. Then solve the equations for $u(x,t)$.

9.1.7 Solution: the two-sheeted Riemann surface (hyperelliptic genus-N Riemann surface)

The variable μ_j's are points lying on the two-sheeted Riemann surface associated with

$$\sqrt{P(\lambda)} = \left(\prod_{k=1}^{2N+2} (\lambda - \lambda_k) \right)^{1/2} \equiv R(\lambda)$$

which has **branch cuts at each of the nondegenerate points** λ_k. To eliminate the indeterminacy related to the square-root singularities (located at λ_j's) the surfaces must be cut and reglued, obtaining a complex manifold of complex dimension one, a hyperelliptic Riemann surface. We need some constructions on this surface.

9.1.8 Holomorphic differential forms and cycles on the Riemann surface

On this hyperelliptic Riemann surface (denoted M) it is possible to define N linearly independent holomorphic (regular) differentials. The following is the canonical basis of differentials one-forms defined on the manifold M:

$$dU_1 \equiv \frac{d\lambda}{R(\lambda)}$$

$$dU_2 \equiv \frac{\lambda\, d\lambda}{R(\lambda)}, \quad ...$$

$$dU_j \equiv \frac{\lambda^j\, d\lambda}{R(\lambda)}, \ ...$$

$$dU_N \equiv \frac{\lambda^{N-1}\, d\lambda}{R(\lambda)}$$

On the surface M there are $2N$ topologically distinct closed curves (**cycles**). On a torus ($N = 1$, called *elliptic surface*) there are two curves which cannot be deformed one into another. A more general Riemann surface, with $N > 1$ can be shown to be equivalent to a sphere with N handles and is called *hyperelliptic Riemann surface*. The number N is called the *genus* of the surface. We have

$N =$ genus of the surface $=$ number of independent holomorphic differential forms

There are $2N$ cycles which are split into two classes: a_j cycles and b_j cycles. Each of these cycles has a specified direction (an arrow) attached to it. To construct the cycles, the rules to be applied are:

1. a_j cycles do not cross any other a_j cycle; b_j cycles do not cross any other b_j cycle;
2. the cycle a_k intersects b_k only once and intersects no other b cycle;
3. the intersection is such that, at the point of intersection, the vector tangent to the cycle a_k, the vector tangent to the cycle b_k and the normal to the tangent plane of M represent a system of three vectors compatible with the *orientation* of M.

This can be represented as:

$$a \circ a = 0\ ,\ \ b \circ b = 0\ ,\ \ a_j \circ b_k = \delta_{jk}$$

9.1.9 Periods and matrices of periods

With the set of holomorphic differential forms and the set of cycles we can define the **matrices of periods**, A and B

$$A_{kj} \equiv \int_{a_j} dU_k$$

$$B_{kj} = \int_{b_j} dU_k$$

and the matrices A and B are **invertible**. A change of basis of holomorphic differential forms is a linear transformation represented by the matrix C:

$$d\psi_j \equiv \sum_{k=1}^{N} C_{jk}\, dU_k$$

We can choose the matrix C as

$$C = A^{-1}$$

Then the periods in the new basis becomes

$$\int_{a_n} d\psi_j = \sum_{k=1}^{N} C_{jk} \int_{a_n} dU_k = \sum_{k=1}^{N} C_{jk} A_{kn} = \delta_{jn}$$

$$\int_{b_n} d\psi_j = \tau_{jn}$$

where

$$\tau = A^{-1} B$$

It can be shown that the matrix τ is symmetric $\tau_{jk} = \tau_{kj}$ and has a positive-definite imaginary part $\operatorname{Im} \tau > 0$.

9.1.10 The Abel map

The Abel map is defined from the Riemann surface M to the space C^N and associates to a hyperelliptic Riemann surface (a sphere with N hadles) a N-torus. Using the differentials $d\psi_j$ one constructs a change of variables by the following procedure:

- choose a base point on M and call it p_0;
- define the variables $W_j(x,t)$ as integrals on the Riemann λ-surface of the *differential forms* from the arbitrary point p_0 to the point which is the **hyperelliptic function** $\mu_k(x,t)$:

  $$\begin{aligned} W_j(x,t) &= \sum_{k=1}^{N} \int_{p_0}^{\mu_k(x,t)} d\psi_j \\ &= \sum_{k=1}^{N} \sum_{m=1}^{N} C_{jm} \int_{p_0}^{\mu_k(x,t)} \frac{\lambda^{m-1} d\lambda}{R(\lambda)} \end{aligned}$$

  These variables have the remarkable property that their x and t dependence is trivial

  $$\frac{d}{dx} W_j(x,t) = \sum_{k=1}^{N} \sum_{m=1}^{N} C_{jm} \frac{\mu_k^{m-1} \frac{d\mu_k}{dx}}{\sigma_k R(\mu_k)}$$

  and using

  $$\frac{d\mu_k(x,t)}{dx} = -2i\sigma_k \frac{R(\mu_k)}{\prod_{n\neq k} (\mu_k - \mu_n)}$$

  it results

  $$\frac{d}{dx} W_j(x,t) = \sum_{m=1}^{N} C_{jm} \sum_{k=1}^{N} \frac{-2i\mu_k^{m-1}}{\prod_{n\neq k} (\mu_k - \mu_n)}$$

To calculate the sum

$$\sum_{k=1}^{N} \frac{\mu_k^{m-1}}{\prod_{n\neq k}(\mu_k-\mu_n)}$$

one can use the contour integral

$$I_m \equiv \frac{1}{2\pi i}\int_C \frac{\lambda^{m-1}d\lambda}{\prod_{n\neq k}(\mu_k-\mu_n)}$$

where the contour C encloses all of the poles μ_n counterclockwise. By the residue theorem

$$\sum_{k=1}^{N} \frac{\mu_k^{m-1}}{\prod_{n\neq k}(\mu_k-\mu_n)} = I_m$$

One can evaluate this integral by compactifying the λ plane into a sphere and noticing that the contour encloses the pole at $z=\infty$. Evaluating the residuu at $z=\infty$

$$\sum_{k=1}^{N} \frac{\mu_k^{m-1}}{\prod_{n\neq k}(\mu_k-\mu_n)} = \delta_{m,N}$$

and from this we obtain

$$\frac{d}{dx}W_j(x,t) = -2iC_{j,N} \equiv \frac{1}{2\pi}k_j$$

A similar calculation gives

$$\begin{aligned}\frac{d}{dt}W_j(x,t) &= -4i\left[C_{j,N-1}+\left(\frac{1}{2}\sum_{k=1}^{2N+2}\lambda_k\right)C_{j,N}\right]\\ &= \frac{1}{2\pi}\Omega_j\end{aligned}$$

It results from these calculations

$$W_j(x,t) = \frac{1}{2\pi}(k_j x+\Omega_j t+d_j)$$

where d_j is a phase which is determined by the initial condition on μ_k.

The Abel map has *linearized* the motion of the points $\mu_k(x,t)$. Now, in order to determine the function $u(x,t)$ we must invert the Abel map, returning from the variables W to μ. This is the *Jacobi inversion problem*.

9.1.11 The Jacobi inversion problem

The return from the variables $W_j(x,t)$ of the N-torus back to the variables $\mu_k(x,t)$ of the Riemann surface M (notice we will only need particular combinations of the functions μ) can be done in an exact way using the Riemann θ function.

The argument of θ is the N-tuple of complex numbers $\overline{z} \in C^N$.

$$\theta\left(z|\tau\right) \equiv \sum_{m_1=-\infty}^{\infty} \cdots \sum_{m_N=-\infty}^{\infty} \exp\left(2\pi i \overline{m} \cdot \overline{z} + \pi i \overline{m} \cdot \tau \cdot \overline{m}\right)$$

with the notations $\overline{m} \cdot \overline{z} = \sum_{k=1}^{N} m_k z_k$ and $\overline{m} \cdot \tau \cdot \overline{m} = \sum_{k,j=1}^{N} \tau_{jk}\, m_j\, m_k$. Adding to the vector z a vector with a single nonzero element in the position k we obtain from the definition:

$$\theta\left(z + e_k|\tau\right) = \theta\left(z|\tau\right)$$

which means that the θ -function has N real periods**.** Adding the full k-th column of the matrix τ, we obtain:

$$\theta\left(z + \tau_k|\tau\right) = \exp\left(-2\pi i z_k - \pi i \tau_{kk}\right) \theta\left(z|\tau\right) \tag{31}$$

where τ_{kk} is the diagonal element of the $\underline{\tau}$ matrix.

There are two quantities which must be determined in order to have the explicit form of the solution. The space and respectively time equations for $u\left(x,t\right)$ depend on the quantity $\sum_{j>k} \mu_j \mu_k = \frac{1}{2}\left[\left(\sum_{m=1}^{N} \mu_m\right)^2 - \sum_{m=1}^{N} \mu_m^2\right]$. So we must determine the following combinations of the variables μ_j:

$$\sum_{m=1}^{N} \mu_m\left(x,t\right) \quad \text{and} \quad \sum_{m=1}^{N} \mu_m^2\left(x,t\right)$$

In order to calculate these two quantities, we shall start by introducing a series of functions , $\psi_j\left(p\right)$ defined on the Riemann surface M:

$$\psi_j\left(p\right) = \int_{p_0}^{p} d\psi_j$$

where p is a point on M , $d\psi_j$ is the j -th holomorphic differential form defined on M and p_0 is an arbitrary fixed point on M. The functions $\psi_j\left(p\right)$ are multiple valued since the contour is defined up to addition of any combination of the cycles on M.

Now consider the function, $F\left(p\right)$:

$$F\left(p\right) \equiv \theta\left(\psi\left(p\right) - K|\tau\right)$$

where θ is the N-dimensional Riemann *theta* function associated with the surface M; $\psi\left(p\right)$ is the N-dimensional column of complex functions $\psi_j\left(p\right)$; K is an N-dimensional column of complex numbers independent of the point p. The function $F\left(p\right)$ is multivalued on the Riemann surface M for the same reason as ψ: moving the point p on the surface such as to turn around one of the b cycle and returning to the initial position, the functions ψ_j will add elements of the τ matrix. This will make the function F to change as imposed by the properties of the θ function, shown in Eq.(31).

In order to render the function F single-valued, we replace its domain M by a new surface, obtained from M by dissecting it in a canonical fashion, along a basis of cycles. This new surface, M^* is simply connected and has the bord composed of a number of arcs equal to $4N$.

This operation is necessary in order to render the function $F(p)$ entire and allow us define the inegral

$$I_0 = \frac{1}{2\pi i} \int_{\partial M^*} d \ln F(p) \tag{32}$$

around the contour of the surface M^*. Applying Cauchy theorem, the integral is the *number of zeroes* of $F(p)$ on the surface M^*. This number is N (Riemann).

We now impose the condition that the N zeros of $F(p)$ coincide with the N points $\left(\mu_j(x,t), \sigma_j\right)$. This fixes the values of the N complex numbers K_j $(j = 1, ..., N)$. By doing so the contour integral (32) calculated with the reziduum theorem will involve the variables μ_j.

Calculation of the two combination of μ's We must remember that the complex λ plane is covered by the two-sheeted Riemann surface whose compact version is M. Further this is mapped by Abel map onto the Jacobi N-torus. To a variable on the Riemann surface M (formally also on M^*), say p, corresponds a certain $\lambda(p)$. One defines the following integrals, which are proved to be real constants :

$$I_1 = \frac{1}{2\pi i} \int_{\partial M^*} \lambda(p)\, d \ln [F(p)] \equiv A_1$$

$$I_2 = \frac{1}{2\pi i} \int_{\partial M^*} \lambda^2(p)\, d \ln [F(p)] \equiv A_2$$

They can be evaluated by the residuu theorem. By the choice of the constants K's, the zeroes of the of the function $F(p)$ are located at μ_j. Then the residues are just the integrand (λ) calculated in μ plus the residuu at the infinite, $\pm\infty$:

$$I_1 = \sum_{m=1}^{N} \mu_m + \underset{\lambda\to\infty^+}{\operatorname{Re}s} [\lambda(p)\, d \ln F(p)] + \underset{\lambda\to\infty^-}{\operatorname{Re}s} [\lambda(p)\, d \ln F(p)]$$

$$I_2 = \sum_{m=1}^{N} \mu_m^2 + \underset{\lambda\to\infty^+}{\operatorname{Re}s} \left[\lambda^2(p)\, d \ln F(p)\right] + \underset{\lambda\to\infty^-}{\operatorname{Re}s} \left[\lambda^2(p)\, d \ln F(p)\right]$$

The reason to write **two residues** at infinity is that $\lambda = \infty$ is not a branching point which means that there are **two points** on the manifold M corresponding to $\lambda = \infty$, one on each sheet of the surface. We note $r^{\pm}$ the value of $\psi(p)$ when p is the point on M^* corresponding to $\lambda \to \infty^{\pm}$.

The result is

$$A_1 = \sum_{j=1}^{N} \mu_j + \frac{i}{2} \frac{\partial}{\partial x} \ln \left[\frac{F(r^- - K)}{F(r^+ - K)}\right]$$

or

$$A_1 = \sum_{j=1}^{N} \mu_j + \frac{i}{2} \frac{\partial}{\partial x} \ln \left[\frac{F(r^- - K)}{F(r^+ - K)} \right]$$

with $\theta^{\pm} = F(r^{\pm} - K)$. The expression of I_2

$$A_2 = \sum_{j=1}^{N} \mu_j^2 - \frac{1}{4} \frac{\partial}{\partial x} \ln \left[F\left(r^+ - K\right) F\left(r^- - K\right) \right] + \frac{i}{4} \frac{\partial}{\partial t} \ln \left[\frac{F(r^- - K)}{F(r^+ - K)} \right]$$

Return to the equations for the function u With these expressions we come back to the two equations for the two partial derivatives of u.

$$\frac{\partial}{\partial x} \ln u = \frac{\partial}{\partial x} \ln \frac{F(r^+ - K)}{F(r^- - K)} + 2iA_1 - i \sum_{m=1}^{2N+2} \lambda_m$$

$$\frac{\partial}{\partial t} \ln u = \frac{\partial}{\partial t} \ln \frac{F(r^+ - K)}{F(r^- - K)} + i\text{const}$$

The solution Let us note

$$\omega_0 = Q$$

$$k_0 = 2A_1 - \sum_{m=1}^{2N+2} \lambda_m$$

which are "external" frequency and wavelength.

The solution is

$$u(x,t) = u(0,0) \exp(ik_0 - i\omega_0 t) \frac{\theta(W^- | \tau)}{\theta(W^+ | \tau)}$$

where

$$W_j^{\pm} = \frac{1}{2\pi} \left(k_j x + \Omega_j t + \delta_j^{\pm} \right)$$

The phases $\delta_j^{\pm}$ are the part of $r^{\pm} - K_j$ which is independent of (x,t).

9.2 Notes on the principal and auxiliary spectra

In the work by **birnir blow up of KdV existence beyond blow-up** the auxiliary spectrum is introduced without reference to the function g see above.

$$\begin{aligned} \frac{\partial \mu_k}{\partial t} &= \left(\sum_{i=0}^{2n} \frac{\lambda_i}{2} - \sum_{j \neq i}^{n} \mu_j \right) \frac{y_k}{\prod_{j \neq k} (\mu_k - \mu_j)} \\ k &= 1, ..., n \end{aligned}$$

where

$$y_k = \left(-\prod_{j=0}^{2n} (\mu_k - \lambda_j) \right)^{1/2}$$

When these functions are calculated the solution to the KdV eq is

$$u(x,t) = \sum_{j=0}^{2n} \lambda_j - 2 \sum_{k=0}^{n} \mu_k (x,t)$$

and the equation

$$\frac{\partial u}{\partial t} = -\frac{\partial^3 u}{\partial x^3} + 6u \frac{\partial u}{\partial x}$$

and the initial condition

$$\begin{aligned} u(x,0) &= u_0(x) \\ &= \sum_{j=0}^{2n} \lambda_j - 2 \sum_{k=0}^{n} \mu_k (x,0) \end{aligned}$$

10 Stability of the envelope solutions

In this Section we concentrate on a single NSE in order to examine the properties of stability of the solutions. The equation has the generic form

$$i \frac{\partial \phi}{\partial t} + \frac{\partial^2 \phi}{\partial x^2} + 2 |\phi|^2 \phi = 0$$

and represents the equation for the envelope of a fast oscillation arising as solution of the barotropic equation. On infinite spatial domain this equation has soliton as solutions. On a periodic spatial domain (which is our case) the solutions of this equations are modulationally unstable. The instability can be examined in the neighbourhood of a solution by imposing a slight deviation and studying its time behaviour by linearization. Consider $\phi(x,t)$ is a solution and perturb it at the initial moment $t = 0$ as

$$\phi'(x,t) = \phi(x,0) + \varepsilon h(x)$$

The question is find what are the relative behaviour of the two functions $\phi(x,t)$ and $\phi'(x,t)$. It they depart exponentially for time close to the initial moment, then $\phi(x,t)$ is an unstable solution. Suppose the initial solution $\phi(x,t)$ is the envelope of an exact plane wave *i.e.* it is constant in space. As a function of time it must have the form

$$\phi(x,t) = a \exp\left(2ia^2 t\right) \tag{33}$$

where a is a real constant. To this solution we add a perturbation

$$\phi'(x,t) = a\exp\left(2ia^2t\right)\{1+\varepsilon\left[A_1\exp\left(ikx-i\Omega t\right)+A_2\exp\left(-ikx+i\Omega t\right)\right]\}$$

where A_1 and A_2 are real coefficients. Linearising the equation for small ε we obtain:

$$\begin{aligned} & \left\{\left(2a^2+\Omega-k^2\right)\exp\left(ikx-i\Omega t\right)+2a^2\exp\left(-ikx+i\Omega t\right)\right\}A_1+ \\ & \left\{\left(2a^2-\Omega^*-k^2\right)\exp\left(-ikx+i\Omega t\right)+2a^2\exp\left(ikx-i\Omega t\right)\right\}A_2 \\ = \; & 0 \end{aligned}$$

Equating with zero the coefficients of the two functions of (x,t) we obtain a system of equations for A_1 and A_2. The compatibility condition gives us the dispersion relation

$$\Omega = \pm k\left(k^2-4a^2\right)^{1/2}$$

which shows that for

$$|k| < 2|a|$$

(*i.e.* for long wavelengths) the two solutions $\phi(x,t)$ and $\phi'(x,t)$ diverge exponentially in time and the function $\phi(x,t)$ is modulationally unstable.

For a particular class of equations (of which the Nonlinear Schrodinger Equation is an example) it is possible to follow the perturbed solution much further in time in an exact manner and, conversely, to return toward the initial time and to recover the resultas of the linear analysis. This is possible because the equation can be exactly solved for initial conditions starting arbitrary close to the envelope of the plane wave. For other types of initial solutions (different of the envelope of plane wave) the analysis can be more cumbersome and numerical methods may be necessary.

10.1 Modulation instability of the envelope of a plane wave

10.1.1 General method

Consider the solution of the NSE describing the spatially uniform envelope of a plane wave. This is the Eq.(33) and we shall take $a=1$,

$$\phi(x,t) = \exp(2it)$$

The spatial domain is of length $d=1$ after normalization.

The method developed by Tracy and Chen for examining the stability of the solutions of the NSE is based on the algebraic-geometric setting of IST on periodic domains and consists of the following steps.

- One starts with an **exact N-band solution** of the NSE (in particular, the envelope of the plane wave is the $N=0$ -band solution).

- It is calculated the **main spectrum** of this solution; since it is an N-band wave only $2N-2$ eigenvalues from the main spectrum can be nondegenerate;
- all other pairs of *degenerate eigenvalues* will be considered as potential degrees of freedom which have been “frozen out” by the **special choice of the initial condition**;
- under the perturbation of the initial condition the degeneracies will be removed and new degrees of freedom will become active;
- the effect on the stability of these new degrees of freedom will be examined by **constructing the new exact solution** of the NSE which includes many of the new degrees of freedom; not all degrees of freedom will be necessary to be considered; most of them has little effect and can be neglected.

10.1.2 Calculation of the main spectrum associated to the uniform amplitude solution

The unperturbed solution is the envelope of the plane wave, Eq.(33). To detremine the main spectrum, the solution is considered **initial contition** (at $t=0$) and introduced in the Lax operator. The eigenvalue problem of the Lax operator has the form

$$\begin{pmatrix} i\frac{\partial}{\partial x} & 1 \\ -1 & -i\frac{\partial}{\partial x} \end{pmatrix} \Phi = \lambda \Phi$$

This equation can have two independent one-column solutions corresponding respectively to the boundary conditions

$$\Phi_1 \left(x=0;\lambda \right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$\Phi_2 \left(x=0;\lambda \right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The solutions are:

$$\Phi_1 \left(x;\lambda \right) = \begin{pmatrix} \cos kx - \lambda i \frac{\sin kx}{k} \\ i\frac{\sin kx}{k} \end{pmatrix}$$

and

$$\Phi_2 \left(x;\lambda \right) = \begin{pmatrix} i\frac{\sin kx}{k} \\ \cos kx + \lambda i \frac{\sin kx}{k} \end{pmatrix}$$

where $k^2 = 1 + \lambda^2$.

The **fundamental solution matrix** Φ is a 2×2 matrix having as column the two above solutions.

From the fundamental solution matricx one can calculate **the monodromy matrix**

$$\begin{aligned} M(\lambda) &= \Phi(d;\lambda) \\ &= \Phi(x=1;\lambda) \end{aligned}$$

The **discriminant** is the **trace of the monodromy matrix**.

$$\Delta(\lambda) = 2\cos\left[d\left(\sqrt{1+\lambda^2}\right)\right]$$

We can proceed to the calculation of the *spectrum*.
In any discussion of spectrum

- the **monodromy matrix** and
- the **discriminant**,

are examined in the complex λ plane.

10.1.3 The spectrum of the unperturbed initial condition (case of plane wave)

The stability or instability of the Bloch fuctions (*i.e.* the values of the quantity $m(\lambda)$ which is the eigenvalue of the monodromy matrix) is decided by the *discriminant* $\Delta(\lambda)$.
We have

$$-2 \leqslant \Delta(\lambda) \leqslant 2$$

for all **real** λ and for $\lambda = i\alpha$ with $-1 \leqslant \alpha \leqslant 1$.
The **main spectrum** corresponds to the values of λ for which

$$\Delta(\lambda) = 2$$

or

$$\lambda_n^0 = \pm\left(\frac{n^2\pi^2}{d^2} - 1\right)^{1/2}$$

We conclude:

- for all $n \neq 0$ values, (*i.e.* $n = \pm 1, \pm 2, ...$) the **eigenvalues are doubly degenerate**; they are two values of λ with module different of 1; they can be real or complex;
- as long as d is finite only **a finite number of degenerate pairs will appear on the imaginary axis**; given d, for sufficiently high n, the quantity under the square root becomes positive;
- at $n = 0$ there are two eigenvalues $\lambda_0 = \pm i$. These eigenvalues are *nondegenerate* in the sense that **there is only one Bloch function for each of these values of** λ. This can be shown directly, constructing the Bloch functions form the matrix of fundamental solutions with $k = 0$ (because $\Delta(\lambda) = 2$ implies $k = 0$) and showing that only **one Bloch function** can be obtained.

10.1.4 The spectrum of the perturbed initial condition (case of plane wave)

Starting from an initial condition slightly different of the *plane wave's envelope*

$$u(x,0) \rightarrow u(x,0) + \varepsilon h(x) \tag{34}$$

we shall have to repeat all steps in the determination of the spectrum. This must be done in a **comparative** manner, identifying the degenerate eigenvalues λ_j of the **main spectrum** which becomes nondegenerate after perturbation. It may be useful to chose the form of the perturbation such as to be able to calculate exactly the new positions of the λ_j from the main spectrum.

The time behaviour of the exact solution developing from the perturbed initial condition is dominated by the imaginary part of the variables W which are obtained on the basis of the new (perturbed) quantities μ_j and this can be calculated. The rate of instability is given by this quantity.

Expressed in physical terms this will give the growth rate of the new solution, *i.e.* the rate of increase of the **asymmetry on the poloidal direction**.

Knowing the rate of growth of the asymmetry, we can estimate the asymmetry of the diffusion fluxes and further the rate of poloidal plasma spin-up. This should be compared with the rate of decay due to the magnetic pumping.

Unsing the new initial condition Eq.(34) we shall calculate the new **main** spectrum, *i.e.* the new values of the parameter λ for which the discriminant is ± 2. All the spectrum of the simple initial condition $u(x,0)$ is perturbed by oredr-ε quantities. The main effect of this perturbation is that the **degeneracies are broken**.

In order to construct the *exact* solution corresponding to the new position of the **nondegenerate** λ values, we need to specify the cycles on the two-sheeted Riemann surface. The steps are as usual:

- choose:
    1. the cycles on the Riemann surface;
    2. the base of the holomorphic differentials
- compute the matrices of periods
    1. the matrix of a periods, and
    2. the matrix of the b periods;
- compute the $\boldsymbol{\tau}$ matrix; (it appears a particular case: in the case of the plane wave, *i.e.* $u(x,t) = a \exp\left(2ia^2 t\right)$ it is very easy to compute the $\boldsymbol{\tau}$ matrix since $\boldsymbol{\tau}$ is the limit of B when $\varepsilon \rightarrow 0$;
- construct the Abel map, which maps the points of the compact two-sheeted Riemann surface onto the Jacobi torus; find the **wavenumbers** and the **frequencies** in the jacobi manifold;

- make the Jacobi inversion, using the **theta** functions;

Finding the base of the holomorphic differentials The new basis is found starting from the old, "unperturbed" base and it is called the "modulation basis". It is a linear combination of the **old basis**

$$dU_j = \frac{1}{2\pi i}\sqrt{1+\left(\lambda_j^0\right)^2}\frac{\prod_{m\neq j}\left(\lambda-\lambda_m^0\right)d\lambda}{R\left(\lambda\right)}, \text{ for } j=1,2,...,N$$

where

$$R\left(\lambda\right)=\sqrt{1+\lambda^2}\prod_{k=1}^{N}\left[\left(\lambda-\lambda_k^0-\varepsilon_k\right)\left(\lambda-\lambda_k^0+\varepsilon_k\right)\right]^{1/2}$$

To understand the choice of basis it is useful to look at the limit

$$\varepsilon \to 0$$

where the basis of holomorphic differential becomes

$$\lim_{\varepsilon\to 0} dU_j = \frac{1}{2\pi i}\sqrt{1+\left(\lambda_j^0\right)^2}\frac{1}{\sqrt{1+\lambda^2}}\frac{d\lambda}{\left(\lambda-\lambda_j^0\right)}$$

This expression shows that, at the limit $\varepsilon \to 0$, each differential "sees" **only one pole**, at λ_j (which is a *double point* at this limit), and the **square-root branch points** $\pm i$.

The loop (cycle) a_j encircles the pole λ_j^0 and the **period** is

$$\begin{aligned}\lim_{\varepsilon\to 0} A_{kj} &= \lim_{\varepsilon\to 0}\int_{a_j} dU_k = \int_{a_j}\lim_{\varepsilon\to 0} dU_k \\ &= \delta_{kj}\end{aligned}$$

This is because the cycle a_j surrounds the pole λ_j^0 or it does not surrounds any other pole.

We find that, under the *limit operation* $\varepsilon \to 0$,

- the matrix **A** goes over to the identity matrix, as $O\left(\varepsilon\right)$;
- the matrix **B** becomes

$$\lim_{\varepsilon\to 0} B_{jk} = \tau_{jk} + O\left(\varepsilon\right)$$

To see what happens with the **B** matrix in this limit, we must examine the *off-diagonal* terms and the **diagonal** terms.

The *off-diagonal* terms are well-behaved and can be integrated analytically:

$$\lim_{\varepsilon\to 0} B_{jk} = \int_{b_j} \lim_{\varepsilon\to 0} dU_k \quad \text{for} \quad j \neq k$$

and

$$\tau_{kj} = \frac{1}{2\pi i}\sqrt{1+\left(\lambda_j^0\right)^2}\int_{b_j}\frac{1}{\sqrt{1+\lambda^2}}\frac{d\lambda}{\left(\lambda-\lambda_k^0\right)} + O\left(\varepsilon\right)$$

The contour b_j passes through the middle of the segment relaying the two new λ_j and goes around the pole at $\lambda = -i$, for example. If this contour does not surrounds the pole at λ_k^0 then the integral is elementary. If the contour intersecting the cut from $-i$ to infinity encircles the pole at λ_k^0 then it may be chnaged such that to intersect the other cut, relaying the branch point $+i$ to infinity, and so it does not encircle any pole.

Conclusion Since the off-diagonal elements of the matrix $\boldsymbol{\tau}$ depend (to $0\left(\varepsilon\right)$) only of the positions of the original double points, which are determined only by the parameter d, it results that:

the off-diagonal terms of the $\boldsymbol{\tau}$ matrix does not carry any information on the *perturbed initial condition.*

Only the diagonal terms of the τ matrix are affected by the initial condition.

The diagonal terms of the matrix $\boldsymbol{\tau}$ The diagonal terms of the $\boldsymbol{\tau}$ -matrix are *singular* at the limit $\varepsilon \to 0$. Consider

$$\tau_{kk} = \frac{1}{2\pi i}\sqrt{1+\left(\lambda_j^0\right)^2}\int_{b_k}\frac{1}{\sqrt{1+\lambda^2}}\frac{d\lambda}{\sqrt{\left(\lambda-\lambda_k^0\right)^2-\varepsilon_k^2}} + O\left(\varepsilon\right)$$

Suppose that

λ_k^0 is on the **imaginary axis**

Then we have

$$\tau_{kk} = \frac{1}{2\pi i}\sqrt{1+\left(\lambda_j^0\right)^2}\left(2\int_{-i}^{\lambda_k^0}+\frac{1}{2}\int_{a_k}\right)\frac{1}{\sqrt{1+\lambda^2}}\frac{d\lambda}{\sqrt{\left(\lambda-\lambda_k^0\right)^2-\varepsilon_k^2}} + O\left(\varepsilon\right)$$

$$\tau_{kk} = \frac{1}{2} + \frac{1}{\pi i}\sqrt{1+\left(\lambda_j^0\right)^2}\int_{-i}^{\lambda_k^0}\frac{1}{\sqrt{1+\lambda^2}}\frac{d\lambda}{\sqrt{\left(\lambda-\lambda_k^0\right)^2-\varepsilon_k^2}} + O\left(\varepsilon\right)$$

The integral can be made with *elliptic functions* but an approximation is (near $\varepsilon_k = 0$)

$$\tau_{kk} = \frac{1}{2} - \frac{i}{\pi}\ln\left|\varepsilon_k\right| + O\left(1\right) + O\left(\varepsilon_k\right)$$

Consider now

λ_k^0 is on **real axis**

then the integral defining the diagonal elements of the matrix τ (the integral along the b_k cycles) is

$$\int_{-i}^{\lambda_k^0} = \int_{-i}^{0} + \int_{0}^{\lambda_k^0}$$

where the second integral is along the **real axis**.

The first integral is done with the residuu theorem after taking safely the limit $\varepsilon_k = 0$ and gives a finite **imaginary** contribution which is well-behaved for this limit. The second can be done and the result is

$$\tau_{kk} = -\frac{i}{\pi} \ln |\varepsilon_k| + iO(1) + O(\varepsilon_k)$$

We then conclude that for $\varepsilon_k \to 0$ the b - period matrix τ becomes logarithmically singular on the main diagonal.

We return to the main purpose of this analysis: the question to which we try to answer is what happens with the cycles and the periods when the small perturbation goes to zero $\varepsilon \to 0$ *i.e.* the eigenvalues of the main spectrum collapse to a simpler structure in the complex plane, specific to the main spectrum of the plane wave solution of the Nonlinear Schrodinger Equation.

As far as the solution is concerned, the limit from a slightly pertubed function (taken as initial condition) toward the plane-wave main spectrum ($\varepsilon \to 0$) has the following effect.

We have to look to the functions W_j which serves to define the arguments of the Jacobi *theta* function (necessary to make the Jacobi inversion of the Abel map).

$$W_j(x,t) = \sum_{k=1}^{N} \int_{p_0}^{\mu_k(x,t)} dU_j$$

where here we shall take as the **basis of the differential forms on the genus-N hyperelliptic Riemann surface** the basis defined before relative to the simple structure of the plane-wave spectrum.Then

$$W_j(x,t) = \frac{\sqrt{1+\left(\lambda_j^0\right)^2}}{2\pi i} \sum_{k=1}^{N} \int_{p_0}^{\mu_k(x,t)} \frac{\prod_{m\neq j}\left(\lambda - \lambda_m^0\right) d\lambda}{R(\lambda)}$$

Repeating the calculations which gives the space derivatives of the $W_j(x,t)$ functions as depending on the space derivatives of the functions $\mu_k(x,t)$ we obtain

$$\frac{\partial W_j(x,t)}{\partial x} = -\frac{1}{\pi}\sqrt{1+\left(\lambda_j^0\right)^2}$$

and since we use a special notation for this space-derivative, we have

$$k_j = -2\sqrt{1+\left(\lambda_j^0\right)^2}$$

Analogously

$$\frac{\partial W_j(x,t)}{\partial t} = \frac{-2\sqrt{1+\left(\lambda_j^0\right)^2}}{\pi}\left[\frac{1}{2}\sum_{k=1}^{2N}\lambda_k - \sum_{m\neq j}\lambda_m^0\right]$$

When going to the limit $\varepsilon \to 0$ most of the terms in the braket cancel each other remaining

$$\frac{\partial W_j(x,t)}{\partial t} = \frac{-2\sqrt{1+\left(\lambda_j^0\right)^2}}{\pi}\lambda_j^0 + O(\varepsilon)$$

Using again the standard notation we obtain

$$\Omega_j = -4\lambda_j^0\sqrt{1+\left(\lambda_j^0\right)^2} + O(\varepsilon)$$

Then we look to the relation between the "wavenumber" k_j and the frequnecy Ω_j:

$$\Omega_j = \pm k_j\sqrt{k_j^2-4} + O(\varepsilon) \quad \text{for } \; j = 1,\cdots,N$$

which is precisely the **linear "dispersion" relation** governing the stability of the small perturbations of the plane-wave solution.

The main conclusion is

The real axis degenaracies have Ω_j real so they are linearly stable.

The imaginary axis degeneracies have Ω_j imaginary so they are linearly unstable.

10.2 The case of non-plane wave initial condition

10.2.1 General procedure

The steps are similar as above.

We must start with a solution of the CNSE having a main spectrum composed of a finite number of **nondegenerate** eigenvalues (which will serve to define the *squared functions* and the Wronskian as a polynomial in the spectral parameter). In the main spectrum there are also a number on **degenerate** eigenvalues.

In order to define this situation as an **initial condition** for a problem of stability to a small perturbation, we organize as usual the complex plane of the spectral variable λ: make **branch cuts** on the complex plane between pairs of conjugate **non-degenerate eigenvalues**. These cuts must avoid the **degenerate** eigenvalues;

Compactify the spactral λ-plane to obtain a Riemann surface with handles.

Draw the cycles on the Riemann surface arising from the branched covering of the complex plane of the spectral surface. The **zeroth-order branch cuts and the basis curves (cycles)** represent the initial configuration, before any perturbation. To this configuration we can associate the **basis of the holomorphic differential forms** and the two matrices of **periods**.

When a perturbation is applied to the **initial function** the *degenerate eigenvalues are splitting* (and new degrees of freedom appear: particular case periodic deformation **Grinevich**). First of all when the degeneracies are splitting a new configuration of cycles is necessary. For each degenerate eigenvalue splitting into two eigenvalues, we need:

- a new branch cut (between these eigenvalues)
- two new cycles, one (a) encircling the pair and another (b) passing on the higher sheaf "through" this cut.
- renumber the eigenvalues of the main spectrum, such as after the already present non-degenerate eigenvalues, first appear the new non-degenerate eigenvalues that have a finite *imaginary* part (since they will be proved to be active in the instability) then the new non-degenerate eigenvalues that are on the real axis (which will be proved to not be unstable). New M degenerate band pairs are open up and the total number of **non-degenerate** eigenvalues is now $2(N+M+1)$;
- redefine and **extend** the basis of the holomorphic differential forms;
- calculate the two matrices of periods, A and B which are now of order $(N+M)\times(N+M)$. The structure of this matrix is

  $$\mathbf{A}^{((N+M)\times(N+M))} = \begin{pmatrix} \mathbf{A}^{0\ (N\times N)} & \mathbf{0}^{N\times M} \\ \mathbf{S}^{M\times N} & \mathbf{1}^{M\times M} \end{pmatrix}$$

  where $\mathbf{S}$ is the matrix of the *singular periods*. This is actually connected to the *vanishing cycles*.

10.2.2 Splitting of the degenerate eigenvalues and excitation of new degrees of freedom

10.2.3 The growth rate of the instability

The inverse of the matrix $\mathbf{A}$ is the matrix $\mathbf{C}=\mathbf{A}^{-1}$ which is the matrix whose elements are used for the definition of the *normalized differential forms* and then of the functions $W_j(x,t)$, arguments of the *theta* function. We have

$$\mathbf{C}^{((N+M)\times(N+M))} = \begin{pmatrix} \mathbf{C}^{0\ (N\times N)} & \mathbf{0}^{N\times M} \\ \mathbf{\Gamma} & \mathbf{1}^{M\times M} \end{pmatrix}$$

where the matrix $\mathbf{\Gamma}$ is an $M\times N$ matrix

$$\mathbf{\Gamma} = -\mathbf{S}\,\mathbf{C}^0$$

Concerning the problem of stability it is shown that the temporal behaviour of the functions $W_j(x,t)$ is determined by the reality conditions on $\mathbf{\Gamma}$.

$$\frac{\partial W_j(x,t)}{\partial t} = \frac{1}{\pi}\left(\frac{1}{2}\sum_{k=1}^{2N+2}\lambda_k\right)k_j - 4i\Gamma_{j-N,N-1} \quad \text{for} \quad N+1\leq j\leq N+M$$

Since the first term is real, the reality of the left hand side (which is Ω_j) is determined as

$$\gamma_j = 8\pi \left| \mathrm{Re}\, \Gamma_{j-N,N-1} \right|$$

It follows that the stability of the perturbation which opened the degenerate eigenvalues is governed by the elements of the matrix $\mathbf{\Gamma}$.

11 Calini Ivey: The deformations of the spectrum for suppression of a handle and change of the genus

This is **Calini Ivey**.

Saffman Local Induction Approximation (LIA) → VFEq → Hasimoto NSEq.
Part of this is in *lines strings vortices*.
Also Landau Lifshitz Gilbert Eq. in *Materials*.

11.1 The system for NSEq (from VFEq after Hasimoto)

This is

$$\mathcal{L}_1 \boldsymbol{\phi} = \lambda \boldsymbol{\phi}$$

and

$$\frac{\partial \boldsymbol{\phi}}{\partial t} = \mathcal{L}_2 \boldsymbol{\phi}$$

The NLSeq. is the *zero curvature* condition of compatibility of these two equations.

$$\mathcal{L}_1 = i\sigma_3 \frac{\partial}{\partial x} + \left(\begin{array}{cc} 0 & q \\ -\overline{q} & 0 \end{array} \right)$$

$$\mathcal{L}_2 = i \left(|q|^2 - 2\lambda^2 \right) \sigma_3 + \left(\begin{array}{cc} 0 & 2i\lambda q - \frac{\partial q}{\partial x} \\ 2i\lambda \overline{q} + \frac{\partial \overline{q}}{\partial x} & 0 \end{array} \right)$$

Here

$$\lambda \equiv \text{spectral parameter}$$

11.2 Reconstruction of the space line from the solution of the NSEq.

Sym Polhmeyer for inverse Hasimoto.

Grinevich.

Bobenko.

After calculation of a fundamental solution of the NSEq

$$\Phi(x, t; \lambda)$$

such that

$$\Phi(0, 0; \lambda)$$

is a fixed element of $SU(2)$, the operation for determination of the "inverse-Hasimoto" curve $\boldsymbol{\gamma}$ is

$$\boldsymbol{\gamma}(x, t) = \Phi^{-1} \frac{d\Phi}{d\lambda}\bigg|_{\lambda=0}$$

The quantity

$$\boldsymbol{\gamma}(s, t) \equiv (x, y, z)$$

is a 3-vector, representing the coordinates of the points of the curve, with s (noted x) the length along the line.

This $\boldsymbol{\gamma}(x, t)$ verifies the equation VFE

$$\frac{\partial \boldsymbol{\gamma}}{\partial t} = \frac{\partial \boldsymbol{\gamma}}{\partial x} \times \frac{\partial^2 \boldsymbol{\gamma}}{\partial x^2}$$

and is with q in relation

$$\begin{aligned} q(x, t) &= \frac{1}{2} \kappa \exp\left(i \int \tau \; ds\right) \\ \kappa &\equiv \text{curvature} \\ \tau &\equiv \text{torsion} \end{aligned}$$

This Sym Phlmeyer reconstruction formula is based on the identification

$$su(2) \sim \mathbf{R}^3$$

in which the Lie bracket corresponds to

$$-2 \times (\text{cross product})$$

This is explained for example in **Battye Sutcliffe** in connection with the Skyrme-Faddeev model. This is in *Notes_ sci*, *topological*.

We can consider a solution of the focusing NSEq.

Choose for example the cosh solution.

Fix the λ of this solution.

Insert the cosh solution and the λ in the AKNS operators $\mathcal{L}_1$ and $\mathcal{L}_2$.

Then solve the AKNS system to find two solutions

$$\phi_1 = \begin{pmatrix} \phi_1^u \\ \phi_1^d \end{pmatrix}$$

and

$$\phi_2 = \begin{pmatrix} \phi_2^u \\ \phi_2^d \end{pmatrix}$$

Then generate a matrix

$$\Phi = \begin{pmatrix} \phi_1^u & \phi_2^u \\ \phi_1^d & \phi_2^d \end{pmatrix}$$

and calculate the points of a curve

$$\boldsymbol{\gamma} = \Phi^{-1} \frac{\partial \Phi}{\partial \lambda}$$

Convert the $SU(2)$ function $\boldsymbol{\gamma}$ in a $\mathbf{R}^3$ vector field. This will be the curve corresponding to cosh.

Or, start with the most elementary solution of NSEq as in **Tracy**

$$\exp(it)$$

11.3 The Floquet spectrum of a finite gap solution.

The Floquet discriminant

$$\Delta(q;\lambda) = \mathrm{Tr}\left(\Phi(x+L,t;\lambda)\,\Phi^{-1}(x,t;\lambda)\right)$$

The product $\Phi(x+L,t;\lambda)\,\Phi^{-1}(x,t;\lambda)$ is the *transfer matrix* across one period L.

See also **Tracy**.

The matrix Φ is a 2×2 matrix whose columns are the fundamental solutions.

After a "tour" on the period L it changes.

In general one defines a *transfer matrix* $\Phi(x+L,t;\lambda)\,\Phi^{-1}(x,t;\lambda)$.

The transfer matrix has *eigenvalues* and eigenfunctions.

The *eigenvalues* of the transfer matrix are the Floquet multipliers.

These Floquet multipliers are

$$\rho^{\pm}(\lambda)$$

In particular each of the two fundamental solutions, after a complete period, can result as multiplied by a phase factor. Then $\rho^{\pm}$ will be exponentials.

The Floquet spectrum

$$\sigma(q) = \{\lambda \in \mathbf{C} | \ \Delta(q;\lambda) \in \mathbf{R}, \ -2 \le \Delta \le 2\}$$

is the set of spectral parameter values λ where the *eigenfunctions* of the AKNS system are bounded.

Why would them NOT be bounded ?

Why this condition of "bound eigenvalues" limits severily the *spectrum* and ponts to some sub-sets in the complex plane of λ ?

Essentially we expect that the *fundamental solutions* (and thir matrix Φ) , after a complete turn, will keep the same value (are periodic) except for some factors that are pure phases (exponentials of complex numbers). This means that the Floquet multipliers are $\rho^{\pm}(\lambda)$ pure phases.

This does not happen in every point or on a finite region on the complex λ plane, but on restricted sets.

- finite number of complex bands; the bands consist of limited curved lines in λ plane on which the λ's produce periodicity on Φ, with only phase factors.
- the points where Δ is at the limits of the Floquet spectrum,

$$\Delta(\lambda) = \pm 2$$

  - are *simple* points
  - or *multiple* points, as zeros of $\Delta = \pm 2$.

The *quasimomentum differential.*

For $\lambda \in \mathbf{C}$ the transfer matrix $\Phi(x+L,t;\lambda)\,\Phi^{-1}(x,t;\lambda)$ has two eigenvalues

$$\rho^{+}(\lambda)$$
$$\rho^{-}(\lambda)$$

$$\rho^{\pm}(\lambda) = \frac{\Delta(\lambda) \pm \sqrt{\Delta(\lambda)^2 - 4}}{2}$$

These eigenvalues are the *Floquet multipliers.*

The two functions $\rho^{\pm}(\lambda)$ are branches of a holomorphic function ρ defined on a two-sheeted Riemann surface Σ. For this two-sheeted Riemann surface one has a projection

$$\pi : \Sigma \to \mathbf{C}$$

and this projection is *branched* at the simple points λ's, which are the simple zeros of $\Delta = \pm 2$.

We would like the Floquet multipliers to be phase factors. For - at least, a weak periodicity.

Because we would like the function Φ to be periodic *up to phase factors*.

So we are interested in the situations where $\rho^{\pm}$ are *exponentials* of purely imaginary functions.

Take a point P on the Riemann suraface Σ and

$$\rho(P) = \exp[iL\Omega_1(P)]$$

The differential

$$d\Omega_1 = \frac{1}{iL} d\log\rho$$

is the *quasimomentum* differential.

From sheet to sheet Ω_1 changes by -1.

Each pair of the zeros of Ω_1 are projected to a single λ.

The result :

for the Vortex Filament to be a closed curve in space, it is necessary that, in the NSEq version of the theory,

$$\lambda = \Lambda_0$$

is a

- real double point
- a zero of the quasi-momentum differential $d\Omega_1$.

A double point which is real (it is on the real axis and is a double root of $\Delta(\lambda) = \pm 2$) is deformed into a new pair of branch points (un-pinching).

(thus increasing the genus of the Riemann surface by one).

11.4 The hyperelliptic Riemann surface is Σ

It is a complex one-dimensional manifold with a *surjective holomorphic map* π which is 2 to 1, branched at a finite number of points. These points are on the spectral plane, λ.

$$\begin{array}{ccc} \lambda_j & \in & \mathbf{C} \\ 1 & \leq & j \leq 2g+2 \end{array}$$

where

$$g \equiv \text{genus of } \Sigma$$

Then

$$\zeta^2 = \prod_{j=1}^{2g+2} (\lambda - \lambda_j)$$

The surface Σ is a two-point compactification of the set

$$(\lambda, \zeta) \in \mathbf{C}^2$$

where the additional points are mapped by π to

$$\lambda = \infty$$

There are two points

$$\begin{array}{l} \infty_+ \text{ and} \\ \infty_- \end{array}$$

according to

$$\frac{\zeta}{\lambda^{g+1}} \to +1 \text{ or } -1$$

On this surface the Floquet multiplier is defined, ρ.

The homology basis

$$a_1, ..., a_g$$

Each a_k encloses a complex-conjugate pair of branch points.

There is an additional cycle

$$a_0$$

which is homotopic to the linear combination of the others.

The differentials

$$\begin{aligned} & \frac{\lambda^k}{\zeta} d\lambda \\ \text{for } 0 \quad \leq & \quad k < g \end{aligned}$$

are holomorphic.

This means that in the neighborhood of any branch point and of the two points $\infty_\pm$ the differential can be written

$$f(w)\, dw$$

where

- the function $f(w)$ is holomorphic
- w is a coordinate in a patch around that point

But the differential forms $\left(\lambda^k/\zeta\right) d\lambda$ for higher k, $k \geq g$, the differential forms are *meromorphic* since there are poles.

Another essential component: the *set of cycles*

$$\begin{aligned} & a_k \ , \ b_k \\ k \quad = & \quad 1, g \text{ plus } a_0 \end{aligned}$$

a basis of the homotopy of the Riemann surface.

From the set of differentials $\left(\lambda^k/\zeta\right) d\lambda$ to the set of the normalized basis of differential one-forms.

The transition is made through the normalization

$$\oint_{a_k} \omega_j = 2\pi i \ \delta_{jk}$$

This must be supplemented with three meromorphic differentials

$$d\Omega_1 = \frac{\lambda^{g+1} - \frac{c}{2}\lambda^g + ...}{\zeta} d\lambda$$

$$d\Omega_2 = 4\frac{\lambda^{g+2} - \frac{c}{2}\lambda^{g+1} - d\ \lambda^g + ...}{\zeta} d\lambda$$

$$d\Omega_3 = \frac{\lambda^g + ...}{\zeta} d\lambda$$

with the definitions

$$c \equiv \sum_{j=1}^{2g+2} \lambda_j$$

(this is real, since $\lambda' s$ come in conjugate pairs)

$$d \equiv -\frac{c}{8} + \frac{1}{4} \sum_{j=1}^{2g+2} \lambda_j^2$$

The Abelian integrals

$$\Omega_i\left(P\right) = \int_{\lambda_{2g+2}}^{P} d\Omega_i$$

The Riemann matrix

$$B_{jk} = \oint_{b_k} \omega_j$$

has *negative definite real part.*

One considers separately the set of *columns* of the matrix B.

The *theta* function

$$\theta(\mathbf{z}) = \sum_{\mathbf{n}\in\mathbf{Z}^g} \exp\left[\mathbf{n}^T\left(\mathbf{z}+\frac{1}{2}B\mathbf{n}\right)\right]$$

where B are the columns of the matrix B_{ij}. And $\mathbf{n} \in \mathbf{Z}^g$ is a set of g integers.

The Abel map

$$A : \Sigma \to \mathbf{C}^g$$

$$\begin{aligned} A(P) &= \int_{P_0}^{P} \begin{pmatrix} \omega_1 \\ \dots \\ \omega_g \end{pmatrix} \\ &\in \mathbf{C}^g \end{aligned}$$

the finite-gap solution of the Nonlinear Schrodinger Equation

$$\begin{aligned} q(x,t) &= A \ \exp(-iEx + iNt) \\ &\times \frac{\theta(i\mathbf{V}x + i\mathbf{W}t - \mathbf{D} - \mathbf{r})}{\theta(i\mathbf{V}x + i\mathbf{W}t - \mathbf{D})} \end{aligned}$$

where

$$\mathbf{V}, \mathbf{W}, -\mathbf{r}$$

are vectors of the b-periods of the one-forms

$$d\Omega_1 \ , \ d\Omega_2 \ , \ d\Omega_3$$

The constants

$$\Omega_1(P) \sim \pm\left(\lambda - \frac{E}{2} + O(1)\right)$$

$$\Omega_2(P) \sim \pm\left(2\lambda^2 + \frac{N}{2} + O(1)\right)$$

$$\exp[\Omega_3(P)] \sim \pm\left(\frac{2i}{A}\lambda + O(1)\right)$$

when

$$P \to \infty_{\pm}$$

NOTE

There are differences relative to the treatment of **Tracy** where there is also an *auxiliary spectrum*, functions defined on the Riemann surface.

11.5 The idea of the *deformation*. Grinevich

This is in **9703020 Grinevich**.

Notes are in *lines strings vortices.*

One starts with a single pair of conjugate branch points
and add a new pair, enclosed by a new $a-$cycle, in the homology basis

Calini Ivey choose to keep a_0 as the cycle that encircles the first, primary, pair of conjugated branch points.

11.6 The period doubling bifurcation

In fluids one can see a closed vortex line that extends itself by doubling the length via a double cover of the circle. If it exists it should be seen in cross section of a smoke ring, as a breaking out and separation of two vortices.

This is similar to the period-doubling bifurcation in nonlinear systems. **Ott**.

12 Change of Genus of the Riemann surface

Calini Ivey, Grinevich, preservation of periodicity.

Eynard birth of a cut.

optics, effect through the NSEq for lasers

13 Singularities

The paper **bettelheim zabrodin wiegmann hele shaw.**

Separation (detachment) of a blob from a finger.

They refer to **Gurevich Pitaevskii** on KdV

$$\frac{\partial u}{\partial t_3} = \frac{3}{2} u \frac{\partial u}{\partial t_1} + \varepsilon \frac{\partial^3 u}{\partial t_1^3}$$

[**note** the presence of two time variables, instead of x, t]

with *step-like* initial condition on t_1,

$$\begin{aligned} u\left(t_1 \to -\infty\right) &= u_0 \\ u\left(t_1 \to +\infty\right) &= 0 \end{aligned}$$

The system at $t_3 = 1$ the solution $u\left(t_3, t_1\right)$ is smooth, at $t_3 = 0$ is infinite slope, at $t_3 = -1$ is overhang.

The solution of KdV

$$
\begin{aligned}
u\left(t_3,t_1\right) &\approx 2\alpha\; dn^2\left[\frac{5\sqrt{\alpha}}{12\sqrt{6}\varepsilon}\left(t_1+Vt_3\right),\; m\right] \\
&\quad +\gamma
\end{aligned}
$$

where α, m, V, γ all depend on the two times.

There is a *link between modulated periodic solutions of the KdV equation and the growth of planar domains* in the Hele Shaw problem.

The periodic solutions of the nonlinear integrable equation are represented by the *spectral curve*, a Riemann surface.

For KdV the spectral curve is a hyperelliptic curve

$$
\begin{aligned}
y^2 &= R_m\left(z\right) \\
m &= 2l+1
\end{aligned}
$$

where

$$
\begin{aligned}
R_m &\equiv \text{ polynomial of odd degree} \\
&\quad \text{with REAL roots}
\end{aligned}
$$

The spectral curve that corresponds to the solution given above (of KdV) is a polynomial of degree

$$
m=5
$$

A *nonperiodic solution is described by a slowly varying spectral curve.*

A shock wave is a singular behavior that corresponds to a change of *genus* of the spectral curve. [**Comment** but change of genus occurs in **Calini Ivey** VFEq at un-pinching of eigenvalues. What about **Fournier** Burgers pole dynamics as interaction of strings ?]

They show that for Hele Shaw (interface dynamics) *an increase of the genus corresponds to bubble break-off since the interface is a real section of the curve when the coordinates* z and y are real. [useful for ejection of filaments/blobs from plasma]

The form of a generical finger is

$$
y^2=R_m\left(x\right)
$$

$$
\begin{aligned}
R_m &\equiv \text{polynomial with real coefficients} \\
\text{has at least one REAL root } x &= u \\
&\quad \text{ at the tip of the finger}
\end{aligned}
$$