1 Introduction skyrmions

Skyrmions are topological structures of fields that map the real base space to a compact internal space.

2 Introduction

From **Kharzeev chiral and vortical**

"the θ-vacuum is a superposition of an infinite number of topological states connected by tunneling instanton transitions"

θ is the coefficient of the Chern Simons term in the Lagrangian, the first term being Maxwell.

The confinement mechanism in QCD, proposed by t'Hooft.

A pair of magnetic monopole and anti-monopole are connected to form an Abrikosov vortex.

I.E. a tube (vortex) of magnetic flux.

Due to the Meissner effect the magnetic field is expelled from the rest of the superconductor.

[question: why Abrikosov ? - possibly because it is a vortex of Abelian-Higgs-type, ANO, in superconductors]

Dual system

the magnetic and the electric charges are interchanged

two charges of opposite sign are connected by a tube of electric flux. This flux is expelled from the bulk by an effect which is dual to Meissner: there is a condensation of magnetic charges in the bulk, similar to the Cooper pairs which are electrical. Then there is a dual Meissner effect expelling the electric flux from the bulk. The flux between the electric charges is compressed by an analog Meissner effect, with the "magnetic-cooperons" (which are two monopoles tied by a ANO vortex) repelling the electric field.

[**NOTE**. The tube of magnetic flux that connects two magnetic monopoles in the model of quark confinement is compressed from all sides by the repulsion of the superconductor which consists of cooperons = two electric charges coupled.

This looks like the

- vorticity sheet in atmosphere, strong atmospheric fronts **Ohkitani**, Δ^α
- current sheet **Biskamp**
- meandering stream **DiFrancesco**.

(in *plasma current sheet*).

Then two questions:

- in confinement by two magnetic monopoles tied by an ANO vortex does-it exist the meandering of the tube and the continuum thinning,
- do we have in general, for ANO vortex in superconductors, a description of the vortex acted upon by the Meissner repulsion ?]

Topological theories include *sigma* models, *Yang-Mills*, VLE, etc.

Very important lesson from *sphaleron*

the transition between two different topological sectors is NOT possible with continuous deformation of only pure gauge configurations, $U\partial_\mu U^{-1}$.

But it is possible if the sequence of configurations includes non-pure-gauge fields.

The example of **Volkov** consists simply of the rise in time, parametrized by a parameter h between the initial pure gauge state with topological degree 0 and the final pure gauge state with topological degree κ.

The path in the space of configurations does NOT consist of *only* solutions of the equations of motion.

[this aspect of *sphalerons* is essential: the initial and final state are connected simply by a path in the function space but not every state is solution.]

So, there is no miracle.

The amplitude starts from 0 and arrives at the final state, which has a $\neq 0$ topological degree.

It is NOT that the system has naturally (i.e. according to the equations of motion) has evolved.

This is only formal, since one can ask how topological degree is created.

2.1 Mathematical methods and instruments

Topological degree. Winding. Links and knots.

Homotopy. Coverings. Monodromy (like in the Integrability on periodic domains with Lax operators of order 2).

Homology and cohomology. Cohomology of differential forms, the De-Rahm cohomology; cohomology of Complexes of chains and co-chains. Complexes of simplexes. Cellular complexes (see Novikov geometry).

Characteristic classes. Chern, Pontryagin

Holonomy of a connection (gauge field A^μ) along a closed path, a loop. Leading to Wilson line operator.

Anholonomy is the appropriate name for the situation that are the end of the complete cycle along the loop, the object that was transported is not the same as the initial object.

Text of **Berry** in the **book Shapere Wilczek**.

2.2 Classical systems

Chiral model. See **Ward, Uhlenbeck**. *Harmonic mappings.*

[the file of Notes is *chiral model unitons*].

$O(n)$ model. Topological properties. Stereographic transformation and exact solution. **Ward**: the solitons shrink or expand (Derrick), **Zakrzewski**, numerical study. Terms that stabilize the solitons, Skyrme in $SU(2)$, Faddeev from Hopf. Moduli spaces. Skyrme. Fadeev-Niemi.

CP^n model. The topological equivalences:

$$\begin{aligned} \mathbf{R}^2 &\simeq S^2 \\ CP^1 &\simeq S^2 \end{aligned}$$

Two-dimensional Grassmannian model **Zakrzewski**, without the Skyrme or Faddeev term.

Sigma model.

The Thirring model.

The sine-Gordon model.

The bosonization. The Abelian bosonization (Thirring-Sine-Gordon). The Non-Abelian bosonization (Witten, Bralic).

The Coulombian model in $2D$.

Note the review of **Novikov, Shifman, Veinshtein and Zakharov**.

Instantons, Rajaraman. Shuryak QCD.

Particle on circle.

Particle in periodic potential.

[possible connection with the cycles on the Riemann surface of non-trivial genus. Every value on a cycle results from instanon transitions. They may be distributed as a Gaussian and this explains the Jacobi θ functions, solutions of the diffusion equation in imaginary time - or, the instantons are transitions in imaginary time. This is a *subject*].

3 Solitons, links knots Battye Sutcliffe

The only way to find a new solution for the Skyrme and Faddeev models is NOT to use restrictions based on symmetries, but to calculate numerically, from the equations.

3.1 The standard mapping $SU(2)$ to $\mathbf{R}^3$

A remark made by **Meissner** and cited,

a field with Hopf charge Q can be obtained by applying the standard Hopf mapping to a map between two S^3 spheres which has winding number Q.

Consider a mapping $U(\mathbf{x})$ from the space $\mathbf{R}^3$ to the group $SU(2)$,

$$\begin{aligned} U(\mathbf{x}) &= \text{Skyrme field} \\ U(\mathbf{x}) &: \mathbf{R}^3 \to SU(2) \\ \text{with } U(|\mathbf{x}| \to \infty) &= I \end{aligned}$$

By boundary condition it is possible to compactify the base space

$$\mathbf{R}^3 \to \text{compactified to} \to S^3$$

The topology of $SU(2)$ is S^3. Then

$S^3 \overset{U(\mathbf{x})}{\to} S^3$
topological degree of the Skyrme field, $B =$ Hopf charge
($B \equiv$ baryon number of the Skyrme field)

Now, $U \in SU(2)$ can be represented using two complex numbers

$$Z_0 \ , \ Z_1$$

as

$$\begin{aligned} U &= \begin{pmatrix} Z_0 & -\overline{Z}_1 \\ Z_1 & \overline{Z}_0 \end{pmatrix} \\ \text{with} \quad |Z_0|^2 + |Z_1|^2 &= 1 \end{aligned}$$

Now we use these objects to describe the *Hopf mapping.*
The standard Hopf mapping consists of the unit vector

$$\mathbf{n}$$

and we have to find a possibility to define $\mathbf{n}$ on the basis of the previous Skyrme field.
here it is

$$\mathbf{n} = Z^\dagger \boldsymbol{\tau} Z$$

It can be shown that

$$Q = B$$

In contrast to the **Hopf mapping** that maps $\mathbf{R}^3 \left(\sim S^3\right)$ to S^2 where we cannot say simply which is the analytical form of the mapping,
for the mapping of the **Skyrme** model, which maps $3-$sphere on $3-$sphere $S^3 \to SU(2) \left(\sim S^3\right)$, there is an analytical formula.
One first takes

$$U \in SU(2)$$

Then one choses a
- parameter f function
- a direction $\mathbf{v}$ in the algebra space of $SU(2)$

The expression for U is

$$U = \exp\left(if \ \mathbf{v} \cdot \boldsymbol{\tau}\right)$$

for the **Skyrme** mapping $\mathbf{R}^3 \to SU(2)$, and from this one finds expressions for the $\mathbf{n}$ mapping

$$\begin{aligned} n_1 &= 2\left(v_3 v_1 \sin f - v_2 \cos f\right)\sin f \\ n_2 &= 2\left(v_3 v_2 \sin f + v_1 \cos f\right)\sin f \\ n_3 &= 1 - 2\left(1 - v_3^2\right)\sin^2 f \end{aligned}$$

Ansatz
the function f only depends on r
the *direction in the algebra space,* $\mathbf{v}$, does NOT depend on radius r.

Then v depends on the angles (θ, φ).

It becomes a mapping from the real space (θ, φ), which is a sphere S^2, to the space of $\mathbf{n}$, which is also a sphere S^2.

Then $\mathbf{v}$ is a topological mapping $S^2 \to S^2$.

[For the following see below]

We now look for an expression for $\mathbf{v}$.

Introduce a complex coordinate in the physical space (only angles, without r),

$$\begin{aligned} (\theta, \varphi) &\to z \\ z &= \exp(i\varphi)\tan\left(\frac{\theta}{2}\right) \end{aligned}$$

Define a function $R(z)$, with which one describes the target sphere

$$\mathbf{v} = \frac{1}{1 + |R|^2}\left(R + \overline{R} \quad , \quad i\left(\overline{R} - R\right) \quad , \quad |R|^2 - 1\right)$$

Now, to generate a *configuration* for the field $\mathbf{n}$ one must simply choose some function $R(z)$, for example

$$R(z) = z^Q$$

The above calculation shows how to pass from the $SU(2)$ Skyrme model to the $O(3)$ model with unit versor $\mathbf{n}$.

Note

In **Battye Sutcliffe Proc Roy Soc 1999** it is shown how to pass from the **Skyrme** model where the target is *algebraic* $SU(2)$ to the model $O(3)$ with its unit vector $\mathbf{n}$.

They offer a formula that includes

- a profile function $f(r)$
- a direction in the space of the algebraic target space $SU(2) \sim S^3$, *i.e.* a unit vector $\mathbf{v}$.

We note that the $su(2)$ algebra has three generators E_-, H, E_+. Then any element of the algebra will have three components and this is *like* in the $3D$ space, *i.e.* as a vector. This is the direction in the algebraic target space.

This is

$$
\begin{aligned}
U &= \exp[if\ \mathbf{v}\cdot\boldsymbol{\tau}] \\
\boldsymbol{\tau} &\equiv \text{Pauli}
\end{aligned}
$$

The element U of the $SU(2)$ group is a matrix 2×2 with complex entries, with determinant $= 1$ and unitary.

We now look for the expression of this 2×2 matrix $U = \exp[if\ \mathbf{v}\cdot\boldsymbol{\tau}]$ in terms of two complex numbers.

Only *two* complex numbers are sufficient but since they come with *four* independent variables and since we have the $\det U = 1$, the number of independent variables is reduced to 3.

$$
\begin{aligned}
U &= \begin{pmatrix} z_0 & -z_1^* \\ z_1 & z_0^* \end{pmatrix} \\
\text{where } z &= \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} \\
\text{with} \quad |z_0|^2 + |z_1|^2 &= 1
\end{aligned}
$$

and the *standard Hopf map*

$$\mathbf{n} = z^\dagger\ \boldsymbol{\tau}\ z$$

It results

$$
\begin{aligned}
n_1 &= ... \\
& \quad \text{etc., in terms of } \mathbf{v} \text{ and } f
\end{aligned}
$$

[as above

$$
\begin{aligned}
n_1 &= 2(v_3 v_1 \sin f - v_2 \cos f)\sin f \\
n_2 &= 2(v_3 v_2 \sin f + v_1 \cos f)\sin f \\
n_3 &= 1 - 2\left(1 - v_3^2\right)\sin^2 f
\end{aligned}
$$

The expressions for $Q = 1$ are given below.]

See **Wu Zee** and **Hietarinta**.

3.2 Notes on compactification $\mathbf{R}^3 \to S^3$

This plays an important role and must be understood. Is-it a real change ? or it is a different expression?

The reference is here the *sterographic projection* in $2D$. It is the one-point compactification of the plane $\mathbf{R}^2$ into S^2 and can be made if the field defined on $\mathbf{R}^2$ has the same asymptotic value on all points of a circle of large radius.

NOTE

The compactification is done with only *topological* purposes.

It does NOT modify the *metric*.

The two spaces, $\mathbf{R}^3$ and S^3 are very different and no conclusion that depends on the metric can be transferred from one to the other.

Theories are different. Especially when one looks for the instability of a solution of the FS model that is unstable and leads to a concentrated soliton (how much we would need that in fluids; but all demonstrations - **Ward, Jackson** - works with S^3 not the real flat space).

END

Important note from **Manton Rybakov**

- there is *topological compactification*, which replaces $\mathbf{R}^3$ with S^3 only from the point of view of the topology, but does not affect the metric (in **Jackson Manton** the metric is used explicitely)
- there is simple replacement of $\mathbf{R}^3$ with S^3, motivated by the fact that solutions of the equations for the solitons are trivial in the real flat space

3.3 Notes and comments to Battye Sutcliffe

The **Faddeev** model.

$$\begin{aligned}\mathcal{L} &= \frac{\partial \mathbf{n}}{\partial x^{\mu}} \cdot \frac{\partial \mathbf{n}}{\partial x^{\nu}} \\ &\quad -\frac{1}{2}\left(\frac{\partial \mathbf{n}}{\partial x^{\mu}} \times \frac{\partial \mathbf{n}}{\partial x^{\nu}}\right) \cdot \left(\frac{\partial \mathbf{n}}{\partial x_{\mu}} \times \frac{\partial \mathbf{n}}{\partial x_{\nu}}\right)\end{aligned}$$

The boundary condition

$$\mathbf{n} = (0, 0, 1)$$

then

$$\mathbf{R}^3 \text{ (base space) is compactified to } S^3$$

and

$$S^3 \rightarrow S^2$$

and

$$\pi_3\left(S^2\right) = \mathbf{Z}$$

$$\begin{aligned}Q &\equiv \text{Hopf charge} \\ &= \text{soliton charge} \\ &= \text{linking number}\end{aligned}$$

Take the element of *area* on the target space, the space of internal symmetry of $\mathbf{n}^2 = 1$

$$\omega \equiv \text{element of area in } S^2$$

(it is the *solid angle* in S^2 **Nagaosa**) and

$$\begin{aligned} F &= \mathbf{n}^* \omega \ \in S^3 \\ &= \text{pullback of the area } \omega \\ &\quad \text{under the mapping } \mathbf{n} \end{aligned}$$

Note that this introduces the field F as the pullback of the area ω on S^2.
F is like the usual electromagnetic tensor.
Due to the triviality of the second cohomology group of the sphere S^3

$$H^2 \left(S^3\right) = 0$$

this pullback must be exact

$$F = dA$$

Therefore F is an exact differential form, which means it is obtained as external derivation of a differential form of lower degree. In this case it is question of a differential $1-$form A.

Note that this introduces the "gauge" potential A.

The potential A can be obtained from the field tensor F by integration (inversion of the equation $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$) , by Biot-Savart integral. Therefore, since we start from the *area* ω [solid angle] on the target sphere S^2 and take the pullback to find F, to further find A we make an operation which is NOT local.

Therefore we are NOT in the setting of the **Bogomolnyi** type Lagrangians.

As **Battye Sutcliffe** note, the fact that the energy bound that can be derived for the

$$S^3 \rightarrow SU\left(2\right)$$

model is NOT of the type Bogomolnyi, leads to a bound which is *fractionar*

$$E \geq c\ Q^{3/4}$$

This is the explanation that the lower bound is fractionar not topologically integer like for example $O\left(3\right)$ model.

NOTE what about **Niemi Semenoff** $N_R - N_L = \frac{e}{2\pi}\Phi$, where $\Phi \equiv$ magnetic flux, which also gives non-integer value? See *axial anomaly* text. **END**

We must note that this is a serious departure relative to Bogomolnyi as it is used in the $2D$ *Euler model and so it is a strong difference to the* $2D$ *matter* ϕ , A_μ *field and interaction of the Field Theoretical model for Euler.*

Therefore if we want to use Field Theory of mixed spinors in fluid $3+1$ dimensions, we must abandon the essential source that was present in $2D$: the point-like vortices interacting in plane by a self-generated potential.

The FT was just a description (reformulation) of the discrete model.

Then the Hopf charge is the integral of the Chern-Simons term over the base space $\mathbf{R}^3$,

$$Q = \frac{1}{4\pi^2} \int d^3x \; F \wedge A$$

3.4 Construction of the mappings with a given Hopf charge

The construction intends to generate functions (mappings) that have a given Hopf charge. It is NOT a solution of the : Skyrme, Faddeev or similar model, whose equations of motion are not solved in this way. But, as shown by **Battye Sutcliffe** the functions so constructed are taken as *initial states* for the numerical solution of the equations of motion, leading to solutions.

One starts from a mapping between S^3 spheres

$$S^3 \to S^3$$

that has winding number Q.

And it is possible to use this map to construct a Hopf field with topological charge Q.

For this, one starts from a Skyrme field, which is a mapping

$$\mathbf{R}^3 \to SU(2)$$

Taking the boundary condition

$$\begin{array}{rcl} U & \to & \mathbf{I} \\ \text{for } |\mathbf{x}| & \to & \infty \end{array}$$

this mapping is between

$$S^3 \text{ compactified } \mathbf{R}^3$$

and

$$S^3 \text{ the space of } SU(2)$$

The Skyrme field is

$$S^3 \to S^3$$

and has *winding number* B . This is the *baryon number* of the Skyrme model.

Now starts some transformations that will allow to use this Skyrme field $\left(S^3 \to S^3 \text{ mapping}\right)$ to build other mappings, Hopf.

Let us write the element of $SU(2)$ in terms of two complex numbers

$$U = \begin{pmatrix} Z_0 & -\overline{Z}_1 \\ Z_1 & \overline{Z}_0 \end{pmatrix}$$

Now, we know that $SU(2)$ to which U belongs, is a space of dimension 3. We introduce by Z_0 and Z_1 four numbers. But we ask, for the unitarity of U, that

$$|Z_0|^2 + |Z_1|^2 = 1$$

and this returns at 3 dimensions.

Now we construct a Hopf mapping, using this U represented in terms of Z_0 and Z_1.

Consider the column matrix

$$Z \equiv \begin{pmatrix} Z_0 \\ Z_1 \end{pmatrix}$$

With this we will define a versor $\mathbf{n}$ constructed as follows

$$n^a = Z^\dagger \ \tau^a \ Z$$

with τ^a the Pauli matrices.

The vector has unit length

$$|\mathbf{n}| = 1$$

and has a boundary value that is given by the boundary value of $U(\mathbf{x})$

$$\mathbf{n}(\infty) = \mathbf{n}_\infty \ \ \text{fixed, the same for all boundar } \mathbf{R}^3$$

This mapping between two 3-spheres,

$$S^3 \xrightarrow{\mathbf{n}} S^3$$

is a HOPF map.

It has a topological charge.

This topological charge Q (Hopf charge) is equal to the winding number B of the Skyrme field, which is the *bayon number*

$$Q^{Hopf} = B^{Skyrme} \ \ (baryon)$$

Conclusion until here: *one can use Skyrme fields* $\left[S^3 \rightarrow SU(2)\right]$ *to construct field configurations with Hopf charge,* $\left[S^3 \rightarrow S^3\right]$.

Therefore we need Skyrme fields.

There is a procedure to construct Skyrme fields (**Houghton**).

To construct Skyrme fields $S^3 \rightarrow SU(2)$ one must use *rational maps between Riemann spheres.*

A step of preparation: express the Skyrme field in terms of

f a profile function

$\mathbf{v}$ a vector in the $SU(2)$ space

Take

$$\begin{aligned} \mathbf{v} &\equiv (v_1, v_2, v_3) \\ |\mathbf{v}| &= 1 \end{aligned}$$

The matrix U of a Skyrme field is expressed as

$$U = \exp\left[i\ f\ \mathbf{v} \cdot \boldsymbol{\tau}\right]$$

We use this form to find the two complex numbers (Z_0, Z_1) that represent an alternative representation of U.

After finding

$$Z \equiv \begin{pmatrix} Z_0 \\ Z_1 \end{pmatrix}$$

in terms of f and $\mathbf{v}$ one can construct the field $\mathbf{n}$ of the Hopf mapping, using $\mathbf{n} = Z^{\dagger} \boldsymbol{\tau} Z$.

The connection is explicit

$$\begin{aligned} n_1 &= 2\left(v_3 v_1 \sin f - v_2 \cos f\right) \sin f \\ n_2 &= 2\left(v_3 v_2 \sin f + v_1 \cos f\right) \sin f \\ n_3 &= 1 - 2\left(1 - v_3^2\right) \sin^2 f \end{aligned}$$

This is not all.

We need more about the Skyrme field, supposed of the form $\exp\left[i\ f\ \mathbf{v} \cdot \boldsymbol{\tau}\right]$.

Assumption

$$\begin{aligned} f &\equiv f(r) \\ \mathbf{v} &\equiv \mathbf{v}(\theta, \varphi) \end{aligned}$$

By this assumption $\mathbf{v}$ becomes a mapping between sphere (θ, φ) of the base space to a sphere $(v_1, v_2 v_3)$ with $|\mathbf{v}| = 1$, of the target space

$$S^2 \xrightarrow{\mathbf{v}} S^2$$

and this mapping *has a winding number.*

A concrete assumption:

take in the base space

$$(\theta, \varphi) \to z = \exp(i\varphi) \tan\left(\frac{1}{2}\theta\right)$$

and take a function $R(z)$ to construct the components of $\mathbf{v}$,

$$\mathbf{v} = \begin{pmatrix} \frac{R+\overline{R}}{1+|R|^2} \\ \frac{i\left(\overline{R}-R\right)}{1+|R|^2} \\ \frac{|R|^2-1}{1+|R|^2} \end{pmatrix}$$

Now, choose a *rational function*

$$R = z^Q$$

NOTE
For

$$Q = 1$$

we have

$$\begin{aligned} R &= z = \exp(i\varphi)\tan\left(\frac{\theta}{2}\right) \\ \overline{R} &= \overline{z} = \exp(-i\varphi)\tan\left(\frac{\theta}{2}\right) \end{aligned}$$

$$\begin{aligned} \overline{R} + R &= 2\cos\varphi\ \tan\left(\frac{\theta}{2}\right) \\ \overline{R} - R &= -2i\sin\varphi\ \tan\left(\frac{\theta}{2}\right) \\ |R|^2 &= \tan^2\left(\frac{\theta}{2}\right) \\ |R|^2 + 1 &= 1 + \tan^2\left(\frac{\theta}{2}\right) = \frac{1}{\cos^2\left(\frac{\theta}{2}\right)} \end{aligned}$$

The vector becomes

$$\begin{aligned} v_1 &= \frac{R + \overline{R}}{1 + |R|^2} = \frac{2\cos\varphi\tan\left(\frac{\theta}{2}\right)}{1/\cos^2\left(\frac{\theta}{2}\right)} = 2\cos\varphi\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) \\ &= \cos\varphi\sin\theta \end{aligned}$$

$$\begin{aligned} v_2 &= \frac{i\left(\overline{R} - R\right)}{1 + |R|^2} = \frac{i\left(-2i\sin\varphi\ \tan\left(\frac{\theta}{2}\right)\right)}{\frac{1}{\cos^2\left(\frac{\theta}{2}\right)}} = 2\sin\varphi\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) \\ &= \sin\varphi\sin\theta \end{aligned}$$

$$v_3 = \frac{|R|^2 - 1}{1 + |R|^2} = \frac{\tan^2\left(\frac{\theta}{2}\right) - 1}{\frac{1}{\cos^2\left(\frac{\theta}{2}\right)}} = -\cos\theta$$

Finally

$$\mathbf{v} = \begin{pmatrix} \cos\varphi\sin\theta \\ \sin\varphi\sin\theta \\ -\cos\theta \end{pmatrix}$$

END

Note In **Volkov winding diffusion** using

$$U(\mathbf{x}) : S^3 \to SU(2)$$

the *winding number* is

$$\kappa[U] = \frac{1}{24\pi^2} \mathrm{tr} \int_{S^3} UdU^{-1} \wedge UdU^{-1} \wedge UdU^{-1}$$

Taking

$$\begin{aligned} U^{(k)}(\mathbf{x}) &= U^{(k)}(\xi, \theta, \varphi) \\ &= \exp\left[-i\kappa\ \xi n^a \tau^a\right] \end{aligned}$$

where

$$n^a \equiv (\sin\theta\cos\varphi\ ,\ \sin\theta\sin\varphi\ ,\ \cos\theta)$$

is on the sphere, then

$$\kappa[U] = \kappa$$

End.

4 Hopf vortices unwinding Hietarinta

The paper **0904.1305 unwinding Hopf vortices.**

Essentially, a configuration of vortices is initialized. Then the dynamics evolves according to the equation derived from $O(3) +$ Faddeev Skyrme Lagrangian. The evolution goes in the sense of minimizing the energy.

In this setting the basis space in NOT compactified to S^3 but is
(1) $S^2 \times T^1$, i.e. one of the directions is periodic, or
(2) T^3, all directions in the basis space are periodic.

5 The Belavin Prasad Sommerfeld states

There is in supersymmetric theories **Duality Hoffman** an operator commuting with all other symmetry generators and represented by an antisymmetric complex matrix called **central charge matrix**

$$Z^{ij}$$

which for $N = 2$ supersymmetric theories is

$$Z^{ij} = \varepsilon^{ij} Z$$

The importance of the central charges relies on the fact that it is the lower bound of the masses in the theory

$$M \geq |Z|$$

where Z is **the largest eigenvalue of the matrix Z of central charges.** This is the Belavin-Prasad-Sommerfeld bound and the states which saturates this bound are called BPS states.

The BPS states are very important since they cannot decay anymore. They preserve **half** of the supersymmetry.

See also discussion of **Bonora** on the Matrix String Theory, on string interactions described by solutions of the MST susy equations for (X^{μ}, A), reduced to simpler structure by projection on lower dimension. The equations look similar to the SD equations for $2D$ Euler fluid (sinh-Poisson).

6 Mapping S^3 to $SU(2)$ Gibbons Steif

In **Gibbons Steif gravitation**.

The imbedding of S^3 into the group manifold of $SU(2)$

$$\begin{aligned} g &= \begin{pmatrix} x^4 + ix^3 & x^2 + ix^1 \\ -x^2 + ix^1 & x^4 - ix^3 \end{pmatrix} \\ &= x^4 + ix^i \tau^i \end{aligned}$$

$$\begin{aligned} \det(g) &= \left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^3 + \left(x^4\right)^2 = 1 \\ & \quad \text{(only 3 independent variables)} \end{aligned}$$

The $SO(4)$ transformations are induced by the transformations of the double cover

$$SU(2)_L \times SU(2)_R$$

The transformations correspond to mukltiplication of g to the left and to the right with matrices $\in SU(2)$.

NOTE the symmetry is also in **Fujikawa** Berry - axial anomaly. **END**

This defines left invariant one forms.

They are the pull back to S^3 (basis space) of the one forms

$$\begin{aligned} \mathbf{e}_i^L &= -i \text{ tr } \left[\tau_i \ g^{-1} dg\right] \\ &= 2\left(x^4 dx^i - x^i dx^4 + \varepsilon_{ijk} \ x^j \ dx^k\right) \\ \text{for} \quad i &= 1, 2, 3 \end{aligned}$$

These left invariant one forms have the property that the *two-forms* resulting from the external differentials, are

$$d\mathbf{e}_i^L = \frac{1}{2} \ \varepsilon_{ijk} \ \mathbf{e}_j^L \ \wedge \ \mathbf{e}_k^L$$

The analogous right invariant one forms

$$\mathbf{e}_i^R = i \text{ tr } \left[\tau^i \ dg \ g^{-1}\right]$$

with

$$d\mathbf{e}_i^R = \frac{1}{2} \ \varepsilon_{ijk} \ \mathbf{e}_j^R \ \wedge \ \mathbf{e}_k^R$$

Other expressions

$$\mathbf{e}_i^L = \varepsilon_{ijk} \ \left(M_{jk} + \widetilde{M}_{jk}\right)$$

$$\mathbf{e}_i^R = \varepsilon_{ijk} \ \left(M_{jk} - \widetilde{M}_{jk}\right)$$

where

$$M_{jk} = x^j \ dx^k - x^k \ dx^j$$

$$\widetilde{M}_{jk} = \frac{1}{2}\varepsilon_{jklp} \ M_{lp}$$

The definitions above have introduced the *left and right invariant one forms*

Now we introduce the *left and right invariant vector fields*

They are the duals to the one-forms.

This means that once we know the *one-forms* $\mathbf{e}_i^L$ and $\mathbf{e}_i^R$ $\left(\sim dx^i\right)$ the vector fields are operators that contract with them

$$E_i^L = \frac{1}{2}\left(x^4 \frac{\partial}{\partial x^i} - x^i \frac{\partial}{\partial x^4} + \varepsilon_{ijk} \ x^j \ \frac{\partial}{\partial x^k}\right)$$

$$E_i^R = -\frac{1}{2}\left(x^4 \frac{\partial}{\partial x^i} - x^i \frac{\partial}{\partial x^4} - \varepsilon_{ijk} \ x^j \ \frac{\partial}{\partial x^k}\right)$$

We note a similarity with the triple formulation from $O(3)$ model

$$\mathbf{n} \to z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \to A_\mu = z^\dagger \partial_\mu z$$

The new variables in the S^3 case are

$$z^1 = x^4 + ix^3$$

these are the z (may be seen as longitudinal) and the time

$$z^2 = x^1 + ix^2$$

these are the coordinates in the plane transversal to $z \equiv x^3$.

The vector fields $E_i^{L,R}$ are operators.
The 3rd component is an operator of rotation as follows.

The operator (vector field) E_3^L,

$$\begin{aligned} z_1 &\rightarrow \exp\left(i\frac{\theta}{2}\right) z_1 \\ z_2 &\rightarrow \exp\left(i\frac{\theta}{2}\right) z_2 \end{aligned}$$

The operator (vector field) E_3^R,

$$\begin{aligned} z_1 &\rightarrow \exp\left(-i\frac{\theta}{2}\right) z_1 \\ z_2 &\rightarrow \exp\left(i\frac{\theta}{2}\right) z_2 \end{aligned}$$

They are replaced with operator

$$\begin{aligned} L_i &= -iE_i^L \\ R_i &= -iE_i^R \end{aligned}$$

These operators verify

$$[L_i, L_j] = i\varepsilon_{ijk} L_k$$

$$[R_i, R_j] = i\varepsilon_{ijk} R_k$$

$$[L_i, R_j] = 0$$

Natural ansatz for a gauge field on S^3 is

$$\begin{aligned} A &= \frac{i}{e} f(\eta)\ g^{-1}\ dg \\ &= -\frac{1}{e} f(\eta)\ \mathbf{e}_i^L\ \frac{\tau^i}{2} \end{aligned}$$

$$F = F^i\ \frac{\tau^i}{2}$$

For a non-vacuum configuration, the Chern Simons

$$\begin{aligned} N_{CS} &= \int_{S^3} \omega_3 \\ \omega_3 &= \frac{e^2}{8\pi^2} \mathrm{tr}\left(A \wedge dA - \frac{2}{3} ie\ A \wedge A \wedge A\right) \end{aligned}$$

Here ω_3 is a *three-form.*

ω_3 is $\sim \mathbf{A} \cdot \mathbf{B}$. It is density of helicity.

The derivative of ω_3 (the derivative to time of the helicity) is $\mathbf{E} \cdot \mathbf{B} \sim F\widetilde{F}$, which is the divergence of a current.

$$\begin{aligned}
d\omega_3 &= \frac{e^2}{8\pi^2} \mathrm{tr}\,(F \wedge F) \\
&\sim F\ \widetilde{F} \\
&\sim \text{exact differential, } \partial K \text{, divergence of a current} \\
&\sim d\omega_3 \ \text{ is a scalar in } 4D
\end{aligned}$$

This is the meaning of the anomaly

$$\begin{aligned}
\partial j &= \partial K \\
j &\equiv \text{axial current} \\
K &\equiv \text{current of winding}
\end{aligned}$$

7 The topological degree

Consider the *sigma* model in two dimensions (planar Heisenberg ferromagnet, plane nematic liquid crystals) with the scalar field having three components.

The space is $\mathbf{R}^2$ and in every point $x^\mu = (x, y)$ there is a vector $\boldsymbol{\phi} = (\phi^1, \phi^2, \phi^3)$ of length 1

$$\boldsymbol{\phi} \cdot \boldsymbol{\phi} - 1 = 0$$

The tip of the vector is a point on a sphere S^2 (called space of internal symmetry).

Taking the condition that $\boldsymbol{\phi}$ *is the same on a circle of very large radius in the plane*, the infinite distant "boundary" can be replaced by a point: the plane is compactified to a sphere S^2. The field $\boldsymbol{\phi}$ represents a map:

$$\text{(the plane } \mathbf{R}^2 \text{ compactified)} \rightarrow \text{(the space of internal symmetry)}$$

$$S^2 \xrightarrow{\phi} S^2$$

The field has a topological nature. Any realization of the field $\boldsymbol{\phi}$ is a map which cover one sphere (internal space) with the other sphere (the basis space) once, twice, ..., **an integer number of times**. In formal language, the fields $\boldsymbol{\phi}$ are classified into topological classes given by the second homotopy group of the internal space

$$\boldsymbol{\pi}_2\left(S^2\right) = \mathbf{Z}$$

There is no smooth deformation which could allow to pass from one class to another. This is identical to *ideal fluids*, where reconnections (change of topology) are not possible: the motions of the *ideal fluid* are homotopic deformations preserving the topological content.

Define the density of Lagrangian

$$L = \frac{1}{2}\partial_\mu \boldsymbol{\phi}\cdot\partial^\mu \boldsymbol{\phi}$$

The *action* of the static field

$$S = \int \frac{1}{2}\partial_\mu \boldsymbol{\phi}\cdot\partial^\mu \boldsymbol{\phi}\, dx^1 \wedge dx^2$$

$$\begin{aligned} S &= \frac{1}{2}\int dx^1 \wedge dx^2 \left\{\partial_\mu \boldsymbol{\phi}\cdot\partial^\mu \boldsymbol{\phi} - 2\varepsilon_{\mu\rho}\left(\boldsymbol{\phi}\times\partial^\rho\boldsymbol{\phi}\right)\cdot\partial^\mu\boldsymbol{\phi}\right. \\ &\quad \left. +\varepsilon_{\mu\rho}\left(\boldsymbol{\phi}\times\partial^\rho\boldsymbol{\phi}\right)\varepsilon_{\mu\sigma}\left(\boldsymbol{\phi}\times\partial^\sigma\boldsymbol{\phi}\right) + 2\varepsilon_{\mu\rho}\left(\boldsymbol{\phi}\times\partial^\rho\boldsymbol{\phi}\right)\cdot\partial^\mu\boldsymbol{\phi}\right\} \end{aligned}$$

$$c_1 = \frac{-1}{8\pi}\varepsilon_{\mu\rho}\left(\boldsymbol{\phi}\times\partial^\rho\boldsymbol{\phi}\right)\cdot\partial^\mu\boldsymbol{\phi} dx^1 \wedge dx^2$$

The **First Chern class**, the integral on S^2 is an integer

$$\int_{S^2} c_1 = -n$$

Then

$$S = 4\pi n + \frac{1}{2}\int dx^1 \wedge dx^2 \left(\partial_\mu\boldsymbol{\phi} - \varepsilon_{\mu\rho}\left(\boldsymbol{\phi}\times\partial^\rho\boldsymbol{\phi}\right)\right)^2$$

Two conclusions:

- The action is bounded from below by a topological limit

$$S \geqslant 4\pi n$$

- The "excess" of action is suppressed for those states which obey

$$\partial_\mu\boldsymbol{\phi} - \varepsilon_{\mu\rho}\left(\boldsymbol{\phi}\times\partial^\rho\boldsymbol{\phi}\right) = 0$$

  which is called the *duality condition* and $\boldsymbol{\phi}$ is *self-dual*.

In plane: the number of vortices in a superfluid is a topological degree and is invariant.

8 The topological mass

The paper **SD topological massive Jackiw** on the equivalence between two models. The model at Self-Duality.

And the model with topological mass which is

$$L_T = -\frac{1}{2}F^\mu F_\mu + \frac{1}{2}mF^\mu A_\mu$$

where

$$F^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho}F_{\nu\rho}$$

and

$$F_{\nu\rho} = \partial_\nu A_\rho - \partial_\rho A_\nu$$

There is a folder *mass generation*.

9 The Abelian-Higgs topological model

This is for supraconductors.

The Lagrangian is relativistic (*i.e.* covariant kinetic energy for the matter field), non-linear scalar self-interaction with condensate (which gives the mass) and Maxwell for the gauge field.

It has as topological solution the *Abrikosov-Nielsen-Olesen* vortex.

The vortex is a tube of magnetic field that can penetrate the superconducting medium.

See **Hindmarsh-Kibble**.

9.1 Kozhevnikov. Study of the perturbation around the basic vortex solution. Quantum stability of the ANO vortex

From **quantum stability Kozhevnikov 1995**

"nontriviality of the
First homotopic group of the three-dimensional space with
an excluded line or, in other words, with the impossibility
of shrinking to a point the loop surrounding such
a line without crossing the region of the normal phase"

The field of the vortex is expressed in terms of the angle representing the phase of the scalar (Higgs) function: it is a discontinuous variable.

Using this variable is appropriate in the London case, where
the screening length of the magnetic field
is much larger compared with
the correlation length of the scalar field

The action of the Abelian Higgs model

$$\begin{aligned} S \;\; = \;\; \int d^4x \; \Big[& \frac{i}{2}\left(\phi^* \frac{\partial \phi}{\partial t} - \frac{\partial \phi^*}{\partial t}\phi\right) - 2eA_0\phi^*\phi \\ & - \frac{1}{2m}\left(\mathbf{D}\phi\right)^* \left(\mathbf{D}\phi\right) \\ & - \alpha \left|\phi\right|^2 - \frac{\beta}{2}\left|\phi\right|^4 \\ & - \frac{1}{16\pi} F_{\mu\nu}F^{\mu\nu} \Big] \end{aligned}$$

where

$$A_\mu = \left(A_0, \; \mathbf{A}\right)$$

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}$$

$$D_\mu = \frac{\partial}{\partial x^\mu} - 2ieA_\mu$$
$$\mathbf{D} = \frac{\partial}{\partial \mathbf{x}} - 2ie\mathbf{A}$$

Why $2e$: it is considered that ϕ results from *pairing*.

It is replaced

$$\begin{aligned} \phi &\rightarrow \Phi + \phi \\ A_\mu &\rightarrow A_\mu + a_\mu \end{aligned}$$

now ϕ and a_μ are small. Then

$$S = \int d^4x \left\{ L_{bkg}\left[\Phi, A_\mu\right] + L_{fluct}\left[\Phi, \phi, A_\mu, a_\mu\right] + L_{int}\left[\Phi, \phi, A_\mu, a_\mu\right] \right\}$$

$$\begin{aligned} L_{bkg} &= \frac{i}{2}\left(\Phi^* \frac{\partial \Phi}{\partial t} - \frac{\partial \Phi^*}{\partial t}\Phi\right) - 2eA_0\Phi^*\Phi \\ &-\frac{1}{2m}\left(\mathbf{D}\Phi\right)^*\left(\mathbf{D}\Phi\right) - \alpha\left|\Phi\right|^2 - \frac{\beta}{2}\left|\Phi\right|^4 \\ &-\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} \end{aligned}$$

for the background fields.

$$\begin{aligned} L_{fluct} &= \frac{i}{2}\left(\phi^* \frac{\partial \phi}{\partial t} - \frac{\partial \phi^*}{\partial t}\phi\right) - 2eA_0\phi^*\phi \\ &-\frac{1}{2m}\left(\mathbf{D}\phi\right)^*\left(\mathbf{D}\phi\right) - \alpha\left|\phi\right|^2 \\ &-\frac{\beta}{2}\left[4\left|\Phi\right|^2\left|\phi\right|^2 + \Phi^{2*}\phi^2 + \Phi^2\phi^{2*}\right] \\ &-\frac{2e^2}{m}\left|\Phi\right|^2\ \mathbf{a}^2 - \frac{1}{16\pi}f_{\mu\nu}f^{\mu\nu} \end{aligned}$$

for the fluctuations interacting with the background fields.

Now, the Lagrangian density for the interaction between the fluctuations, on the background.

$$\begin{aligned} L_{int} &= \frac{e}{m}\mathbf{a}\ \left\{\phi\left[\left(-i\mathbf{D}\right)\Phi\right]^* + c.c \right.\\ &\quad +\Phi\left[\left(-i\mathbf{D}\right)\phi\right]^* + c.c. \\ &\quad +\phi\left[\left(-i\mathbf{D}\right)\phi\right]^* + c.c. \\ &\quad -\frac{\beta}{2}\left[\left|\phi\right|^4 + 2\left|\phi\right|^2\left(\Phi^*\phi + \Phi\phi^*\right)\right] \\ &\quad \left. +\frac{2e^2}{m}\ \mathbf{a}^2\ \left(\left|\phi\right|^2 + \Phi\phi^* + \Phi^*\phi\right)\right\} \end{aligned}$$

It is imposed the condition that the linear terms in the pertrubations ϕ and a_μ be zero.

This is equivalent to taking zero the variations of the action functional to the functions of the model, i.e. the Euler Lagrange equations.

It results the set of equations for the "background" fields.

$$i\frac{\partial \Phi}{\partial t} - 2eA_0\Phi = -\frac{1}{2m}\left(\mathbf{D}\Phi\right)^* \left(\mathbf{D}\Phi\right) + \alpha \left|\Phi\right|^2 + \beta\Phi \left|\Phi\right|^2$$

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{j}$$

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\mathbf{j} = \frac{e}{2m}\left[\Phi^* \left(-i\mathbf{D}\right)\Phi + c.c.\right]$$

Two parameters

$$\lambda = \left(\frac{m\beta}{16\pi e^2 \left|\alpha\right|}\right)^{1/2}$$

London penetration depth

$$\xi = \left(\frac{1}{4m\left|\alpha\right|}\right)^{1/2}$$

correlation length

The London limit

$$\lambda \gg \xi$$

For the string: the modulus

$$\left|\Phi\right| = \left|\Phi_0\right| = \sqrt{\frac{\left|\alpha\right|}{\beta}}$$

is constant over almost all space except the core of width $\sim \xi$ where it goes to 0 on the axis.

The variable is

$$\chi = \arg \Phi$$

If there is static configuration

$$\frac{\partial \Phi}{\partial t} = 0$$

then there is no electrostatic potential

$$A_0 = 0$$

but if there is time variation

$$i\frac{\partial\Phi}{\partial t}-2eA_0\Phi=0$$

then a time variation of the phase $\chi(\mathbf{x},t)$ of the field Φ means that $A_0\neq 0$.

The action of the background fields is

$$S_{bgk}=\int d^4x\left[\frac{1}{8\pi}\left(\mathbf{E}^2-\mathbf{B}^2\right)-\frac{1}{32\pi\ e^2\lambda^2}\left(\boldsymbol{\nabla}\chi-2e\mathbf{A}\right)^2\right]$$

All fields in the theory will be expressed through $\chi(\mathbf{x},t)$.

The background fields are

$$\mathbf{A}(\mathbf{x},t)=\frac{1}{2e\lambda^2}\int d^3x'\ G(\mathbf{x}-\mathbf{x}')\ \boldsymbol{\nabla}_{\mathbf{x}'}\ \chi(\mathbf{x}',t)$$

notation $\boldsymbol{\nabla}_{\mathbf{x}'}\ \chi(\mathbf{x}',t)=\frac{\partial}{\partial\mathbf{x}'}\chi(\mathbf{x}',t)$;

$$\mathbf{B}(\mathbf{x},t)=\frac{1}{2e\lambda^2}\int d^3x'\ G(\mathbf{x}-\mathbf{x}')\ \boldsymbol{\nabla}_{\mathbf{x}'}\times\left[\boldsymbol{\nabla}_{\mathbf{x}'}\ \chi(\mathbf{x}',t)\right]$$

$$\begin{aligned}\frac{\partial\mathbf{E}(\mathbf{x},t)}{\partial t} &= \frac{1}{2e}\boldsymbol{\nabla}\ \frac{\partial\chi(\mathbf{x},t)}{\partial t}\\ &\quad -\frac{1}{2e\lambda^2}\int d^3x'\ G(\mathbf{x}-\mathbf{x}')\ \boldsymbol{\nabla}_{\mathbf{x}'}\ \frac{\partial\chi(\mathbf{x}',t)}{\partial t}\end{aligned}$$

The Green function

$$G(\mathbf{x}-\mathbf{x}')=\frac{\exp\left(-\frac{|\mathbf{x}-\mathbf{x}'|}{\lambda}\right)}{4\pi\ |\mathbf{x}-\mathbf{x}'|}$$

The phase function χ is singular

$$\boldsymbol{\nabla}\times\left[\boldsymbol{\nabla}\ \chi(\mathbf{x},t)\right]=2\pi\ n\ \widehat{\mathbf{e}}_{\parallel}\ \delta(\mathbf{x}_{\perp})$$

9.1.1 Assume a constant external magnetic field

Along the axis of the string.

$$(\mathbf{x})\rightarrow(r,\theta,z)$$

$$\int dz\ G(\mathbf{x})=2K_0\left(\frac{|r|}{\lambda}\right)$$

Then one has explicit forms

$$A_i(\mathbf{r}) = -\frac{n}{2e}\left[\frac{1}{r} - \frac{1}{\lambda}K_1\left(\frac{r}{\lambda}\right)\right]\varepsilon_{ij}\frac{r_j}{r}$$

$$\mathbf{B} = \frac{n}{2e}\frac{1}{\lambda^2}K_0\left(\frac{r}{\lambda}\right)\widehat{\mathbf{e}}_{\parallel}$$

$$E_i = -\frac{1}{2e}\left(\frac{\partial n}{\partial t}\right)\frac{1}{\lambda}K_1\left(\frac{r}{\lambda}\right)\ \varepsilon_{ij}\frac{r_j}{r}$$

and

$$\chi = n\theta$$

$$\nabla_i\chi = -n\ \varepsilon_{ij}\frac{r_j}{r}$$

Note χ is the function-phase of the scalar field Φ. θ is the azimuthal angle. The gradient of the phase function χ is the "velocity" and this contains a vector product, i.e. it is directed perpendicular on the distance between two points.

The action functional with these form of the field functions, dependent on χ which is $n\theta$,

$$\begin{aligned} S_{bkg}[n] &= \frac{L_z}{16e^2}\ln\left(\frac{\lambda}{\xi}\right)\int dt\ \left(\frac{\partial n}{\partial t} - \frac{1}{\lambda^2}n^2\right) \\ &\quad + \frac{L_z}{4e}\int dt\ B_{ext}\ n \end{aligned}$$

The calculations that follow are done in order to find the effect of the emission of a scalar pair in the interaction with the vortex.

this treatment is useful for the stability of a filament of vorticity in fluid.
The condition is to find a set of fields:
- gauge field (velocity) and
- scalar field (filament of vorticity) and
- study the fluctuations around them.

A filament of vorticity is solution of the LIA equation or the Gilbert Landau Lifshitz (or from NSEq via inverse Hasimoto). There is hardly a connection with ANO, except maybe formal analytical. ANO needs magnetic expulsion, cooperons.

9.2 Yang vortices of opposite sign in Abelian gauge theory

There is a metric since the problem is studied in gravity theory.

Gauged $O(3)$ model (i.e. derivations are replaced by covariant derivation operators, A_μ).

$$\begin{aligned} D_j \boldsymbol{\phi} &= \frac{\partial}{\partial x^j}\boldsymbol{\phi} + A_j \;(\mathbf{n}\times\boldsymbol{\phi}) \\ \mathbf{n} &\equiv \text{north pole} \end{aligned}$$

$$\mathbf{R}^2 \to S^2$$

The static energy

$$\int_{\mathbf{R}^2} d^2x \;\left[\frac{1}{2}(F_{12})^2 + \frac{1}{2}\left([D_a\boldsymbol{\phi}]^2 + [D_2\boldsymbol{\phi}]^2\right) + V(\boldsymbol{\phi})\right]$$

Choose

$$V(\boldsymbol{\phi}) = \frac{1}{2}(\mathbf{n}\cdot\boldsymbol{\phi})^2$$

Taking in consideration the gravity, one introducesa metric

$$g_{\mu\nu}$$

with signature

$$(-,+,+,+)$$

and the Lagrangian is

$$\begin{aligned} L = \; & \frac{1}{4} g^{\mu\mu'}\; g^{\nu\nu'}\; F_{\mu\nu}F_{\mu'\nu'} \\ & + \frac{1}{2} g^{\mu\nu}\; (D_\mu\phi)(D_\nu\phi) \\ & + \frac{1}{2}(\mathbf{n}\cdot\boldsymbol{\phi})^2 \end{aligned}$$

In the target space one makes a stereographic projection.
It starts from the South Pole

$$\mathbf{n}_S = (0,0,-1)$$

One defines

$$\begin{aligned} u_1 &= \frac{\phi_1}{1+\phi_3} \\ u_2 &= \frac{\phi_2}{1+\phi_3} \end{aligned}$$

and the complex number

$$u = u_1 + iu_2$$

The gauge potential is redefined

$$A_\mu \rightarrow -A_\mu$$

and the covariant derivative is

$$D_\mu u = \frac{\partial u}{\partial x^\mu} - iA_\mu u$$

The Lagrangian

$$\begin{aligned} L &= \frac{1}{4} g^{\mu\mu'} g^{\nu\nu'} \ F_{\mu\nu} F_{\mu'\nu'} \\ &+ \frac{2}{\left(1+|u|^2\right)^2} g^{\mu\nu} \ \left(D_\mu u\right)\left(D_\nu u\right)^* \\ &+ \frac{1}{2}\left(\frac{1-|u|^2}{1+|u|^2}\right)^2 \end{aligned}$$

Invariance to

$$(u, A_\mu) \rightarrow \left(\frac{1}{u}, -A_\mu\right)$$

showing that the zeros and the poles have similar roles.

NOTE

this property is similar to the one of the SD equations Euler $2D$.

$$\rho_1 \rightarrow \frac{1}{\rho_2}$$

and is related to the property found by **Tracy** for the *sinh*-Poisson equation.

There are discourraging aspects

- in the Lagrangian we have Maxwell term, not Lorentz-type Chern Simons for gauge field

- the symmetry $\rho \rightarrow 1/\rho$ in Euler $2D$ exists after the gauge field has been absorbed via SD equations plus ansatz

- here the potential, both original $V(\boldsymbol{\phi}) = \frac{1}{2}(\mathbf{n} \cdot \boldsymbol{\phi})^2$ and after transformation by stereographic projection inside the target-sphere-space

$$V(u) = \frac{1}{2}\left(\frac{1-|u|^2}{1+|u|^2}\right)^2$$

is two-well, like Higgs

The tensor energy-momentum

$$\begin{aligned} T_{\mu\nu} &= g^{\mu'\nu'} \ F_{\mu\mu'} \ F_{\nu\nu'} \\ &+ \frac{2}{\left(1+|u|^2\right)^2} \left[(D_\mu u)^* (D_\nu u) + (D_\mu u)(D_\nu u)^* \right] \\ &- g_{\mu\nu} \ L \end{aligned}$$

For
- straight
- independent
cosmic strings, one has
- to decide a particular (gravitational) metric and
- to look for simpler dependences
- - reflection in time x^0
- - reflection against a fixed direction x^3;
- - u, A_1, A_2

functions of only $\left(x^1, x^2\right)$

- - the potentials are zero

$$\begin{aligned} A_0 &= 0 \\ A_3 &= 0 \end{aligned}$$

The metric is

$$g^{\mu\nu} = (-U, \exp(\eta), \exp(\eta), V)$$

where

$$U, V, \eta \rightarrow \text{functions of only } \left(x^1, x^2\right)$$

The energy density

$$E = -T_0^0$$

is

$$\begin{aligned} E &= \exp(-\eta) \ F_{12} \\ &+ \exp(-\eta) \ J_{12} \\ &+ \frac{1}{2} \left(\exp(-\eta) F_{12} - \frac{1-|u|^2}{1+|u|^2} \right)^2 \\ &+ \frac{2}{\left(1+|u|^2\right)^2} \exp(-\eta) \ |D_1 u + i D_2 u|^2 \end{aligned}$$

where J_{ik} is the rotational, curl, of the current

$$J_{ik} = \frac{\partial J_k}{\partial x^i} - \frac{\partial J_i}{\partial x^k}$$

and the current is

$$J_k = \frac{i}{1+|u|^2}\left[u\ \ (D_k u)^* - u^*\ \ (D_k u)\right]$$

the first two terms

$$\exp(-\eta)\ \ F_{12} + \exp(-\eta)\ \ J_{12}$$
are topological

The equations are

$$D_1 u + i D_2 u = 0$$

and

$$F_{12} = \exp(\eta)\frac{1-|u|^2}{1+|u|^2}$$

like the SD from Euler.

u is complex and is a function of (x^1, x^2);

there is no need of algebraic ansatz

10 The Georgi-Glashow model

This is the basic element of the *Standard model* of weak and electromagnetic interaction.

It has gauge group

$$SO(3)$$

whose universal covering group is $SU(2)$.

See the reviews by **Ketov, Alvarez-Gaume**.

The topological solutions are *magnetic monopoles* of the *t'Hooft-Polyakov* type.

The hedgehog solution.

The discrete character of the magnetic/electric charge.

10.1 The θ angle and the Witten effect

It starts by adding the CS term in the Lagrangian

$$\theta \frac{e^2}{32\pi^2} F^a_{\mu\nu} \widetilde{F}^{a\mu\nu}$$

Remember that $F\widetilde{F}$ (full contraction, inclusiv in group indices, therefore scalar object) is the divergence of a topological current. Then we have $\theta\left(\partial_\mu j^\mu_{topo}\right)$.

In the series of papers regarding the *axion anomaly* and the *global strings* and the radial inward current to the string, etc., of **Lee**, the angle θ first arises as the phase of the scalar function

$$\phi = \frac{f}{\sqrt{2}} \exp\left(i\theta\right)$$

and the term in Lagrangian is

$$Z_\mu \left(\partial^\mu \theta\right)$$

where Z_μ is the Chern-Simons current of the *gauge field* A_μ.

Note that in the **Lecture II-6 of Witten** on gauge theory in $2D = 1 + 1$ with self-interacting bosons it is added to the Lagrangian of the Maxwell Abelian Higgs model a term

$$-\frac{i\theta}{2\pi} \int F$$

which looks different of the θ term mentioned above. *Later*: this is because the θ term in **Lee** is in $4D$ and the θ term in **Witten II-6** is in $2D$. The integral $\int F$ is the total magnetic field through the area and is a topological quantity, similar to the CS term $\widetilde{F}F$. **End.**

The lagrangian studied by **Witten Lecture II-6** is

$$\begin{aligned} L &= \int d^2x \left[\frac{(*F)^2}{4e^2} \right. \\ &\quad \left. + \frac{1}{2} \left|d\phi\right|^2 + \frac{\lambda}{4} \left(\left|\phi\right|^2 - a^2 \right)^2 \right] \\ &\quad - \frac{i\theta}{2\pi} \int F \end{aligned}$$

where

$$\begin{aligned} \left|\phi\right|^2 &= \sum \left|\phi_i\right|^2 \\ \left|d\phi\right|^2 &= \sum \left|d\phi_i\right|^2 \end{aligned}$$

(This looks like the Abelian-Higgs but with the θ term added)

Note for comparison in the unsuppressed sphaleron transitions in **McLerran (sphaleron strikes back)** the example of pendulum is extended by addition of a θ term. **End.**

The mass of the bosons ϕ_i is

$$m^2 = -\lambda a^2$$

If

$$\lambda \to 0$$

but the mass is kept finite, the theory becomes a gauge theory of *free* bosons, which means that the bosons interact with the gauge field but do not interact between them.

Consider the situation where

$$m^2 \to +\infty$$

which is equivalent to take

$$m^2 \gg e^2$$

In this case the quartic term, $|\phi|^4$ that results from the expansion of the scalar potential term, is negligible. What still matters is the second term

$$-\frac{\lambda}{4} 2a^2 |\phi|^2$$

and this induces a *mass* gap and the interaction between the bosons can be approximated by a 1-dimensional Coulomb potential, causing *confinement*. The space reversal symmetry is preserved for $\theta = 0$ (absence of the last term in the Lagrangian). But the space reversal symmetry is lost when $\theta = \pi$. We **note** that the last term is the integration over the plane of the magnetic field, *i.e.* is the magnetic field flux through the plane. This term creates two vacua.

The case

$$m^2 \to -\infty$$

or

$$m^2 \ll -e^2$$

In this case there is a vacuum manifold

$$|\phi|^2 - a^2 = 0$$

which is

$$S^{2N-1}$$

The gauge group $U(1)$ acts freely on this space therefore the space of vacua is

$$\mathbf{CP}^{N-1} = S^{2N-1}/U(1)$$

Since the space-time of the theory is the Riemann manifold Σ the theory consists of fields that map this Riemann manifold onto the vacuum space

$$\Sigma \to \mathbf{CP}^{N-1} = S^{2N-1}/U(1)$$

This is the *sigma* model.

11 The chiral model

See also *chiral model unitons*.

11.1 Introduction to CHIRAL model

Definition of chiral fields

Consider two arbitrary Riemann manifolds N^q and M^n, and the mappings

$$f : N^q \to M^n$$

Cansider also the *functional* $S_0(f)$ defined on the family of these mappings. This functional $S_0(f)$ has the form of a Dirichlet functional, quadratic in the derivatives of the mapping f, possibly with some additional terms.

Let us take

$$M^n = G : \text{ Lie group with two sided invariant metric}$$

Then the **chiral Lagrangian** for **the principal chiral field** has the form

$$S_0(f) = \frac{1}{2} \int_{N^q} \text{tr}\left(g^{\mu\nu} A_\mu A_\nu\right) \sqrt{g}\, d^q y$$

(**note** is-this A^2 of Zakharov ?) where

$$g^{\mu\nu} = \text{metric on } N^q$$

$$A_\mu = f^{-1}(y) \frac{\partial f(y)}{\partial y^\mu}$$

Note looks like pure gauge.

Note. This looks similar to the form

$$U^{-1} \partial_\mu U \sim \partial_\mu \ln U$$

a substitution that is made for transforming the self-duality equations of the Chern-Simons, φ^4-scalar self-interaction, non-relativistic (Schrodinger) Lagrangian theory [Euler fluid] into the *chiral* model equations (C. N. Yang), to be solved afterwards by Uhlenbeck mappings between spheres. The potential A_μ appears here as the *gradient of a phase* of the complex field $U \sim \exp(A)$. For example, the scalar field in the triplet for anomaly, scalar string + gauge field with non-trivial topology + massless fermions, becomes almost constant in magnitude at large distances from the string, $f(r)\exp(i\theta)$, $f \to v$ =const, at $r \to \infty$, and it only remains θ. **End**.

See **9906236**.This is also in **SelfDuality**.

The *chiral* properties of Massive Thirring (MT). The chiral transformation is a $U(1)$ transformation

$$\begin{aligned} \psi &\to \exp\left(ia\gamma^5\right)\psi \text{ for the fermions} \\ \phi &\to \phi + a/\beta \text{ for the bosons} \end{aligned} \tag{1}$$

with a real arbitrary. The massless Thirring is chiral invariant but adding the mass term breaks the chiral symmetry. Simlarly the free bosonic model is chiral invariant but the term $\cos\beta\phi$ breaks the invariance. There is for both models a residual symmetry for

$$a = 2\pi n \quad , \quad n \in \mathbf{Z}$$

The breaking of the symmetry is

$$U(1) \rightarrow \mathbf{Z}$$

The operators

$$\sigma_{\pm}(x) = \frac{1}{2}\overline{\psi}\left(1 \pm \gamma^5\right)\psi$$

account for the *chiral* properties of the system. Under the chiral transformation Eq.(1), they change as

$$\sigma_{\pm} \rightarrow \exp(\pm 2ia)\,\sigma_{\pm}$$

which means that they have well defined $\pm$ chiral charges, becuse they are *eigenfunctions* of the transformation..

The chiral invariant

$$\sigma_+\sigma_-$$

represents a molecule and is *chiral invariant*. By forming molecules the system tries to restore the chiral invariance.

See **bosonization Witten** and **0105057 Faber Ivanov**.

11.2 Models

The paper **9601096 geom sigma models** contains an explanation of the Supersymmmetric Dual Chiral σ models and Dual sigma models.

The Lagrangian for the Chiral model on the space

$$\begin{aligned} O(4) &\simeq O(3) \times O(3) \\ &\simeq SU(2) \times SU(2) \end{aligned}$$

is

$$L_{CM} = \frac{1}{2} g_{ab} \partial_\mu \varphi^a \partial^\mu \varphi^b$$

where g_{ab} is the metric on the field manifold (three-sphere S^3).

NOTE see parametrization of the Skyrme field $\Phi \equiv \left(\Phi^1, \Phi^2, \Phi^3, \Phi^4\right)$ with $\Phi^\alpha\Phi^\alpha = 1$, $\Phi \in S^3 \sim SU(2)$ in paper **Jackson Manton S^3 Skyrme solitons with** $O(2) \times O(2)$ symmetry. They look for baryon solutions, $B = pq$. **END.**

The group elements $U \in SU(2)$ are parametrized as (using the Pauli matrices)

$$U = \varphi^0 + i\tau^j\varphi^j$$

where the summation is over $j = 1, 2, 3$. (The group $O(3)$ has three generators, the spatial rotations, in particular the Euler angles; like $SU(2)$ which also has three generators, $E_{\pm}, H$).

Remember **Battye Sutcliffe** $U = \exp\left[if\ \mathbf{v}\cdot\boldsymbol{\tau}\right]$ where $|\mathbf{v}| = 1$ and $f \equiv f(r)$.

Here we have

$$\begin{aligned} U^{\dagger}U &= 1 \\ \left(\varphi^{0}\right)^{2} + \varphi^{2} &= 1 \\ \text{where } \varphi^{2} &= \sum_{j=1}^{3}\left(\varphi^{j}\right)^{2} \quad \text{(space part)} \end{aligned}$$

NOTE
In **Jackson Manton Skyrme** the fields are elements of $SU(2)$ with

$$\begin{aligned} U &= \sigma + i\boldsymbol{\tau}\cdot\boldsymbol{\pi} \\ &= \sigma + i\left(\tau^{x}\pi^{x} + \tau^{y}\pi^{y} + \tau^{z}\pi^{z}\right) \end{aligned}$$

and identification

$$\left(\sigma, \pi^{x}, \pi^{y}, \pi^{z}\right) = \left(\Phi^{1}, \Phi^{2}, \Phi^{3}, \Phi^{4}\right)$$

END.

It is eliminated the time component, φ^{0}, as

$$\varphi^{0} = \pm\sqrt{1-\varphi^{2}}$$

and we obtain the metric on the manifold of the fields

$$g_{ab} = \delta^{ab} + \frac{\varphi^{a}\varphi^{b}}{1-\varphi^{2}}$$

Then the Lagrangian becomes ($CM \equiv$ Chiral Model)

$$\begin{aligned} L_{CM} &= \frac{1}{2}g_{ab}\partial_{\mu}\varphi^{a}\partial^{\mu}\varphi^{b} \\ L_{CM} &= \frac{1}{2}\left(g^{ij} + \frac{\varphi^{i}\varphi^{j}}{\sqrt{1-\varphi^{2}}}\right)\partial_{\mu}\varphi^{i}\partial^{\mu}\varphi^{j} \\ &= \frac{1}{2}J_{\mu}^{k}J^{k\mu} \end{aligned}$$

where the current arises from the definition of $U = \varphi^{0} + i\tau^{j}\varphi^{j}$,

$$\begin{aligned} U^{-1}\partial_{\mu}U &= -i\tau^{k}J_{\mu}^{k} \\ & \quad \text{sum over the index } k \\ J^{k} &= \text{projection along the Pauli direction } \tau^{k} \end{aligned}$$

$$J_{\mu}^{k} = -v_{a}^{k}\partial_{\mu}\varphi^{a}$$

where

$$v_{a}^{k} = \sqrt{1-\varphi^{2}}g_{ak} + \varepsilon^{akb}\varphi^{b}$$

The principal chiral model 9307021

This text is also in **Self Duality**.

The compatibility condition for the system of linear first-order differential equations

$$\begin{aligned} \frac{\partial \psi}{\partial x} &= U(\lambda)\psi \\ \frac{\partial \psi}{\partial t} &= V(\lambda)\psi \end{aligned}$$

is

$$\frac{\partial U}{\partial t} - \frac{\partial V}{\partial x} + [U, V] = 0$$

For

$$\begin{aligned} U(\lambda) &= -\frac{u}{\lambda - 1} \\ V(\lambda) &= \frac{v}{\lambda + 1} \end{aligned}$$

and substituting

$$t \rightarrow y$$

the principal chiral model has the variables

$$\begin{aligned} &u(x, y) \\ &v(x, y) \end{aligned}$$

that belong to the Lie algebra g. The equations are

$$\begin{aligned} \frac{\partial u}{\partial y} + \frac{1}{2}[u, v] &= 0 \\ \frac{\partial v}{\partial x} - \frac{1}{2}[u, v] &= 0 \end{aligned}$$

The connection is ω, defined as function of the spectral parameter λ

$$\omega : \mathbf{C} \rightarrow \wedge^1 (M, g)$$

which means that $\omega(\lambda)$ is a g-valued 1-form over the two-dimensional manifold M, having the local coordinates (x, y).

$$\omega(\lambda) = U(\lambda)\, dx + V(\lambda)\, dy$$

and the condition of compatibility is the vanishing of the curvature

$$\Omega(\lambda) = d\omega(\lambda) - \frac{1}{2}[\omega(\lambda), \omega(\lambda)]$$

In this work (where it is mentioned **Uhlenbeck)** it is given the *connection* for the *sinh*-Gordon equation

$$\omega(\lambda) = (\lambda L_1 + L_0)\, dx + \left(M_0 + \frac{1}{\lambda} M_1\right) dy$$

with

$$\begin{aligned}
L_0 &= \exp(-\psi) E_- + \frac{1}{2}\frac{\partial\psi}{\partial x} H \\
L_1 &= \exp(\psi) E_+ \\
M_0 &= \exp(-\psi) E_+ + \frac{1}{2}\frac{\partial\psi}{\partial y} H \\
M_1 &= \exp(\psi) E_-
\end{aligned}$$

where ψ satisfies

$$\frac{\partial^2\psi}{\partial x \partial y} = 2\sinh(2\psi)$$

Then

$$\begin{aligned}
\omega(\lambda) &= \left(\lambda \exp(\psi) E_+ + \exp(-\psi) E_- + \frac{1}{2}\frac{\partial\psi}{\partial x} H\right) dx \\
&\quad + \left(\exp(-\psi) E_+ + \frac{1}{2}\frac{\partial\psi}{\partial y} H + \frac{1}{\lambda}\exp(\psi) E_-\right) dy
\end{aligned}$$

This is a differential 1-form, with values in the $SU(2)$ algebra.

These should be compared with the variables

$$\mathcal{A}_\pm = A_\pm + \sqrt{\frac{\kappa}{2}}\psi$$

Indeed it is almost the same thing, since

$$\begin{aligned}
A/H &\sim a^* = \partial_z \ln \phi_1 \\
&= \frac{\partial\psi}{\partial x}
\end{aligned}$$

NOTE
This is the sinh-Gordon equation.
In **Kotlyarov** it is said that this is connected with the surfaces that have constant negative Gaussian curvature. Check from **Bobenko** if the CG curvature is or not for Liouville equation.
END

The connection

principal chiral model
sinh-Gordon equation
surfaces with constant negative Gaussian curvature

12 The two-dimensional Grassmannian models and the CP^N models

From the paper of **Zakrzewski.**

The model is defined on the basis of introduction of the complex Grassmann manifold

$$G(M,N) = \frac{U(N)}{U(M) \times U(N-M)}$$

The first variable to be defined over the

$$(x_1, x_2) \text{ plane}$$

is an element of $U(N)$:

$$g(x_1, x_2) \in U(N)$$

defined as

$$g = (Z, Y)$$

where

$$\begin{aligned} Z &= (Z_1, Z_2, ..., Z_M) \\ Y &= (Z_{M+1}, Z_{M+2}, ..., Z_N) \end{aligned}$$

and the variables Z_k represent *column vectors* of N elements.

The unitarity of g gives

$$Z_i \cdot Z_k = \delta_{ik}$$

The *dynamical variable* of the Grassmann model is the matrix Z

$$Z = N \times M \text{ matrix}$$

with

$$Z^\dagger Z = 1$$

The Lagrangian

$$L = \text{tr}\left[(D_\mu Z)^\dagger (D_\mu Z)\right]$$

$$S = \int d^2 x L$$

$$D_\mu = \partial_\mu - A_\mu$$

where

$$A_\mu \equiv Z^\dagger \partial_\mu Z$$

$$A_\mu^\dagger = -A_\mu$$

The topological charge density is

$$q(x) = i\varepsilon_{\mu\nu}\partial_\mu\left[\text{tr}\left(Z^\dagger \partial_\nu Z\right)\right]$$

This is simply

$$\begin{aligned} q\left(x\right) &= i\varepsilon_{\mu\nu}\partial_{\mu}\left(\mathrm{tr}A_{\nu}\right) \\ &\sim i\mathbf{\nabla}\times\mathbf{A} \\ &= i\mathbf{B} \end{aligned}$$

and it results that the density of the topological charge is the magnetic field. The total charge is the $2D$ integration of $q\left(x\right)$ which means the total magnetic flux through the $2D$ surface.

For comparison this would mean that

$$\begin{aligned} \left|\phi_1\right|^2 &= \rho_1 \\ \phi_1 &= D_{\mu}Z \\ &= \left(\partial_{\mu}-A_{\mu}\right)Z \end{aligned}$$

and we should ask: who is Z?

It is interesting to find that the action and the topological charge are expressed as

$$L = 2\mathrm{tr}\left[\left(D_+Z\right)^{\dagger}\left(D_+Z\right)+\left(D_-Z\right)^{\dagger}\left(D_-Z\right)\right]$$

$$q = 2\mathrm{tr}\left[\left(D_+Z\right)^{\dagger}\left(D_+Z\right)-\left(D_-Z\right)^{\dagger}\left(D_-Z\right)\right]$$

13 Ward CP^1 Manton solitons moving on geodesics

The idea of **Manton** of taking the kinetic energy as defining the metric on a manifold.

The solitons are moving on the geodesics of the manifold.

The idea is close to that of **Jacobs-Rebbi** that have used this approach to study the interaction between ANO vortices of the Abelian-Higgs superconductor, close to the critical λ. Attraction or repulsion.

CP^1 lumps.

Motion close to the manifold of static configurations.

"One can have a ring whose centre remains fixed, but whose radius decreases to a minimum and then increases again."

"there are solutions in which the configuration 'oscillates' between the ring-type and the separate-lump-type"

The CP^1 equivalent to $O(3)$ in $2+1$ dimensions

$$\mathbf{R}^2 \times T\,(time) \rightarrow u \rightarrow CP^1$$

$$\begin{aligned} & \frac{1}{\left(1+|u|^2\right)^2}\left(\frac{\partial u}{\partial x^\mu}\right)\left(\frac{\partial u}{\partial x_\mu}\right)^* \\ \mu &= 0,1,2 \end{aligned}$$

The classical equation of motion

$$\left(1+|u|^2\right)\Box u = 2u^* \;\left(\partial_\mu u\right)\left(\partial^\mu u\right)$$

Note see **Alfieri Zumino** comparison between classical solutions and solutions of YM. **End**

Here

$$\Box = \partial_\mu \partial^\mu$$

The equation of motion can be seen as the condition of solubility of the pair of over-determined equations

$$\begin{aligned} \Box u &= 0 \\ \left(\partial_\mu u\right)\left(\partial^\mu u\right) &= 0 \end{aligned}$$

Solution of this pair of equations
- take two complex analytic functions f and g
- they can have singularities or be multi-valued
- write the equation

$$z - f(u)\;\, t + f(u)^2\;\, z^* = g(u)$$

- solve for u

Here

$$\begin{aligned} z^0 &\equiv t \\ z &= \frac{1}{2}\left(x^1 + ix^2\right) \end{aligned}$$

14 The cosmological string

See also **Lee.**

See **Zee**.

From **superconducting strings**.

It is reminded the approach of Witten.

The Lagrangian

$$
\begin{aligned}
L &= -\frac{1}{2}\left(D_{\mu}\phi\right)^{*}\left(D^{\mu}\phi\right)-\frac{1}{2}\left(D_{\mu}\sigma\right)^{*}\left(D^{\mu}\sigma\right) \\
&-\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu}-\frac{1}{16\pi}H_{\mu\nu}H^{\mu\nu} \\
&-V\left(\phi,\sigma\right)
\end{aligned}
$$

The covariant derivatives

$$D^{\mu}\phi=\frac{\partial}{\partial x_{\mu}}+iqC^{\mu}$$

$$D^{\mu}\sigma=\frac{\partial}{\partial x_{\mu}}+ieA^{\mu}$$

and

$$
\begin{aligned}
V\left(\phi,\sigma\right) &= \frac{1}{8}\lambda_{\phi}\ \left(\left|\phi\right|^{2}-\eta^{2}\right)^{2} \\
&+\frac{1}{4}\lambda_{\sigma}\ \left|\sigma\right|^{4}-\frac{1}{2}m^{2}\left|\sigma\right|^{2} \\
&+f\ \left|\phi\right|^{2}\left|\sigma\right|^{2}
\end{aligned}
$$

The stationary superconducting string carries *uniform* current.
Cylindrical symmetry

$$\phi\left(\mathbf{x}\right)=\left|\phi\left(r\right)\right|\exp\left(in\varphi\right)$$

$$\sigma\left(\mathbf{x}\right)=\left|\sigma\left(r\right)\right|\exp\left[i\psi\left(z\right)\right]$$

$$A^{\mu}\rightarrow A_{z}\left(r\right)$$

$$C^{\mu}\rightarrow C_{\theta}\left(r\right)$$

The string has no net charge

$$\frac{\partial\psi}{\partial t}=0$$

and the current is *uniform* along the string

$$\frac{\partial^{2}\psi}{\partial z^{2}}=0$$

15 The σ model

In the paper **Disoriented Chiral Condensate Phys. Rep**. it is given the following example of **linear σ model** with *chiral symmetry breaking* term

$$S = \int d^4x \left[\frac{1}{2} (\partial_\mu \phi^a) (\partial^\mu \phi^a) - \frac{\lambda}{4} \left(\phi^a \phi^a - v^2 \right)^2 + H^a \phi^a \right]$$

where

$$H_a \equiv (\mathbf{0}, H)$$

points in the σ direction in the internal *chiral* space. Here a is the index of the components in the internal isospin space.

It is seen that it choses one direction in isospin space and forvorizes the alignement of the field ϕ_a with that direction. This breaks the chiral symmetry.

Note. The fact that the components of the field $\phi \equiv (\phi^a)$ are in the *internal isospin space* or in the simple $3D$-space like in the classical model $\boldsymbol{\phi} \equiv \left(\phi^1, \phi^2, \phi^3\right)$ is irrelevant. **End.**

Isovector current

$$\overrightarrow{V}^\mu = \overrightarrow{\pi} \times \partial^\mu \overrightarrow{\pi}$$

and isovector axial current

$$\overrightarrow{A}^\mu = \overrightarrow{\pi} \partial^\mu \sigma - \sigma \partial^\mu \overrightarrow{\pi}$$

In the review **TwoD sigma model Novikov** it is shown that the SUSY extension of the $O(3)$ sigma model has *axial anomaly*. The divergence of the axial current is given by the density of the topological charge of the σ model

$$\partial_\mu j^{5\mu} = \varepsilon^{abc} \varepsilon^{\nu\rho} \sigma^a \left(\partial_\nu \sigma^b\right) (\partial_\rho \sigma^c)$$

NOTE

The form of this expression is the same as the integrand of the Hopf invariant

$$F_{ij} = \varepsilon_{abc} \, n^a \, \left(\partial_i n^b\right) \left(\partial_k n_j^c\right)$$

See **Ward**. The contraction on the group (internal space) is complete, it is a *mixed vector product*. The two spatial indices (i, j) come from the direction of derivative on the base space, S^3 or $\mathbf{R}^3$.

the expression in the RHS is

$$\varepsilon^{ij} F_{ij}$$

This is a version of the more complex expression

$$F\widetilde{F}$$

which is the helicity and it is indeed the *divergence of a topological current.*

END

15.1 Two-dimensional *sigma* models: modelling non-perturbative effects in quantum chromodynamics

The review by **Novikov, Shifman, Veinshtein and Zakharov**.

The model is the $O(N)$ sigma model, with special emphasis on large N.
The dimension of the base space is $1+1$ but it is an Euclidean metric, $2D$.
The Lagrangian

$$L = \frac{N}{2f} \left(\partial_\mu \sigma^a(x)\right) \left(\partial^\mu \sigma^a(x)\right)$$

with the constraint

$$\sigma^a(x) \sigma^a(x) = 1$$

For 1+1 dimensions (and for 2 Euclidean space), and for $N=3$ (which means target $O(3)$) there are INSTANTONS, topological solutions, non-perturbative solutions. This is the $O(3)$ model in *plane* ($2D$) if the compactification is possible by the boundary conditions.

For $2D$ and $N \neq 3$ there are still non-perturbative effects.

The Euclidean action is the *kinetic* part plus the *constraint* of unitarity, with a Lagrange multiplier

$$S_E = \frac{1}{2} \int d^2x \left\{ \left(\partial_\mu \sigma^a(x)\right) \left(\partial^\mu \sigma^a(x)\right) + \frac{\alpha(x)}{\sqrt{N}} \left(\sigma^a(x) \sigma^a(x) - \frac{N}{f} \right) \right\}$$

where $\alpha(x)$ is a Lagrangian multiplier.

The generating functional is

$$Z_E[J] = \int D[\sigma^a(x)] D[\alpha(x)] \exp\left\{ -S_E + \int d^2x J^a(x) \sigma^a(x) \right\}$$

Since the action is *quadratic* in the field variables $\sigma^a(x)$, we can integrate over $\sigma^a(x)$ and get

$$\begin{aligned} Z_E[J] &= \int D[\alpha(x)] \exp(-S_{eff}) \\ &\times \exp\left\{ \frac{1}{2} \int d^2x J^a(x) \left[\frac{1}{-\partial^2 + \alpha(x)/\sqrt{N}} \right] J^a(x) \right\} \end{aligned}$$

where the effective action is the *determinant* resulting after the integration over the quadratic action

$$S_{eff} = \frac{N}{2} \mathrm{Tr} \ln \left[-\partial^2 + \frac{\alpha(x)}{\sqrt{N}} \right] - \int d^2x \frac{\sqrt{N}}{2f} \alpha(x)$$

Now, in order to calculate the functional integral over the function $\alpha(x)$ we can use the saddle-point technique. This means that we look for a *value* of $\alpha(x)$ that makes extremum the effective action. This value is noted

$$\alpha(x) = \sqrt{N} m^2$$

and around it the field $\alpha(x)$ has fluctuations, called quantum

$$\alpha(x) = \sqrt{N}m^2 + \alpha_q(x)$$

The effective action is *expanded* in the small quantum fluctuations. The second line below is the expansion of $\ln(1+...)$ function

$$\begin{aligned} S_{eff} &= \frac{N}{2}\mathrm{Tr}\ln(-\partial^2 + m^2) \\ &+ \frac{N}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\mathrm{Tr}\left[\frac{1}{-\partial^2 + m^2}\frac{\alpha_q(x)}{\sqrt{N}}\right] \\ &- \int d^2x\frac{Nm^2}{2f} - \frac{\sqrt{N}}{2f}\int d^2x\alpha_q(x) \end{aligned}$$

The first term in the first line and the first term on the third line are inessential constants and can be omitted.

Suppression of the linear term since we expect that the Lagrange multiplier is at the saddle point.

Consequences:

1. asymptotic freedom: the coupling constant vanishes for the mass infinite
2. dimensional transmutation: the mass depends on the coupling constant in a non-analytical way

16 The $O(3)$ model, the Skyrme model and the Faddeev model

Papers on skyrmions and domain walls in *biblio, classical systems, skyrmions condensed matter.*

Matryoshka skyrmions.

Ribbons domain walls.

Nagaosa.

16.1 Introduction

The models

- the mapping

$$\mathbf{R}^2 \sim S^2 \to S^2$$

  is the **Belavin Polyakov** system $O(3)$ defined on the plane $\mathbf{R}^2$ compactified to S^2. The base is compactified only topologically, not geometrically. Derrick theorem shrinks solutions.

- the mapping

$$\mathbf{R}^3 \to S^2$$

which maps the $(3+1)$ −dimensional space to S^2 is the $O(3)$ nonlinear sigma model. It has topological solitons if the boundary conditions (i.e. in spatially far points) are adequate.

- **Kundu Rybakov**. The base space is

$$(3+1)\,D$$

and the target is S^2. The invariance is

$$G = diag\left[O(2)_I \times O(2)_S\right]$$

where $I \equiv$ rotation around the n_3 (internal-space) axis; and $S \equiv$rotation around the z (base-space) axis. The mapping

$$\begin{array}{rcl} \mathbf{R} \times \mathbf{R}^3 & \to & S^2 \\ (\text{time, } 3-\text{space}) & \to & (\text{internal space of unitary vectors}) \\ & & N_a(t, \mathbf{x}) \\ \left|N(t, \mathbf{x})\right|^2 & = & 1 \end{array}$$

and

$$N_a(t, \infty) \overset{\text{asymptotic}}{=} \delta_{a3}$$

then

$$S^3 \to S^2$$

where S^3 is the compactification of $\mathbf{R}^3$, is Hopf mapping. Here we have lines in the base space S^3 (compactified $\mathbf{R}^3$) corresponding to given points in S^2. There is topological Hopf index,

$$Q = \pi_3\left(S^2\right) \sim \mathbf{Z}$$

which gives the *linking* of the lines. See below. This may be the **Faddeev** model, with the supplementary term that removes the effects of Derrick theorem.

- the mapping

$$\mathbf{R}^3 \to SU(2) \sim S^3$$

is the Skyrme mapping. The basis space is euclidean $\mathbf{R}^3$. The target space is the group manifold of $SU(2)$, the elements of which are matrices

U depending on the point $\mathbf{x}$ of the base space. The Energy of Skyrme model (**Manton Ruback**)

$$E = \int d^3x \left\{ -\frac{1}{2}\mathrm{Tr}\left(\frac{\partial U}{\partial x^i}U^{-1}\ \frac{\partial U}{\partial x^i}U^{-1}\right) \right.$$
$$\left. -\frac{1}{16}\mathrm{Tr}\left[\frac{\partial U}{\partial x^i}U^{-1}\ ,\ \frac{\partial U}{\partial x^j}U^{-1}\right]^2 \right\}$$

In the Skyrme model the soliton $Q = 1$ is the hedgehog.

- the mapping
$$S^3 \to SU(2) \sim S^3$$
is the Skyrme mapping on the compactification of $\mathbf{R}^3$. There is a *radius* of compactification, L. The limit $L = \sqrt{2}$ separates stable identity mapping from the concentrated solution around a point (**Ward**). Comments in *extreme events* on this.

- the **Faddeev** mapping
$$\left.\begin{array}{c}\mathbf{R}^3\\ S^3\end{array}\right\} \to S^2$$
with fourth-degree term in derivatives, is Faddeev (also called Skyrme Faddeev). Note that the target is no more $SU(2)$ but is the usual *sphere* like in nonlinear sigma $O(3)$. There is no exactly known soliton, but approximations. **Niemi** finds the electron-ion plasma model whose solutions are toroidal vortices. Also **Niemi** finds a straight twisted vortex.

One starts with the $O(3)$ *sigma model* modified with a term of fourth degree in the gradients, to stabilize the soliton solutions of the equations of motion.

The field is

$$\mathbf{n} = (n_1, n_2, n_3)$$
$$\mathbf{n}\cdot\mathbf{n} = 1 \quad \text{(unit vector)}$$

The Lagrangian, in

$$(3+1) \ \text{dimensions}$$
$$(\mathbf{x}, t)$$

is (**Faddeev**)

$$\mathcal{L} = (\partial_\mu \mathbf{n})(\partial^\mu \mathbf{n}) - \frac{1}{2}(\partial_\mu \mathbf{n}\times\partial_\nu \mathbf{n})\cdot(\partial^\mu \mathbf{n}\times\partial^\nu \mathbf{n})$$

The condition imposed to the field, at ∞, is

$$\mathbf{n}_\infty = (0,0,1)$$

Note the nonlinear term that can be written $F_{\mu\nu}F^{\mu\nu}$. This is the way it is defined a *gauge field* whose potential A_μ can further be expressed through z and $z^\dagger$ from $SU(2)$.

Two distinct objects, as discussed by **Battye Sutcliffe**.

- the *Hopf map*. This is related to the $O(3)$ model. It is a map from the Euclidean space $\mathbf{R}^3$ to the unit vector $\mathbf{n}$ with the tip on some point of a sphere S^2,
$$\mathbf{n} : \mathbf{R}^3 \to S^2$$
with the constraint that the field $\mathbf{n}(\mathbf{x})$ takes the same direction for all asymptotic points in $\mathbf{R}^3$. This compactifies $\mathbf{R}^3$ to S^3 and the Hopf map becomes - only topologically, not by the metric,
$$S^3 \overset{\mathbf{n}(\mathbf{x})}{\to} S^2$$
The homotopy is not trivial
$$\pi_3\left(S^2\right) = \mathbf{Z}$$
This map has a *topological degree*, calculated by the *Hopf index*.

- the Skyrme field $U(\mathbf{x})$ is a map from the real space $\mathbf{R}^3$ to the algebra manifold of $SU(2)$, which topologically is S^3
$$SU(2) \sim S^3$$
(remember the algebra of $SU(2)$ has three *generators*, $E_\pm$ plus Cartan subalgebra generator H, therefore every element of the algebra needs three numbers to be defined, the coefficients of these generators. The same is the number of independent variables needed to define the hypersphere S^3). Then the mapping $U(\mathbf{x})$ takes a real space point $\mathbf{x}$ and gives a *matrix* of $SU(2)$
$$\mathbf{R}^3 \sim S^3 \overset{U(\mathbf{x})}{\to} S^3$$
This map has *winding number*.

We can start from a general structure of the Skyrme map,

$$\begin{aligned} U(\mathbf{x}) &= \begin{pmatrix} Z_0 & -\overline{Z}_1 \\ Z_1 & \overline{Z}_0 \end{pmatrix} \\ \text{where } |Z_0|^2 + |Z_1|^2 &= 1 \end{aligned}$$

and look for a construction that would produce a Hopf map.

This means to find an expression for the unit vector $\mathbf{n}(\mathbf{x}) : S^3 \to S^2$.

This is
$$\mathbf{n} = Z^\dagger \boldsymbol{\tau} Z$$

where $\boldsymbol{\tau}$ is Pauli and the complex column matrix

$$Z = \begin{pmatrix} Z_0 \\ Z_1 \end{pmatrix}$$

Then, the *winding number* B of the Skyrme field $U(\mathbf{x})$ is the *topological Hopf index* Q of the Hopf field $\mathbf{n}(\mathbf{x})$

$$B^{Skyrme} = Q^{Hopf}$$

Latter it is constructed a Hopf map using a Skyrme field.

This is because to construct a Skyrme field it is possible using *rational maps* between Riemann spheres.

Mapping $A_i^a \in su(2) \rightarrow \varepsilon^{abc}\partial_i n_b \; n_c$ **Faddeev**. *Studii 19 october 2015.*

Some notes are in *Fluids.*

The possibility to use Faddeev Skyrme model for a unit vector whose components are the three terms in the *relative helicity* (**Lili, Moffat**) is discussed in *fluids.*

Note

The term introduced by **Faddeev** has four derivatives and excludes the shrinking due to Derrick.

This term is given as (**9811176_hopf_solitons_S3_and_R3_Ward**)

$$e_4 = \frac{1}{4} g^{ij} g^{kl} \; F_{ik} F_{jl}$$

$$F_{ik} = \varepsilon_{abc} \; n^a \; \left(\partial_i n^b\right)\left(\partial_k n^c\right)$$

and one notes that the last line is like a tensor of an electromagnetic field (after the indices in the internal space a, b, c are all contracted in a mixed vector product).

Note remember the association or Definition $A_i^a \in su(2) \rightarrow \varepsilon^{abc}\partial_i n_b \; n_c$. **End.**

It is here that a gauge potential may be introduced. One simply seeks the gauge potential which is behind the tensor F.

Then one can calculate the *helicity* $\mathbf{A} \cdot \mathbf{B}$.

And in this way there is a reformulation from

- the topological mapping $\mathbf{R}^3$ $\left(\text{in fact } S^3\right) \rightarrow S^2$ with Hopf number and
- the *helicity, the Chern Simons* term of a gauge theory, A_i^a derived from $\mathbf{n}$.

16.2 The $O(3)$ model on the physical plane, $2D$ (basis) to S^2 (target)

The paper is **Belavin Polyakov**.

The topology is *sphere over sphere.*

$$S^2 \to S^2$$

The Hamiltonian of the $O(3)$ in the absence of the Faddeev nonlinearity

$$H = \int d^2x \sum_{a=1}^{3} (\boldsymbol{\nabla} n^a)^2$$

(just the kinetic term)

The vector in the target space (sphere S^2)

$$\mathbf{n} = (\cos\theta,\ \sin\theta\cos\varphi,\ \sin\theta\sin\varphi)$$

the boundary conditions

$$\mathbf{n}(x) \to (0,0,1) \quad \text{at} \quad |\mathbf{x}| \to \infty$$

which allows compactification of the base manifold.

Then there is a mapping

$$S^2\,(\text{compactified plane}) \to S^2\ \ (\text{space of internal symmetry})$$

This is a q-degree mapping

$$q = \frac{1}{8\pi}\int d^2x\ \varepsilon_{abc}\ \varepsilon_{\mu\nu}\ n^a\ \frac{\partial n^b}{\partial x_\mu}\frac{\partial n^c}{\partial x_\nu}$$

it is the number of times a sphere covers the other sphere.

Note that, being in $2D$ for the domain, we can make full contraction on spatial (*base space)* indices using $\varepsilon_{\mu\nu}$. This is a vector product and the result is along the third, invisible, coordinate.

There is nothing left for a *current*, no supplementary indice. In $3+1$ domensions, the two derivatives of $\mathbf{n}$ of the target (internal) space to two of the coordinates of the real space domain still do not exhaust the number of indices and a tensor $\varepsilon_{\mu\nu\rho}$ preserves an indice for that partial contraction to be a current, J^μ like in **Zee**. **End.**

Note

Taking the product and contraction with $\varepsilon_{\mu\nu}$ means to take the *dual*

$$\varepsilon_{\mu\nu}F^{\mu\nu}$$

where the tensor $F^{\mu\nu}$ results from the *mixed vector product* in the target space.

This *dual* is a topological density.

It is the *divergence of a current.*

In **Aratyn Ferreira** the magnetic field is

$$B_i = \varepsilon_{ijk} F_{jk}$$

and is further used to determine - Biot-Savart - the potential $\mathbf{A}$ since we need the volume integral of the density of helicity $\mathbf{A} \cdot \mathbf{B}$.

End.

Inequality **Belavin Polyakov**

$$\left(\frac{\partial n^a}{\partial x_\mu} - \varepsilon_{abc} \, \varepsilon_{\mu\nu} \, n^b \frac{\partial n^c}{\partial x_\nu} \right)^2 \geq 0$$

Then

$$H = \int d^2 x \, \left(\frac{\partial n^a}{\partial x_\mu} \right)^2 \geq 8\pi q$$

There is a lower value for the energy. Here $q = \frac{1}{8\pi} \int d^2 x \, \varepsilon_{abc} \, \varepsilon_{\mu\nu} \, n^a \, \frac{\partial n^b}{\partial x_\mu} \frac{\partial n^c}{\partial x_\nu}$.

Define the new set of variables (w_1, w_2) and use them to construct a *complex* variable w,

$$\begin{aligned} w_1 &= \cot \frac{\theta}{2} \cos \varphi \\ w_2 &= \cot \frac{\theta}{2} \, \sin \varphi \\ w &= w_1 + i w_2 = \cot \frac{\theta}{2} \exp (i\varphi) \end{aligned}$$

The lowest limit of the energy is attained when the paranthesis is zero

$$\frac{\partial n^a}{\partial x_\mu} - \varepsilon_{abc} \, \varepsilon_{\mu\nu} \, n^b \frac{\partial n^c}{\partial x_\nu} = 0$$

Here one can use the trigonometric expressions of the components of the vector $\mathbf{n}$ and after that one replaces the angles (θ, φ) in terms of variables (w_1, w_2) with the result

$$\begin{aligned} \frac{\partial w_1}{\partial x_1} &= \frac{\partial w_2}{\partial x_2} \\ \frac{\partial w_2}{\partial x_1} &= -\frac{\partial w_1}{\partial x_2} \end{aligned}$$

This is the Cauchy-Riemann system for

$$\begin{aligned} f(z) &= w_1 + i w_2 \\ z &= x_1 + i x_2 \end{aligned}$$

This is similar to **Zee**.

The general form of w as $f(z)$ is determined by *zeros* and *poles* and has the expression

$$\begin{aligned} w &= \cot\frac{\theta}{2}\exp(i\varphi) \\ &= \prod_i \left(\frac{z-z_i}{\lambda}\right)^{m_i} \prod_j \left(\frac{\lambda}{z-z_j}\right)^{n_j} \\ \text{with } \sum m_i &> \sum n_j \quad \text{(more zeros than poles)} \end{aligned}$$

In **Rajaraman** the model $O(3)$

$$\begin{aligned} L &= \frac{1}{2}(\partial_\mu \boldsymbol{\phi})\cdot(\partial^\mu \boldsymbol{\phi}) \\ \text{with the constraint } \boldsymbol{\phi}\cdot\boldsymbol{\phi} &= 1 \end{aligned}$$

in the base space $2D$ is

$$\boldsymbol{\nabla}^2\boldsymbol{\phi} - \left(\boldsymbol{\phi}\cdot\boldsymbol{\nabla}^2\boldsymbol{\phi}\right)\boldsymbol{\phi} = 0$$

which comes from

$$\begin{aligned} \partial_\mu\partial^\mu\boldsymbol{\phi} + \lambda\boldsymbol{\phi} &= \mathbf{0} \\ \text{or } \Box\boldsymbol{\phi} + \lambda\boldsymbol{\phi} &= \mathbf{0} \end{aligned}$$

$$\lambda(\mathbf{x},t) = -\boldsymbol{\phi}\cdot\Box\boldsymbol{\phi}$$

Alfaro Fubini draw a parallel between the scalar classical equation

$$\Box\phi + \lambda\phi = 0$$

and the solution of the Yang Mills equations A_μ, as given by **Belavin Polyakov** **.**

16.3 Linking, spin, statistics Zee Wilczek

The σ model in $(2+1)\,D$

$$\begin{aligned} E &= \frac{1}{2f}\int d^2x \left(\frac{\partial n^a}{\partial x^i}\right)^2 \quad \text{(kinetic)} \\ i &= 1,2 \ \text{(i.e. } x,y) \\ a &= 1,2,3 \quad \text{(internal space Sphere)} \\ \text{and } n^a n^a &= 1 \end{aligned}$$

Ground state

$$n^a = (0,0,1)$$

then the hedgehog

$$\begin{aligned} n^a &= (\widehat{\mathbf{e}}\cos f \ , \ \sin f) \\ \widehat{\mathbf{e}} &= \left(\frac{x^1}{|\mathbf{x}|} \ , \ \frac{x^2}{|\mathbf{x}|}\right) \end{aligned}$$

This means

$$n^a = \left(\frac{x^1}{|\mathbf{x}|}\cos f \ , \ \frac{x^2}{|\mathbf{x}|}\cos f \ , \ \sin f\right)$$

and

$$\begin{aligned} f(\mathbf{x}) &= f(r) \\ \text{and} \ \ f(r=0) &= \pi \\ f(\infty) &= 0 \end{aligned}$$

NOTE compare with **Niemi electron-ion twisted vortex**. Similar reparametrization. **END.**

The topological current

$$J^\mu = \frac{1}{8\pi} \ \epsilon^{\mu\nu\lambda} \ \epsilon^{abc} \ n^a \ \frac{\partial n^b}{\partial x^\nu} \ \frac{\partial n^c}{\partial x^\lambda}$$

[mixed product, full contraction, in target *internal* space, and vector product, flux, in real *base* space, $3D$].

Here we have

$$S^2 \to S^2$$

with topological charge

$$Q = J^0$$

The skyrmion.

Find the spin of the skyrmion

First, change

$$n^a \to z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

(from 3 variables $n^a : n^1, n^2, n^3$ with $n^a n_a = 1$ i.e. effectively two variables,

- to 4 variables in z_1 and z_2 with $|z_1|^2 + |z_2|^2 = 1$, effectively three variables),

$$n^a = z^\dagger \ \sigma^a \ z$$

$$\begin{aligned} |z_1|^2 + |z_2|^2 &= 1 \\ & \quad \text{this is } S^3 \end{aligned}$$

Now we have

$$\pi_3\left(S^2\right)$$

There is invariance

$$z \rightarrow \exp(i\theta)\ z$$

To find the spin (Feynman)
rotate the skyrmion adiabatically by 2π in a time T very long.
at the end of this rotation the wave function aquires a phase factor

$$\exp(iS)$$

where S is the action along this trajectory.

The angular momentum J is

$$\exp(iS) = \exp(i2\pi\ J)$$

If the action is

$$S_0 = \frac{1}{2f}\int d^2x\ \left(\frac{\partial n^a}{\partial x^i}\right)^2 \qquad \text{(kinetic)}$$

then this action is

$$S_0 \sim \frac{1}{T}$$

and

$$\begin{array}{rcl} S_0 & \rightarrow & 0 \\ \text{for } T & \rightarrow & \infty \end{array}$$

But to the action one must add *topological terms*

$$S = S_0 + \theta\ H$$

where

$$H \equiv \text{Hopf invariant}$$

How is constructed the Hopf invariant.

Start with the current J^μ, $\mu = 0, 1, 2$.

This is conserved, i.e. has zero-divergence, $\partial_\mu J^\mu = 0$ so it is possible to be derived from the rotational of a vector.

Then we introduce a vector A_μ, such that the current results as curl

$$\begin{array}{rcl} J^\mu & = & \epsilon^{\mu\nu\lambda}\ \partial_\nu A_\lambda \\ & = & \frac{1}{2}\epsilon^{\mu\nu\lambda} F_{\nu\lambda} \end{array}$$

NOTE

This is similar to the definition of the velocity in plane in terms of the streamfunction ψ in plane,

$$\mathbf{v} = \boldsymbol{\nabla}\times [\widehat{\mathbf{e}}_z \ \psi]$$

and this suggests that

$$F_{\nu\lambda} \sim *\mathbf{v}$$

END

The Hopf invariant is defined as

$$\begin{aligned} H &= -\frac{1}{4\pi}\int d^3x \ \epsilon^{\mu\nu\lambda} A_\mu \ F_{\nu\lambda} \\ &= -\frac{1}{2\pi}\int d^3x \ A_\mu J^\mu \end{aligned}$$

This is the Abelian version of the mass term by **Deser Templeton Jackiw**.
since H is gauge invariant, the coefficient θ is NOT discrete (quantized)

16.4 Skyrmions in flat and curved space Manton Ryback

The base space is spatial euclidean $3D$.
There is no time dependence, static classical solutions.
The target space is

$$SU(2)$$

This means that, instead of the unitary vector $\mathbf{n}(x)$ in the target space, we have a 2×2 matrix unit complex entries [and special].
The energy

$$\begin{aligned} E &= \int d^3x \left\{ -\frac{1}{2}\mathrm{Tr}\left(\frac{\partial U}{\partial x^i}U^{-1} \ \frac{\partial U}{\partial x^i}U^{-1}\right)\right. \\ &\quad \left. -\frac{1}{16}\mathrm{Tr}\left[\frac{\partial U}{\partial x^i}U^{-1} \ , \ \frac{\partial U}{\partial x^j}U^{-1}\right]^2\right\} \end{aligned}$$

where

$$U \in SU(2)$$

This expression of the energy can be rewritten

$$\begin{aligned} E &= \int d^3x \left\{ -\frac{1}{2}\mathrm{Tr}\left(\frac{\partial U}{\partial x^i}U^{-1} \pm \frac{1}{4}\varepsilon_{ijk}\left[\frac{\partial U}{\partial x^j}U^{-1} \ , \ \frac{\partial U}{\partial x^k}U^{-1}\right]\right)^2 \right. \\ &\quad \pm 12\pi^2 \int d^3x \frac{1}{24\pi^2}\varepsilon_{ijk}\mathrm{Tr}\left(\frac{\partial U}{\partial x^i}U^{-1} \ \frac{\partial U}{\partial x^j}U^{-1} \ \frac{\partial U}{\partial x^k}U^{-1}\right) \end{aligned}$$

The second term is the *degree of the map U*.
NOTE. Is-this a tentative to write as in **Bogomolnyi** procedure? **END**

NOTE seems the same approach as **Zee Wilczek** where they wanted to find the spin by (Feynman) turning in very long time T around a closed loop with the action becoming $S = S_0 + \theta\ H$, where H is the Hopf invariant.
END

NOTE
The first part seems the non-Abelian generalization of the **Polyakov Belavin** model
END

It results a topological bound

$$E \geq 12\pi^2 \, |B|$$

16.5 The $O\,(3)$ model in $3D$

Part of the following are connected with *fluids notes.*
See also *Notes_ topological solutions.*

In **Zakrzewski** we have the CP^1 in the base space $(3+1)\,D$

$$\begin{aligned} L &= (D_\mu z)^\dagger \,(D^\mu z) \\ \text{with the constraint } z^\dagger z &= 1 \\ \text{here } \mu &\equiv 0,1,2,3 \end{aligned}$$

where

$$z \equiv (z_1, z_2)$$

The covariant derivative is

$$D_\mu z = \frac{\partial z}{\partial x^\mu} - \left(z^\dagger \frac{\partial z}{\partial x^\mu} \right) z$$

or, shorter notation

$$D_\mu z = \partial_\mu z - \left(z^\dagger \partial_\mu z \right) z$$

This expressions are useful since they seem to connect with the case where a gauge potential is considered, simply by the use of a *covariant derivative* operator. No dynamics of the gauge field exists.

The gauge potential is here

$$A_\mu = z^\dagger \frac{\partial z}{\partial x^\mu}$$

Remember that in **Zee Wu** the gauge field is introduced after definition of the topological current J and assuming a coupling with a potential A_μ for which it is added the Chern Simons Lagrangian.

The mapping takes a base manifold (like $\mathbf{R}^3$ or the $2D$ plane, etc.) into the target or internal space, where we have unitary vector

$$\mathbf{n}\cdot\mathbf{n}=1$$

One may look for various ways to expand or re-express this constraint.

As above, one can introduce instead of $\mathbf{n}$ a *complex* variable (field) z consisting of a *pair* of complex fields, $z=(z_1,z_2)$ and ask $z^\dagger z=1$, which has $4-1=3$ variables, like in **Zee**.

The further substitution is a *parametrization* that solves the constraint $z^\dagger z=1$

$$z=\frac{(1\ ,w)}{\sqrt{1+|w|^2}}$$

$$\left(1+|w|^2\right)\ \partial_\mu\partial^\mu w-2\left(w^*\ \partial^\mu w\right)\ \partial_\mu w=0$$

This parametrization is not restricted to the case of base space $2D$. The *stereographic* projection is in the internal (*i.e.* target) space.

Let us consider the current (see **Zee 2p1**) of the topological charge

$$J^\sigma=\varepsilon^{\sigma\mu\nu}\ \mathbf{n}\cdot\left(\frac{\partial}{\partial x^\mu}\mathbf{n}\times\frac{\partial}{\partial x^\nu}\mathbf{n}\right)$$

(this is the *vorticity* Ω_σ in **Faddeev** and in **Kuznetsov**). Here the base (real) space is $3+1$ since the completely antisymmetric tensor has three indices, $\varepsilon^{\sigma\mu\nu}$.

See **Nagaosa** skyrmions.

The topological current is mentioned in **Zee Wu duality 2p1**, (change of notation here $A\rightarrow F$)

$$\begin{aligned}J^\sigma &= \varepsilon^{\sigma\mu\nu}F_{\mu\nu}\\ &= \text{topological current}\end{aligned}$$

(**Note** that the equation derived from the Chern Simons lagrangian connects the field tensor F (contracted with $\varepsilon^{\sigma\mu\nu}$) directly with J, there is no derivative like in the Maxwell equations. We have the same connection in the above equation. This already suggests that the Lagrangian of $O(3)$ with **Faddeev** term is connected with Chern Simons).

Note normally we expect that the *topological current* to be derived from the Chern Simons winding number $\widetilde{F}F$, equal to the divergence of J.

In the description of the **Faddeev** model, the notation is $A_{\mu\nu}^{here}\rightarrow F_{\mu\nu}^{Faddeev}$ with suggestion of *field tensor*.

The tensor for **Faddeev** is

$$F_{\mu\nu}=\frac{1}{2}\mathbf{n}\cdot\left(\partial_\mu\mathbf{n}\times\partial_\nu\mathbf{n}\right)$$

and looks like the density of a *flux* on the base space (x, y, z). See *Notes topological solutions*. The explanation is also in **Ward**, the surface on the sphere.

See **Nagaosa**, this is the *solid angle*.

This is NOT yet the *vorticity*.

The vorticity **Faddeev** is the contraction in $3+1$ dimensions $\varepsilon^{\rho\mu\nu}$ with the field tensor $F_{\mu\nu}$.

If $F_{\mu\nu}$ is a flux through a surface, the dual $\varepsilon^{\rho\mu\nu}F_{\mu\nu}$ is a vector, a current J^{ρ}.

Then the dual of the flux (which is the current J^{σ} in **Zee** or the vorticity Ω_{α} in **Faddeev**) is a differential third form.

One would be tempted to consider Ω as a differential $2-$form since it is the rotational of the velocity and appears like a flux

$$\omega_{\alpha} dx \wedge dy$$

but this is connected with the *physical* definition of ω. There is also the mathematical definition (or association) of ω in terms of the unitary vector $\mathbf{n}$ which exists in the internal space. It introduces first $F_{\mu\nu}$ which is a flux. Then Ω is defined as the dual of this flux.

This discussion is not without meaning.

If Ω is a differential $3-$form then it is associated to the *current*. And not to the magnetic field.

Then what we must combine is

$$\Omega_{\alpha} + J_{\alpha}$$

both being rotationals of a flux.

This is in contradiction with the **Sagdeev Moiseev Tur Yanovskii** invariant, which combines Ω with B.

See in a paper the current of neutrinos along the vortex **Voloshin, Kharzeev** in *FieldThmodel MHD*

From the point of view of the mathematical explorations around the physical object *vorticity* the introduction of $\mathbf{n}$ as unitary vector in an internal space is arbitrary and does not provide a *necessary* (obligatory, mandatory) message.

However with this occasion we see how Ω can be written in terms of Clebsch variables and a vortex line appears as the intersection of two surfaces of constant λ respectively μ.

Introducing $\mathbf{n}$ we find which is λ and μ.

The same is the case for the current J.

The connection between the Skyrme field ($\mathbf{R}^3 \sim S^3 \rightarrow SU(2) \sim S^3$) with the Hopf mapping $\mathbf{n}(\mathbf{x})$. (discussed above, according to **Battye Sutcliffe**).

Zee makes the mapping

$$
\begin{aligned}
\mathbf{n} &\rightarrow z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \\
n^a &= z^\dagger \sigma^a z
\end{aligned}
$$

or

$$\frac{v_a}{v} = z^\dagger \sigma^a z$$

with the condition

$$z^\dagger z = 1$$

Here we go from two degrees of freedom (θ, φ) that define $\mathbf{n}$ (on the target sphere) to three degrees of freedom, (z_1, z_2) with the condition of unitary norm.

This is an *extension* of the possibilities of the formalism.

Here we also *split* the original, physical function, $\frac{v_i}{v}$, in a product of quantities that have algebraic content.

Now the change of the physical variable $\partial_\mu n^a$ in a displacement on the physical space is described alternatively as a change of the two new *complex* variables z_1 and z_2, change that consists of

- change of magnitudes $|z_1|$ and $|z_2|$;
- change of phases of each
- change of the algebraic function z; the up and down variables can be associated to spin up and spin down. The usual physical state is a mixture of the two spin orientation. The displacement on x^μ modifies the spin distribution.

Here the Pauli matrices are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

With this we write the explicit form of the components n^a,

$$
\begin{aligned}
n^1 &= \frac{v_1}{v} = \overline{\underline{\begin{matrix} z_1^* & z_2^* \end{matrix}}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \\
&= \overline{\underline{\begin{matrix} z_2^* & z_1^* \end{matrix}}} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = z_2^* z_1 + z_1^* z_2
\end{aligned}
$$

$$
\begin{aligned}
n^2 &= \frac{v_2}{v} = \overline{\underline{\begin{matrix} z_1^* & z_2^* \end{matrix}}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \\
&= \overline{\underline{\begin{matrix} iz_2^* & -iz_1^* \end{matrix}}} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = iz_2^* z_1 - iz_1^* z_2
\end{aligned}
$$

$$
\begin{aligned}
n^3 &= \frac{v_3}{v} = \begin{array}{c}\hline z_1^* \quad z_2^* \\ \hline\end{array} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \\
&= \begin{array}{c}\hline z_1^* \quad -z_2^* \\ \hline\end{array} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = |z_1|^2 - |z_2|^2
\end{aligned}
$$

For the last component n^3 we have

$$
\begin{aligned}
z^\dagger z &= 1 \\
\begin{array}{c}\hline z_1^* \quad z_2^* \\ \hline\end{array} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} &= 1 \\
|z_1|^2 + |z_2|^2 &= 1
\end{aligned}
$$

Then

$$
n^3 = \frac{v_3}{v} = |z_1|^2 - |z_2|^2 = 1 - 2|z_2|^2 = 1 - 2\rho_2
$$

(3 + 1 dim)
The topological current **Zee**

$$
J^\sigma = \varepsilon^{\sigma\mu\nu} \, \mathbf{n} \cdot \left(\frac{\partial}{\partial x^\mu} \mathbf{n} \times \frac{\partial}{\partial x^\nu} \mathbf{n} \right)
$$

is explicitely a curl, which is visible when instead of the unitary vector $\mathbf{n}$ we use the single-column-matrix z

$$
\begin{aligned}
J_\mu &= -\frac{i}{2\pi} \varepsilon_{\mu\nu\lambda} \left(\frac{\partial z}{\partial x^\nu} \right)^\dagger \left(\frac{\partial z}{\partial x^\lambda} \right) \\
&= -\frac{i}{2\pi} \varepsilon_{\mu\nu\lambda} \partial_\nu \left(z^\dagger \partial_\lambda z \right)
\end{aligned}
$$

This should be read

$$
J_\mu \text{ (Zee) is similar to the vorticity } \Omega_\mu \text{ (Faddeev)}
$$

i.e.

$$
\begin{aligned}
J_\mu \text{ (Zee) is a curl } &\rightarrow \text{ then we can find a } \mathbf{B} \\
&\quad\;\; \text{and} \\
\Omega_\mu \text{ (Faddeev) is a curl } &\rightarrow \text{ then we can find a } \mathbf{v}
\end{aligned}
$$

It suggests (as for the Ampere's law) that

$$
z^\dagger \partial_\lambda z
$$

is like a *magnetic field*, as

$$
\begin{aligned}
\text{magnetic field} &\sim z^\dagger \partial_\lambda z \\
& \quad\; (?)
\end{aligned}
$$

(since this is similar to $\mathbf{\nabla} \times \mathbf{B} = \mathbf{j}$).

NOTE

Actually we have in previous work **Zee**

$$A_\mu = z^\dagger \frac{\partial z}{\partial x^\mu}$$

and the above equation shows that $J_\mu - \frac{i}{2\pi}\varepsilon_{\mu\nu\lambda}\partial_\nu \left(z^\dagger \partial_\lambda z\right)$ is the rotational of the gauge potential $\mathbf{J} \sim \mathbf{\nabla} \times \mathbf{A} \sim \mathbf{B}$, like the magnetic field.

We have here a kind of force-free

$$\mathbf{J} \sim \mathbf{B}$$

END

Or, it suggests that the formula of **Faddeev** can be written

$$\begin{aligned} \Omega_\mu &= \varepsilon_{\mu\nu\rho}\ \partial_\nu v_\rho \\ &= \mathbf{\nabla} \times \mathbf{v} \end{aligned}$$

can also be written

$$\begin{aligned} \Omega_\mu &= -\frac{i}{2\pi}\varepsilon_{\mu\nu\lambda}\partial_\nu \left(z^\dagger \partial_\lambda z\right) \\ &= -\frac{i}{2\pi}\mathbf{\nabla} \times \mathbf{V} \end{aligned}$$

with the identification

$$\begin{aligned} V_\lambda &\equiv z^\dagger \partial_\lambda z \\ & \quad \text{a kind of "velocity"} \end{aligned}$$

Then the Hamiltonian density for ideal fluid (**Kuznetsov**)

$$\begin{aligned} H &= \frac{1}{2}\mathbf{V}^2 \\ &= \left(z^\dagger \partial_\lambda z\right)^\dagger \left(z^\dagger \partial_\lambda z\right) \end{aligned}$$

Note however that in the Clebsch variables it is the *potential* A which is written as

$$\begin{aligned} \mathbf{A} &= \lambda \mathbf{\nabla} \mu \\ & \quad + \mathbf{\nabla} \phi \end{aligned}$$

where the first term looks like $z^\dagger \partial_\lambda z$. Then: V_λ is-it $\mathbf{B}$ or $\mathbf{A}$?

Note however that λ, μ are scalars, but z is a column matrix.

End

The idea of Clebsch variables is to obtain the *line* of the magnetic field $\mathbf{B}$ as a line of intersection of two surfaces that are defined by the gradient of two scalar functions λ and μ.

Faddeev writes the vorticity in the form of (a constant A) $\times$ (the current of Hopf index).

This introduces the unitary vector $\mathbf{n}$.

It is interesting but NOT requested or necessary or derived from a particular theory.

It is just a possible writting, providing a mathematical representation of a line in $\mathbf{R}^3$.

But it provides the Clebsch variables

$$\begin{aligned} \lambda &\equiv \cos\varphi \\ \mu &= \theta \end{aligned}$$

and we have

$$\begin{aligned} \frac{\partial \lambda}{\partial t} + (\mathbf{V}\cdot\boldsymbol{\nabla})\,\lambda &= 0 \\ \text{or } \frac{\partial \varphi}{\partial t} + (\mathbf{V}\cdot\boldsymbol{\nabla})\,\varphi &= 0 \end{aligned}$$

which means that φ is *constant* along the trajectory, a Lagrangian invariant.

Also for θ.

The function θ is a Lagrangian invariant.

This simply means that moving along the vortex (?) we have φ and θ constants in the space of the internal symmetry of the vector $\mathbf{n}$.

The fixed (constant) (λ, θ) is a *point* in the target space.

The ensemble of points from the *base* space that correspond to a fixed pair (λ, θ) is a *line*.

This *line* is the vorticity line.

[Further, taking two fixed points (λ_1, θ_1) and (λ_2, θ_2) in the *target* space, there are two lines in the base space corresponding to them. The two lines have Gaussian linking that is given by the Hopf index].

This is natural since the mapping

$$S^3 \rightarrow S^2$$

means that a point on S^2:

$$(\varphi, \theta) = \text{fixed}$$

defines a *full* line of vorticity in S^3.

What do we get from this writting ?

We cannot justify at this moment an extremum of a Lagrangian. Why extremum ?

What is the physical meaning of the decomposition

$$\begin{aligned} n^a &\equiv \left(\frac{v_x}{v}, \frac{v_y}{v}, \frac{v_z}{v}\right) \\ &= z^\dagger \sigma^a z \\ &= \begin{array}{cc} z_1^\dagger & z_2^\dagger \end{array} \sigma_1 \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \end{aligned}$$

The Hopf nonlocal invariant

$$H = \int d^3x \; \varepsilon^{\mu\nu\lambda} J_\mu \; \partial_\nu \frac{1}{\partial^2} J_\lambda$$

$$H = \frac{i}{4\pi^2} \int \left(z^\dagger dz\right) \left(dz^\dagger dz\right)$$

A NOTE

Constantin in studying *Camassa-Holm* (in *integrability, studies, Camassa-holm*) introduced

$$G = \frac{1}{1 - \frac{\partial^2}{\partial x^2}} = \frac{1}{1 - \partial^2}$$

operator whose action on a function $f(x)$ is

$$G[f(x)] = \frac{1}{2} \int_{-\infty}^{\infty} dy \; \exp\left(-|x-y|\right) \; f(y)$$

But this example is in $1D$ and the Hopf H is in $3D$.

END

16.6 Variation around the Faddeev model Aratyn Ferreira

The space is $3+1$ dimensional.

The mapping is

$$\mathbf{R}^3 \rightarrow S^2$$

They adopt a Lagrangian

$$L = -\eta_0 \left(H_{\mu\nu}\right)^{3/4}$$

where

$$\eta_0 = \pm 1$$

in the metrics

$$g_{\mu\nu} = diag\left(1, -1, -1, -1\right)$$

The tensor

$$H_{\mu\nu} = \mathbf{n} \cdot \left(\frac{\partial \mathbf{n}}{\partial x^\mu} \times \frac{\partial \mathbf{n}}{\partial x^\nu}\right)$$

is $O(3)$ invariant.

The mixed product is in the *target* space, on the sphere

NOTE

The mixed product in the target space produces a scalar from the point of view of the internal coordinates n^a, $a = 1, 2, 3$.

The result, $H_{\mu\nu}$, remains a tensor in the base (real) space, x^μ.

It is a formula different of **Zee** who defines a current in the base (real) space $J^\sigma = \varepsilon^{\sigma\mu\nu}\ \mathbf{n}\cdot\left(\frac{\partial}{\partial x^\mu}\mathbf{n} \times \frac{\partial}{\partial x^\nu}\mathbf{n}\right)$.

The stereographic projection in the *target* (*internal*) space

$$S^2 \to \mathbf{R}^2$$

is made through the *complex field* u, (the complex field u has two independent components, like $\mathbf{R}^2$; i.e. u is a complex number in the plane which is also $\mathbf{C}$)

$$\mathbf{n} = \frac{1}{1+|u|^2}\left(u + u^* \quad , \quad -i\left(u - u^*\right) \quad , \quad |u|^2 - 1\right)$$

u is a function because just like $\mathbf{n}$ it depends on the coordinates of the base (real) space x, y, z.

The function

$$u\left(x, y, z\right)$$

is arbitrary.

The function $u\left(x, y, z\right)$ is complex, u and u^*.

NOTE

This form should be compared with the **Zee Lee** expression via Z_0, Z_1. Actually this work uses this form later.

END

NOTE

The expressions, with $R \equiv u$, are those detailed above for $R = z^Q$ for $Q = 1$, for the vector $\mathbf{v}$ that occurs in the expression of $U = \exp\left(if\ \mathbf{v}\cdot\boldsymbol{\tau}\right)$.

END

Then the tensor $H_{\mu\nu} = \mathbf{n}\cdot\left(\frac{\partial \mathbf{n}}{\partial x^\mu} \times \frac{\partial \mathbf{n}}{\partial x^\nu}\right)$ becomes, in terms of the complex function u,

$$H_{\mu\nu} = \frac{-2i}{\left(1+|u|^2\right)^2}\left(\frac{\partial u}{\partial x^\mu}\frac{\partial u^*}{\partial x^\nu} - \frac{\partial u}{\partial x^\nu}\frac{\partial u^*}{\partial x^\mu}\right)$$

The tensor is projected along a vector = derivative of u, and it results a vector K_μ

$$K_\mu = \frac{i}{2}\left(1+|u|^2\right)^2\ H_{\mu\nu}\ \frac{\partial u}{\partial x^\nu}$$

This vector K_μ is perpendicular on the vector field $\frac{\partial u}{\partial x^\mu}$, i.e. it verifies

$$K^\mu \frac{\partial u}{\partial x^\mu} = 0$$

because $H_{\mu\nu}$ is antisymmetric and multiplied with a symmetric one $\frac{\partial u}{\partial x^\nu}\frac{\partial u}{\partial x^\nu}$ and contracted gives 0.

NOTE
A tensor like $H_{\mu\nu}$ is a differential two-forms.
It is a flux.
It should be seen as a flow of a fluid through a surface determined by two vectors $\mathbf{e}_1$ and $\mathbf{e}_2$.
When this flux (flow) is projected along a vector field $H_{\mu\nu}\ \frac{\partial u}{\partial x^\nu}$, we obtain the amount of fluid flow crossing a surface resulting from $(\mathbf{e}_1 \times \mathbf{e}_2) \cdot \frac{\partial u}{\partial x^\nu}$. It would be maximum if the vector field $\frac{\partial u}{\partial x^\nu}$ had the direction defined by $\mathbf{e}_1 \times \mathbf{e}_2$. If not, then it is reduced.
END

And

$$\mathrm{Im}\left(K^\mu \frac{\partial u^*}{\partial x^\mu}\right) = 0$$

One defines

$$\mathcal{K}_\mu = \frac{1}{\sqrt[4]{K^\mu \frac{\partial u^*}{\partial x^\mu}}} K_\mu$$

The equations of motion of the particular model of this paper are

$$\frac{\partial \mathcal{K}_\mu}{\partial x^\mu} = 0$$

this is a zero-divergence.

There is an infinity of conservation laws

$$J_\mu = \mathcal{K}_\mu \frac{\delta G}{\delta u} - \mathcal{K}_\mu^* \frac{\delta G}{\delta u^*}$$

where

$$G = f(u, u^*)$$

Remember $u(x,y,z)$ is arbitrary.
The authors take a simple, harmonic expression for u.

$$u(\eta, \xi, \varphi) = f(\eta) \exp\left[i(m\xi + n\varphi)\right]$$

But u is here introduced in terms of the variables toroidal, in the *base* space $\mathbf{R}^3$,

$$(\eta, \xi, \varphi)$$

NOTE the particular form taken for u. It is a toroidal periodic line (m, n) and the field u has a *radial*$\equiv \eta$ variation given by $f(\eta)$

The relation between the euclidean $(x, y, z) \in \mathbf{R}^3$ base (real) space and the toroidal coordinates is

$$\begin{aligned} x &= \frac{a}{q}\sinh\eta\cos\varphi \\ y &= \frac{a}{q}\sinh\eta\sin\varphi \\ z &= \frac{a}{q}\sin\xi \end{aligned}$$

where

$$a > 0$$

$$\begin{aligned} \xi\ ,\ \varphi \text{ are angles} &\in (0, 2\pi) \\ \eta &\in (0, \infty) \end{aligned}$$

$$\begin{aligned} \eta &= \text{const} \\ &\quad \text{are toroids around } z \text{ axis} \end{aligned}$$

$$\begin{aligned} \xi &= \text{const} \\ &\quad \text{are spheres} \end{aligned}$$

$$\begin{aligned} \varphi &= \text{const} \\ &\quad \text{are half-planes} \end{aligned}$$

and

$$q = \cosh\eta - \cos\xi$$

Define the versors along toroidal coordinates

$$\begin{aligned} \widehat{\mathbf{e}}_i \cdot \widehat{\mathbf{e}}_j &= \delta_{ij} \\ i, j &= \eta, \xi, \varphi \end{aligned}$$

then

$$\begin{aligned} \partial\cdot\mathbf{u} &= \frac{q}{a}\exp\left[i\left(m\xi + n\varphi\right)\right] \\ &\quad \times \begin{pmatrix} \frac{df}{d\eta}\widehat{\mathbf{e}}_\eta \\ +imf\left(\eta\right)\widehat{\mathbf{e}}_\xi \\ +\frac{in\ f(\eta)}{\sinh\eta}\widehat{\mathbf{e}}_\varphi \end{pmatrix} \end{aligned}$$

One defines a vector field from the antisymmetric tensor $H_{\mu\nu}$ and ε_{ijk}

$$B_i = \frac{1}{2}\varepsilon_{ijk}H_{jk}$$

The Hopf index

$$Q = \frac{1}{4\pi^2} \int d^3x \; \mathbf{A} \cdot \mathbf{B}$$

(which is the volume integral of the helicity)

or

$$\begin{aligned} Q &= \frac{nm}{2} \left[\left(\Phi_1^2 + \Phi_2^2 \right) \Big|_0^\infty - \left(\Phi_3^2 + \Phi_4^2 \right) \Big|_0^\infty \right] \\ &= -nm \end{aligned}$$

The form of the *baryon* number Q as product of two periodicity numbers

$$Q = pq$$

also appears at **Jackson Manton** S^3.

NOTE

This is a static situation.

It is similar to the axial anomaly where the amount of fermion zero modes is equal to the amount of winding of the gauge field that has been changed from the remote past to the future.

END

Note

The form $Q = pq$ suggests a possible connection with the subject $n_+ n_-$ =const, i.e. the non-blowup but cusp.

End

16.7 The application of the $O(3)$ in $\mathbf{R}^3$ (compactified to S^3) model to fluids

The paper is **Kuznetsov Mikhailov.** Also in *fluids* and partly in *topological solutions* .

The equation for the *vorticity* is

$$\frac{\partial \mathbf{\Omega}}{\partial t} = \mathbf{\nabla} \times (\mathbf{V} \times \mathbf{\Omega})$$

from Navier Stokes non-dissipative and with no source, after taking the *curl.*

A Hamiltonian structure can be seen

$$\begin{aligned} \frac{\partial \mathbf{\Omega}}{\partial t} &= \{\mathbf{\Omega}, H\} \\ H &= \int d^3x \; \frac{1}{2} \mathbf{V}^2 \\ \{F, G\} &= \int d^3x \; \mathbf{\Omega} \cdot \left[\left(\mathbf{\nabla} \times \frac{\delta F}{\delta \mathbf{\Omega}} \right) \times \left(\mathbf{\nabla} \times \frac{\delta G}{\delta \mathbf{\Omega}} \right) \right] \\ F, G &\equiv \text{functionals} \end{aligned}$$

The Clebsch variables

$$\begin{aligned}\mathbf{V} &= \lambda \boldsymbol{\nabla} \mu + \boldsymbol{\nabla} \phi \\ \boldsymbol{\Omega} &= \boldsymbol{\nabla} \lambda \times \boldsymbol{\nabla} \mu\end{aligned}$$

the second equation shows that the vorticity is a vector tangemt to the curve resulted from the intersection of the surfaces

$$\begin{aligned}\lambda &= \text{const} \\ \mu &= \text{const}\end{aligned}$$

This "line", *i.e.* the vortex, is the pullback to the real space of a single point (φ, θ) on the sphere S^2 where $\mathbf{n}$ exists (the target space).

The Helicity

$$Hel = \int d^3x \ \mathbf{V} \cdot \boldsymbol{\Omega}$$

Representation of the vorticity field **Faddeev**

$$\Omega_\alpha = A \ \varepsilon_{\alpha\beta\gamma} \ \mathbf{n} \cdot \left(\frac{\partial \mathbf{n}}{\partial x^\beta} \times \frac{\partial \mathbf{n}}{\partial x^\gamma} \right)$$

in terms of the unit vector $\mathbf{n}(x, y, z)$ with

$$\mathbf{n}^2 = 1$$

It is like a topological current J^μ (in $3+1$ dim).

NOTE

In the work of **Aratyn Ferreira** it is defined the antisymmetric tensor

$$H_{\mu\nu} = \mathbf{n} \cdot \left(\frac{\partial \mathbf{n}}{\partial x^\mu} \times \frac{\partial \mathbf{n}}{\partial x^\nu} \right)$$

and later it is contracted with the completely antisymmetric tensor ε_{ijk}, to obtain

$$B_i = \varepsilon_{ijk} H_{jk}$$

which is a *magnetic field*. Obtaining $\mathbf{A}$ from this $\mathbf{B}$ one has $Q = \int d^3x \ \mathbf{A} \cdot \mathbf{B}$, the Hopf index.

So,

$$\Omega_i \sim B_i$$

END

A vortex (a line in space) is

$$\mathbf{n} = \mathbf{const}$$

in the target space S^2

which means that, whatever is the point on the line specified by the tangent vector Ω_α in the real space $\mathbf{R}^3$ the unit vector $\mathbf{n}$ points in the same direction, a unique point on the sphere of internal symmetry (θ, φ).

All points of S^3 that are mapped to a single unit vector $\mathbf{n} = \mathbf{const}$ in the space of internal symmetry, with the same constant $\mathbf{n}_0$ are on a line of $\mathbf{R}^3$.

A point on the sphere S^2 is $\mathbf{n}_0$.

To this point there correspond in S^3 a set of points which make a line, the vortex.

This is the mapping

$$S^3 \rightarrow S^2$$

that is realized by the formula for Ω_α above.

There is NO physical discovery regarding the vortex line. It is just a formalism allowing to represent the line.

Two fixed constant unit vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ in the sphere S^2 have associated to them two lines of $\mathbf{R}^3$.

The *linking* of the two lines in the space $\mathbf{R}^3$ is given by the Hopf invariant calculated using the unit vector $\mathbf{n}$.

On the other hand the Hopf index is connected with the Chern-Simons action, or, the integral of the density of helicity.

The Hopf index is connected with the total helicity in the volume

$$\int d^3x \; \mathbf{V} \cdot \mathbf{\Omega} \quad = \quad 64\pi^2 \; A^2 \; \times \overbrace{(Hopf)}^{\text{Hopf index}}$$

where A is the constant used by **Faddeev**.

This form is general, it is NOT as particular as the result of **Aratyn Ferreira**. This is because u has been specified.

Kuznetsov Mikhailov give a solution consisting of a field of $\mathbf{n}$ which has the property that for any two lines the linking is 1.

NOTA

Again comparison with **Aratyn Ferreira**.

Here

$$\int d^3x \; \mathbf{V} \cdot \mathbf{\Omega} \quad = \quad 64\pi^2 \; A^2 \; \times \overbrace{(Hopf)}^{\text{Hopf index}}$$

This form is general, it is NOT as particular as the result of **Aratyn Ferreira**. This is because u has been specified.

The Hopf index in **Aratyn Ferreira**

$$Q = -nm$$

is derived from an assumed *harmonic* form for the function u (an arbitrary function used to represent the components of $\mathbf{n}$)

END

16.8 Toroidal helix Kundu Rybakov 1981

From **Kundu Rybakov**.

The base space is $\mathbf{R}\times\mathbf{R}^3$, (time and full $3D$ space)

$$(3+1)\,D$$

and the target is S^2 i.e. $|N(t,\mathbf{x})|^2=1$. The invariance is

$$G=diag\left[O(2)_I\times O(2)_S\right]$$

where

1. $I\equiv$ rotation around the n_3 axis; n_3 belongs to the *target* space, i.e. it is a unit vector with the tip on the sphere, the North pole; and
2. $S\equiv$ rotation around the z axis. The z axis belongs to the *base* space.

The mapping

$$\begin{aligned}\mathbf{R}\times\mathbf{R}^3 &\to S^2\\ &\quad N_a(t,\mathbf{x})\\ |N(t,\mathbf{x})|^2 &= 1\end{aligned}$$

and

$$N_a(t,\infty)=\delta_{a3}$$

then the full $3D$ Euclidean space can be compactified to S^3,

$$S^3\to S^2$$

where S^3 is the compactification of $\mathbf{R}^3$,

this is Hopf mapping.

Here we have lines in the base space S^3 (compactified $\mathbf{R}^3$) corresponding to given points in S^2. There is topological Hopf index,

$$\begin{aligned}Q &= \pi_3\left(S^2\right)\sim\mathbf{Z}\\ &\quad \left(\text{explore the sphere } S^2 \text{ with the sphere } S^3\right)\end{aligned}$$

which gives the *linking* of the lines.

The tensor

$$F_{\mu\nu}=2\varepsilon_{abc}\;n^a\;\left(\partial_\mu n^b\right)\;\left(\partial_\nu n^c\right)$$

(mixed product, i.e. full contraction in the target a,b,c space, the sphere)

can be written as the *rotational* of a gauge field A_μ,

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$$

The degree of knottedness of the vector-field lines

$$\mathbf{B}=\mathrm{curl}\,\mathbf{A}$$

is given by the Hopf index

$$Q = -\frac{1}{(8\pi)^2} \int d^3x \; \mathbf{A} \cdot \mathbf{B}$$

The Hopf index, this expression is simply the total amount of *helicity*.

Careful, in **Ward** the integrand has the *dual* of the tensor F_{ij} and takes the scalar product with A_k.

This is the time component of a current

$$Q = \int d^3x \; J^0$$

Here the Hopf index Q is calculated as Chern Simons or Helicity density. It was necessary to introduce the gauge field $\mathbf{A}$ by formally solving the Biot Savart equation starting with $\mathbf{B}$ or $F_{\mu\nu}$. This is explicitely shown by **Kundu Rybakov**.

The current J^μ that leads to the Hopf index Q as the integral of the "0" time-component is defined as

$$J^\mu = -\frac{1}{128\pi^2} \; \varepsilon^{\mu\nu\lambda\rho} \; F_{\nu\lambda} \; A_\rho$$

which is a Chern-Simons *current* (not contracted as in $2D$).

Configurations of the lines in $\mathbf{R}^3$ (flat real space) that are toroidal helices.

They correspond to the symmetry

$$G = diag\left[O(2)_I \bigotimes O(2)_S\right]$$

where

$$\begin{aligned} O(2)_I \quad \equiv \quad & \text{group of rotations} \\ & \text{in internal (target) } I \text{ space of } \mathbf{n} \\ & \text{around the axis } n_3 \end{aligned}$$

$$\begin{aligned} O(2)_S \quad \equiv \quad & \text{group of rotations} \\ & \text{in real (base) } S \text{ space } \mathbf{x} \\ & \text{around the axis } z \end{aligned}$$

The configurations that we look for are characterized by the symmetry around two axes.

(1) rotations around the vertical axis z in the *base* space and reflection relative to this plane;

(2) rotations around the north-pole axis n_3 in the *target* space and reflection relative to the transversal plane.

Introduce

$$(T_3)_S \equiv \text{generator of rotations } O(2)_S$$

$$(T_3)_I \equiv \text{generator of rotations } O(2)_I$$

The invariant field $\mathbf{n}$ must satisfy

$$(T_3)_S \, n_a + [(T_3)_I \, \mathbf{n}]_a = 0$$

Note the difference relative to Skyrme baryons with symmetry $O_L(2) \times O_R(2)$ is *only* the *target* space S^3, as in **Jackson**. **End.**

Ansatz
Two functions are introduced

$$w(r, \theta) \quad \text{and} \quad v(r, \theta)$$

depending on the cylindrical real (base) space coordinates (r, θ, φ).
Using these w and v one expresses the components of $\mathbf{n}$ in the target space

$$\begin{aligned} n_3 &= \cos\beta = w(r, \theta) \\ \arctan\left(\frac{n_2}{n_1}\right) &= \gamma = m\varphi + v(r, \theta) \end{aligned}$$

The variables (β, γ) are polar angles for the $\mathbf{n}$ field in the internal space S^2.
After "extraction" of a periodicity $m\varphi$ the function $v(r, \theta)$ is still an angle and should be limited to $(0, 2\pi)$.

To calculate the energy of the solution, one has first to find the gauge potential $\mathbf{A}$ from the magnetic field $\mathbf{B}$. Biot Savart.
Then obtain Q as

$$Q = -\frac{2}{(8\pi)^3} \iint d^3x d^3x' \, \frac{\mathbf{B}(\mathbf{x}') \cdot [\mathbf{R} \times \mathbf{B}(\mathbf{x})]}{R^3}$$

where

$$\mathbf{R} = \mathbf{x} - \mathbf{x}'$$

$$R = |\mathbf{R}|$$

$$\mathbf{B} = -\mathbf{2} \quad (\boldsymbol{\nabla} w \times \boldsymbol{\nabla} \gamma)$$

Clebsch

Inserting the adopted form of n_3 and of $\arctan(n_2/n_1)$ with w and v,

$$\mathbf{B} = -\frac{2m}{r^2 \sin\theta} \left(\widehat{\mathbf{e}}_r \, \frac{\partial w}{\partial \theta} - \widehat{\mathbf{e}}_\theta \, r \frac{\partial w}{\partial r} \right) - \frac{2K}{r} \widehat{\mathbf{e}}_\varphi$$

where

$$K=\frac{\partial w}{\partial r}\frac{\partial v}{\partial \theta}-\frac{\partial w}{\partial \theta}\frac{\partial v}{\partial r}$$

From this, one finds that

$$\boldsymbol{\nabla} w=0 \text{ on the } z \text{ axis}$$

then

$$w\rightarrow 1 \quad \text{for} \quad r\sin\theta \rightarrow 0$$

and this means that

$$\begin{aligned} w &= \text{const} \\ &\sim \quad \text{(is homeomorphic with)} \quad T^2 \end{aligned}$$

which means that the surface w =const is a torus in the real (base) space.

The charge (Hopf index)

$$Q=\frac{m}{4\pi}\int_0^{\infty}dr\int_0^{\pi}d\theta \ (1-w)\left(\frac{\partial w}{\partial r}\frac{\partial v}{\partial \theta}-\frac{\partial w}{\partial \theta}\frac{\partial v}{\partial r}\right)$$

Change to other cylindrical coordinates in the real space

$$(r,\theta,\varphi)\rightarrow(\rho,z,\varphi)$$

$$w(\rho,z)=w(\rho,-z) \ \text{ even up-down on } z$$

$$v(\rho,z)=-v(\rho,-z) \quad \text{odd up-down on } z$$

The function v is NOT single valued

$$v\in\left[-\frac{\pi n}{2},+\frac{\pi n}{2}\right]$$

there are *jumps*

$$\begin{aligned} [v] &= \varepsilon n\pi \\ \varepsilon &\equiv \pm 1 \end{aligned}$$

at a line

$$C(\rho,z)$$

the jump-line.

The line $C(\rho,z)$ has in its very close neighborhood two lines, on one side and on the other, two lines $C_{+,-}$.

Remark

$$\begin{aligned} & 2(w-1)\left(\frac{\partial w}{\partial \rho}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial z}\frac{\partial v}{\partial \rho}\right) \\ = & \ \text{curl}\left[v\boldsymbol{\nabla}(1-w)^2\right]\Big|_{\varphi} \end{aligned}$$

Apply the Stokes theorem in the integrand

$$\begin{aligned} Q &= \frac{m}{4\pi}\int_{C_+\cup C_-} v\ (\mathbf{dl}\cdot\boldsymbol{\nabla})\,(1-w)^2 \\ &= \frac{m}{4\pi}\,[v]\int_{C(\rho,z\geq 0)} (\mathbf{dl}\cdot\boldsymbol{\nabla})\,(1-w)^2 \end{aligned}$$

Finally

$$Q = \pm nm$$

NOTE

This is important since we know that the Hopf degree for a helical line in real space which is the pullback of a point in S_I^2 it is the product of the integer numbers (n,m).

See also **Nitta**.

See also **Jackson Manton** S^3 where the *baryon* number is pq.

END

NOTE

The *baryon* number of a Skyrme soliton is

$$B = pq$$

where p is in the periodic expression for (Φ^1,Φ^2) and q is in the periodic expression (Φ^3,Φ^4).

These periodicities exist in the *target* space.

See **Jackson Manton Skyrme**

See also **Ward (?)** for the symmetry $SO(2)\times SO(4)$. Also pq.

END

This model offers an explicit expression for the helical line (n,m).

16.9 The Faddeev model

Start at random from **0303092 elliptic Hirayama**.

The Lagrangian

$$L = c_2\,(\partial_\mu \mathbf{n})\,(\partial^\mu \mathbf{n}) - 2c_4 F_{\mu\nu}F^{\mu\nu}$$

where

$$\mathbf{n}^2 = 1$$

and the term (**Faddeev**) with four derivatives

$$F_{\mu\nu} = \frac{1}{2}\mathbf{n}\cdot(\partial_\mu \mathbf{n}\times\partial_\nu \mathbf{n})$$

The equation for the field

$$\partial_\mu \left(c_2 \ \mathbf{n} \times \partial^\mu \mathbf{n} - 2c_4 \ F^{\mu\nu} \partial_\nu \mathbf{n} \right) = 0$$

Two new vectors, similar to the electromagnetic field.
These are vectors in the space of internal symmetry, target

$$\mathbf{A}_\mu = \left(A_\mu^1, A_\mu^2, A_\mu^3 \right)$$

$$\mathbf{B}_\mu = \left(B_\mu^1, B_\mu^2, B_\mu^3 \right)$$

where

$$B_\mu^a = \varepsilon^{abc} \ n^b \partial_\mu n^c$$

as a *vector product in the internal space* (a, b, c).

The equations for A and B,

$$\partial_\mu \left(c_2 \ \mathbf{A}^\mu + c_4 \ \left(\mathbf{A}^\mu \times \mathbf{A}^\nu \right) \times \mathbf{A}_\nu \right) = 0$$

$$\partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu = 2 \mathbf{A}_\mu \times \mathbf{A}_\nu$$

and

$$\partial_\mu \left(c_2 \ \mathbf{B}^\mu + c_4 \ \left(\mathbf{B}^\mu \times \mathbf{B}^\nu \right) \times \mathbf{B}_\nu \right) = 0$$

and it results that **A** and **B** are *parallel*.
Note if **A** and **B** are parallel they are in *force-free* configuration. **End.**

Consider **Ward**.
Ward prepares the discussion for the extension to real spaces that are NOT flat as $\mathbf{R}^3$ but have a metric. This will appear if the real space is compactified to S^3.
This is why **Ward** uses g^{ij}, metric on the real space.

Important note from **Manton Rybakov**

- there is *topological compactification*, which replaces $\mathbf{R}^3$ with S^3 only from the point of view of the topology, but does not involve the metric
- there is simple replacement of $\mathbf{R}^3$ with S^3, motivated by the fact that solutions of the equations for the solitons are trivial in the real flat space

R.S. Ward
This distinction is important

$$E = \int dV \ \left(c_2 e_2 + c_4 e_4 \right)$$

$$\begin{aligned} dV &= \sqrt{g}\ dx^1 \wedge dx^2 \wedge dx^3 \\ g &= \det(g_{ij}) \end{aligned}$$

and

$$\begin{aligned} e_2 &= g^{ij}\ (\partial_i n^a)\ (\partial_j n^a) \\ &\quad \text{kinetic energy density} \end{aligned}$$

$$e_4 = \frac{1}{4} g^{ij} g^{kl}\ F_{ik} F_{jl}$$

This is like *Maxwell* Lagrangian for the electromagnetic field.

$$F_{ij} = \varepsilon_{abc}\ n^a\ \left(\partial_i n^b\right) \left(n_j^c\right)$$

The integer Q Hopf index

$$\begin{aligned} Q &= \frac{1}{32\pi^2} \int_M d^3x\ \eta^{jkl}\ F_{jk}\ A_l \\ &\quad \text{integral of the helicity} \end{aligned}$$

where

$$\eta^{jkl} = \frac{1}{\sqrt{g}} \varepsilon^{jkl}$$

The fact that the integrand contains the completely antisymmetric tensor $\eta \sim \varepsilon^{jkl}$ means that there is a scalar product of A_l with the *dual* of the tensor, *i.e.* with $\varepsilon^{jkl} F_{kl}$.

In **Aratyn Ferreira** it is defined

$$B_i = \varepsilon_{ijk} H_{jk}$$

and this confirms that the integrand in the Hopf index is indeed the *helicity* $\mathbf{A} \cdot \mathbf{B}$.

The inequality

$$E \geq \text{const}\ Q^{3/4}$$

In the definition of the energy for a mapping = solution of the Lagrange equations for $e_2 + e_4$ one has to make a space integration over the full spatial extension in the base manifold (real space (x, y, z)) of the mapping field $\mathbf{n}(x, y, z)$.

The region in the real space where the points are mapped into points $\mathbf{n}(x, y, z)$ of $O(3)$ is limited. Actually, at some distance far, the asymptotic regime is manifested and the points where

$$\mathbf{n} \rightarrow (0, 0, 1)$$

are to be found beyond a certain distance in $\mathbf{R}^3$.

Now there is NOT a compactification but a *replacement* of the flat real space $\mathbf{R}^3$ with a space with the metric of S^3.

After replacement we say that this region is characterized by a radius R. Inside the radius R the field has many values.

The integrals over the volume are limited to the region non-asymptotic.

Therefore we have an integral until the distance R.

Beyond the distance R the mapping leads to vectors that are fixed to one direction, the asymptotic vector.

The conclusion of **Ward**: the extension of the solution of the equations of motion over a region that has the radius R is unstable if R is large. This means that, assuming a fixed Hopf topological degree Q, if the space region where the mapping leads to a non-fixed $\mathbf{n}(x, y, z)$ is larger than a limit, $R > \sqrt{2}$ then the solution is unstable and an energetically better state for the system exists, consisting of concentration of the mapping in a small region, a lump.

More detailed, according to **Ward.** The Hopf map send a point of S_R^3 : (Z^0, Z^1) of complex numbers to the point of $CP^1 = S^2$ with homogeneous coordinates $[Z^0, Z^1]$. The energy density for this map is that of the identity and is constant on S_R^3,

$$E = 16\pi^2 \left(R + \frac{1}{R}\right)$$

(this map is identity but not isometry).

A perturbation of the map is assumed. The energy is then perturbed

$$E[\phi + \delta\phi] = E[\phi] + \int_M d^3x \; G[\delta\phi]$$

where $G[\delta\phi]$ is quadratic in $\delta\phi$ and $\partial_j(\delta\phi)$.

We must determine the eigenmodes of the operator G that are negative.

In **Manton geometry** for a map of degree 1 the energy lower limit is given by

$$\begin{aligned} \lambda_1 &= \lambda_2 = \lambda_3 \\ & \text{everywhere constant} \end{aligned}$$

where λ_i are th eigenvalues of $D = JJ^T$ with J the Jacobian of application between the two spaces: basis to internal.

In **Manton force THoft Polyakov** it is shown that the 3×3 matrix $\partial_i n_k$ has determinant 0. Then one of the eigenvalues is zero.

Kundu Rybakov solutions of the Hopf mapping.

16.10 Gladikowski1997 static solitons

It is a mapping

$$\mathbf{R}^3 \times S \rightarrow S^2$$

and it is found that stable Hopf mapping are realized for closed vortices in the base space.

Two solitons superposed are better than distant. Therefore there is attraction.

16.11 Beyond skyrmions (condensed matter)

Phys Rep.

17 Skyrme model of baryons

See **9408168 Sutcliffe instanton**.
Manton skyrmions geometry.

Note Sutcliffe has a work on extension around a line (filament) such as to produce a mapp from a full space to the target space. And he introduces an energy for this.

Nonlinear scalar field theory.

The scalar field is a mapping π from a domain of the *physical space* S to a *target space* Σ.

$$\begin{array}{ccc} & & \text{(physical space)} \xrightarrow{\pi} \text{(target space)} \\ S & \rightarrow & \Sigma \end{array}$$

Riemannian manifolds with metrics

$$\begin{array}{ll} g & \text{metric of base space } S \\ \gamma & \text{metric of target space } \Sigma \end{array}$$

"*The energy functional will be a measure of the extent to which the map π is metric preserving.*"
Like in the theory of elasticity.
Note in **Jackson $\mathbf{S}^3$** the isometry at $L = 1$ should have no elastic energy. But the energy of bifurcation is $L = \sqrt{2}$. **End.**

Consider a point in the physical space

$$p \in S$$

and its image under π in the target space

$$\pi(p) \in \Sigma$$

The neighborhoods of p and $\pi(p)$ are Euclidean (on manifolds) and the metrics g and γ are diagonal, they are unit matrices.

Orthonormal frames of versors exist locally at p and at $\pi(p)$.

The map is

$$\pi^{\alpha}(p_1, p_2, p_3)$$

The Jacobian matrix

$$J_{i\alpha} = \frac{\partial \pi^{\alpha}}{\partial p^{i}}$$
$$\text{(at the point } p\text{)}$$

is a measure of the deformation introduced by the map π.

Since the ortonormal systems defined locally (in small neighborhood) at p and respectively $\pi(p)$ can be modified by rotations with a matrix O at p and Ω at $\pi(p)$, the Jacobian changes

$$J \to O^{-1} J \Omega$$

with - however, no change of the physical energy.

Instead, one introduces the *strain tensor*

$$D = J J^{T}$$

It is symmetric and under the rotation O at p it changes

$$D \to O^{-1} D O$$

Consider the invariants of D (the strain tensor).

They are symmetric combinations of the eigenvalues. The eigenvalues are positive, this is why they are adopted as squares, λ^2. They are

$$\lambda_1^2 \ , \ \lambda_2^2 \ , \ \lambda_3^2$$

and the invariants

$$\begin{aligned} Tr(D) &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \\ \frac{1}{2}\left[Tr(D)\right]^2 - \frac{1}{2} Tr\left(D^2\right) &= \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 \\ \det(D) &= \lambda_1^2\lambda_2^2\lambda_3^2 \end{aligned}$$

Any can be used as an "energy" at the point p.

See also **new skyrmion solution Jackson Manton Wirzba** S^3.

Considerations for the case that the systems of versors of the frame at p and of the frame at $\pi(p)$ are not orthonormal.

Definition of an invariant which is *the sum over the squares (i.e. power two) of the areas of parallelograms formed by pairs of vectors obtained by derivation of the versors of the* $\pi(p)$ *frames with respect to* p. This is the Faddeev term in the Lagrangian, e_4.

1202.3988 helical buckling.

(**Note** Is-it coiling instability?)

This makes the *static energy* to be

$$E = \text{const} \times \int_{R^3} d^3x \left[(\partial_i \phi^a) \left(\partial^i \phi^a\right) + \frac{1}{2} F_{ij} F^{ij} \right]$$

where

$$F_{ij} = \varepsilon_{abc}\ \phi^a\ \partial_i \phi^b\ \partial_j \phi^c$$

(like in **Faddeev**).

The mapping is

$$S^3 \rightarrow S^2$$

The Hopf degree is the linking number of two curves that represent preimages of two versors in the target sphere S^2.

Alternatively it is calculated as integral over the two spaces of the density of helicity.

The Skyrme solution for

$$\begin{aligned} L &= \frac{1}{16} F_\pi^2 \ \text{Tr} \left[\frac{\partial U}{\partial x^\mu} \ \frac{\partial U^\dagger}{\partial x^\mu} \right] \\ &\quad + \frac{1}{32 e^2} \text{Tr} \left[\left(\frac{\partial U}{\partial x^\mu} \right) U^\dagger \ , \left(\frac{\partial U}{\partial x^\nu} \right) U^\dagger \right]^2 \end{aligned}$$

17.1 Static properties skyrmions Witten

Also in *research, topological solutions.*

This is the paper, 1983.

"baryons are solitons in the non-linear sigma model"

$SU(2)$.

Take

$$U = 1 + iA + O\left(A^2\right)$$

and replace this in $\text{Tr}\left[\left(\frac{\partial U}{\partial x^\mu}\right) U^\dagger\ , \left(\frac{\partial U}{\partial x^\nu}\right) U^\dagger\right]^2$.

The Wess Zumino term is

$$\begin{aligned} n\Gamma &= n \frac{2}{15\pi^2 F_\mu^5} \int d^4x\ \epsilon^{\mu\nu\sigma\rho}\ \text{Tr}\left[A\ \ (\partial_\mu A)\ \ (\partial_\nu A)\ \ (\partial_\sigma A)\ \ (\partial_\rho A)\right] \\ &\quad + \text{higher orders} \end{aligned}$$

The function A (the distance from the identity for an arbitrary element U of $SU(2)$) must be expressed in terms of a basis of operators

$$A = a_a \tau_a$$

then

$$n\Gamma = n\frac{2}{15\pi^2 F_\pi^5}\int d^4x\ \epsilon^{\mu\nu\sigma\rho}\ a_a\ \left(\partial_\mu a_b\right)\ \left(\partial_\nu a_c\right)\ \left(\partial_\sigma a_d\right)\ \left(\partial_\rho a_e\right)\ \mathrm{Tr}\left[\tau_a\tau_b\tau_c\tau_d\tau_e\right]$$

"This term is completely anti-symmetrical in the Lorentz indices $[\mu,\nu,\sigma,\rho]$, *so it needs to be*

completely anti-symmetrical in the isospin indices b, c, d *and* e. [otherwise is 0]

But that is impossible

because there are only three independent $SU(2)$ *generators."*

The Skyrme model includes a term that prevents the shrinking of the solitons.

$$\begin{aligned} L &= \frac{1}{16}F_\pi^2\ \mathrm{Tr}\left(\partial_\mu U\ \partial_\mu U^\dagger\right) \\ &\quad +\frac{1}{32e^2}\mathrm{Tr}\left[\left(\partial_\mu U\right)U^\dagger\ ,\ \left(\partial_\nu U\right)U^\dagger\right]^2 \end{aligned}$$

where U is an $SU(2)$ matrix that transforms as

$$U \to AUB^{-1}$$

under chiral

$$SU(2)\times SU(2)$$

transformation.

NOTE

Looks close to the mixed spinors.

END

Extremum of the action leads to an equation for which it is adopted the Skyrme ansatz

$$U = \exp\left[i\ f(r)\ \boldsymbol{\tau}\cdot\widehat{\mathbf{e}}_x\right]$$

with

$$\begin{aligned} f(r=0) &= \pi \\ f(r\to\infty) &= 0 \end{aligned}$$

This gives the mass

$$\begin{aligned} M &= 4\pi\int dr\ r^2\ \left\{\frac{1}{8}F_\pi^2\ \left[\left(\frac{\partial f}{\partial r}\right)^2 + 2\frac{\sin^2 f}{r^2}\right]\right. \\ &\quad \left. +\frac{1}{2e^2}\frac{\sin^2 f}{r^2}\left[\frac{\sin^2 f}{r^2} + 2\left(\frac{\partial f}{\partial r}\right)^2\right]\right\} \end{aligned}$$

Introduce the dimensionless radial variable

$$r \to \rho \equiv eF_\pi \ r$$

and the variational equation is

$$\left(\frac{1}{4}\rho^2 + 2\sin^2 f\right) \ f'' + \frac{1}{2}\rho f' + (\sin 2f)\, f'^2 - \frac{1}{4}(\sin 2f) - \frac{\sin^2 f \ \sin 2f}{\rho^2} = 0$$

Taking

$$U_0 = \exp\left[i \ f(r) \ \boldsymbol{\tau}\cdot\widehat{\mathbf{e}}_x\right]$$

then

$$AU_0A^{-1}$$

is also a solution.

It is adopted

$$A(t)$$

and $U = A(t)\, U_0 A^{-1}(t)$.

$$L = -M + \lambda \mathrm{Tr}\left[(\partial_0 A)\left(\partial_0 A^{-1}\right)\right]$$

where

$$\lambda = \frac{4\pi}{6}\frac{1}{e^2 F_\pi}\Lambda$$

$$\Lambda = \int d\rho \ \rho^2 \ \sin^2 f\left[1 + 4\left(f'^2 + \frac{\sin^2 f}{\rho^2}\right)\right]$$

Writting

$$\begin{aligned} A(t) &= a_0 + i \ \mathbf{a}\cdot\boldsymbol{\tau} \\ a_0^2 + \mathbf{a}^2 &= 1 \end{aligned}$$

then

$$L = -M + 2\lambda\sum_{i=0}^{3}\left(\frac{da_i}{dt}\right)^2$$

Introduce the conjugated momenta

$$\pi_i = \frac{\delta L}{\delta\left(\frac{da_i}{dt}\right)} = 4\lambda\left(\frac{da_i}{dt}\right)$$

the Hamiltonian

$$\begin{aligned} H &= \pi_i\left(\frac{da_i}{dt}\right) - L \\ &= 4\lambda\left(\frac{da_i}{dt}\right)\left(\frac{da_i}{dt}\right) - L \\ &= M + \frac{1}{8\lambda}\sum_{i=0}^{3}\pi_i^2 \end{aligned}$$

The usual quantification

$$\pi_i \to -i\frac{\partial}{\partial a_i}$$

$$\begin{aligned} H &= M + \frac{1}{8\lambda}\sum_{i=0}^{3}\left(-\frac{\partial^2}{\partial a_i^2}\right) \\ \sum_{i=0}^{3} a_i^2 &= 1 \end{aligned}$$

Then we have the Laplacian on the 3-sphere.

Wavefunctions: traceless symmetric polynomials in a_i (like harmonic polynomials). Example

$$(a_0 + ia_1)^l$$

solution of

$$\begin{aligned} & \left(-\nabla^2\right)\left[(a_0 + ia_1)^l\right] \\ = & \ l\,(l+2)\ \left[(a_0 + ia_1)^l\right] \end{aligned}$$

For such solution, spin and isospin

$$I_k = \frac{i}{2}\left(a_0\frac{\partial}{\partial a_k} - a_k\frac{\partial}{\partial a_0} - \varepsilon_{klm}\ a_l\frac{\partial}{\partial a_m}\right)$$

$$J_k = \frac{i}{2}\left(a_k\frac{\partial}{\partial a_0} - a_0\frac{\partial}{\partial a_k} - \varepsilon_{klm}\ a_l\frac{\partial}{\partial a_m}\right)$$

Spin = isospin = $\frac{1}{2}l$.

"two consistent
ways to quantize the soliton; one may require $\psi(A) = \psi(-A)$ for all solitons, or
one may require $\psi(A) = -\psi(-A)$ for all solitons. The former choice corresponds
to quantizing the soliton as a boson. The latter choice corresponds to quantizing it
as a fermion."

The eigenvalues of the Hamiltonian are

$$E = M + \frac{1}{8\lambda}l\,(l+2)$$

where

$$l = 2J$$

The nucleon mass is

$$M_n = M + \frac{1}{2\lambda}\frac{3}{4}$$

and of Δ meson

$$M_\Delta = M + \frac{1}{2\lambda}\frac{15}{4}$$

18 Toroidal solutions in the internal (target) space

Is this the experimentally observed toroidal skyrmion in *light* ? (if yes, then it looks like a toroidal vortex formed when a fluid is ejected from a hose, smoke ring. Does it has swirl ? axial flow).

See paper **toroidal vortices light** in *optica*.

18.1 In Jackson Manton S^3.

First the coordinates are adopted on S^3 base space, of radius L.

$$(\mu, \phi_1, \phi_2)$$

with

$$\begin{aligned} 0 &\leq \mu \leq \frac{\pi}{2} \\ 0 &\leq \phi_1, \phi_2 \leq 2\pi \end{aligned}$$

The *cartesian coordinates* are

$$\begin{gathered} L \sin\mu \cos\phi_1 \\ L \sin\mu \sin\phi_1 \\ L \cos\mu \cos\phi_2 \\ L \cos\mu \sin\phi_2 \end{gathered}$$

with the infinitesimal distance

$$ds^2 = L^2 \left(d\mu^2 + \sin^2\mu \; d\phi_1^2 + \cos^2\mu \; d\phi_2^2 \right)$$

and the volume

$$dV = L^3 \sin\mu \cos\mu \; d\mu \; d\phi_1 \; d\phi_2$$

We **note** that here it is a compactification that is not only topological but also geometrical.

And now, in the target (internal) space

The sigma model components

$$\begin{aligned} & \left(\sigma,\ \pi^{z}\ ,\ \pi^{x},\pi^{y}\right) \\ \equiv\ & \left(\Phi^{1},\Phi^{2},\Phi^{3},\Phi^{4}\right) \end{aligned}$$

The configuration is chosen to have $O\left(2\right)_{L}\times O\left(2\right)_{R}$ symmetry.

$$\begin{aligned} \Phi^{1} &= \sin\left[f\left(\mu\right)\right]\ \cos p\phi_{1} \\ \Phi^{2} &= \sin\left[f\left(\mu\right)\right]\ \sin p\phi_{1} \\ \Phi^{3} &= \cos\left[f\left(\mu\right)\right]\ \cos q\phi_{2} \\ \Phi^{4} &= \cos\left[f\left(\mu\right)\right]\ \sin q\phi_{2} \end{aligned}$$

The first group factor $O\left(2\right)_{L}$ involves $\left(\Phi^{1},\Phi^{2}\right)$ and the second involves the components $\left(\Phi^{3},\Phi^{4}\right)$.

The map has

$$Q=pq$$

NOTE regarding the topological degree of the toroidal configuration

$$Q=pq$$

Since we have here a product of two topological numbers, p and q, we look for similarity with the property of the SD states in $2D$ Euler

$$n_{+}n_{-}=\text{const}$$

(discrete, classical, statistical, **Edwards, Montgomery**) or

$$\phi_{1}\phi_{2}=1$$

in the FT formulation.

We have considered this property as more general, expressing the fact that two opposite components must be in this relation at the SD. Should be at the origin of the property "no blowup but Cusp" . If true it should arise in *Camassa Holm*, Sivashinsky, stationary $2D$ Euler, with the cusp solution **Chow Gurarie.**

But

what is the meaning of such property in the present case, if realised

$$pq=\ \text{const}\ \ (\text{integer})$$

Why the toroidal and poloidal turns would be in the relation

$$p=\frac{\text{const}}{q}$$

? And, which solution should show CUSP instead of blowup when this is realised?

END

18.2 The Kuznetsov Mikhailov configuration $Q=1$

In the base space we have the coordinates

$$\mathbf{r}=(x,y,z)$$

In the target space we have the unitary vector

$$\begin{aligned}\mathbf{n} &= (n_x,n_y,n_z)\\ |\mathbf{n}|^2 &= 1\end{aligned}$$

The mapping is provided by the connection (it is for fluids)

$$\mathbf{n}\cdot\boldsymbol{\sigma}=q^{\dagger}\sigma_3 q$$

$$q=\frac{1-i\mathbf{r}\cdot\boldsymbol{\sigma}}{1+i\mathbf{r}\cdot\boldsymbol{\sigma}}$$

or

$$q=(1-i\mathbf{r}\cdot\boldsymbol{\sigma})(1+i\mathbf{r}\cdot\boldsymbol{\sigma})^{-1}$$

Adopt toroidal coordinates in the *base space*

$$(x,y,z)\rightarrow(u,v,\chi)$$

(like in the **Jackson Manton** text, above)

$$x+iy=\frac{\sec h(u)}{\cosh(u)+\cos(v)}\exp(i\chi)$$

$$z=\frac{\sin(v)}{\cosh(u)+\cos(v)}$$

Then the components of the unitary vector

$$(n_x,n_y,n_z)$$

in the internal (*target)* space are, according to the formula connecting $\mathbf{n},\boldsymbol{\sigma}$ and $\mathbf{r}$,

$$\begin{aligned}\arctan\left(\frac{n_y}{n_x}\right) &= \chi-v\\ n_z &= 1-\frac{4\sec h^2(u)}{\cosh(u)\left[\cosh(u)+\cos(v)\right]}\end{aligned}$$

The whole space is stratified into tori

$$U = \text{const}$$

The vortex lines, running once along the azimuthal direction are closed.
The link between any two is 1.

Then the vorticity is

$$\begin{aligned}\Omega_\alpha &= A \\ &\quad \times \varepsilon_{\alpha\beta\gamma}\, \mathbf{n} \cdot \left(\frac{\partial \mathbf{n}}{\partial x^\beta} \times \frac{\partial \mathbf{n}}{\partial x^\gamma} \right)\end{aligned}$$

The *parametrization of the flows by the* **n**-*field is a transition to CLEBSCH variables*

$$\boldsymbol{\Omega} = 2A \ \left[\boldsymbol{\nabla} \cos\varphi \times \boldsymbol{\nabla}\theta \right]$$

Then the evolution of the unitary vector **n** of the internal space is

$$\begin{aligned}\frac{\partial \mathbf{n}}{\partial t} + (\mathbf{V} \cdot \boldsymbol{\nabla})\, \mathbf{n} &= 0 \\ \text{or } \frac{d\mathbf{n}}{dt} &= 0\end{aligned}$$

In Clebsch variables

$$\boldsymbol{\Omega} = \boldsymbol{\nabla}\lambda \times \boldsymbol{\nabla}\mu$$

for

$$\mathbf{V} = \lambda \boldsymbol{\nabla}\mu + \boldsymbol{\nabla}\phi$$

For the Hamiltonian

$$H = \int d^3x \ \frac{1}{2} |\mathbf{V}|^2$$

the Euler equation for **V** and for **Ω** are expressed in terms of λ and μ,

$$\begin{aligned}\frac{\partial \lambda}{\partial t} &= \frac{\delta H}{\delta \mu} \\ \frac{\partial \mu}{\partial t} &= -\frac{\delta H}{\delta \lambda}\end{aligned}$$

or

$$\begin{aligned}\frac{\partial \lambda}{\partial t} + (\mathbf{V} \cdot \boldsymbol{\nabla})\, \lambda &= 0 \\ \frac{\partial \mu}{\partial t} + (\mathbf{V} \cdot \boldsymbol{\nabla})\, \mu &= 0\end{aligned}$$

Now we can write these equations for the versor $\mathbf{n}$ of the internal space

$$\frac{\partial \mathbf{n}}{\partial t} + (\mathbf{V} \cdot \boldsymbol{\nabla})\, \mathbf{n} = 0$$

Taking

$$\widetilde{H} = \frac{H}{A}$$

the equations for the versor can be written as Hamiltonian

$$\frac{\partial \mathbf{n}}{\partial t} = \mathbf{n} \times \frac{\delta \widetilde{H}}{\delta \mathbf{n}}$$

These are the equations Landau Lifshitz Gilbert, with $\widetilde{H}$.

19 Skyrmions on a 3 - sphere

19.1 Manton Hopf solitons in base space S^3(compactified $\mathbf{R}^3$)

The Skyrme model uses *algebraic space* $SU(2)$ as target space. This is $\sim S^3$.

For this reason the mapping can become

$$S^3 \rightarrow S^3$$

and it can be *identity.*

If we are in the **Faddeev** model the topological map is $S^3 \rightarrow S^2$ and the pre-image of a point in the target space is a *line* in the real space (**Hopf**).

Important remark regarding the occurence of a new important parameter, the radius R of compactification of the space domain.

This is first introduced in **Skyrme** model. See **new skyrmion solution Manton**.

Since the compactification is made when

$$\mathbf{n} \rightarrow (0, 0, 1)$$

we understand that the radius R is the limit of the real space domain beyond which the real space points are all mapped to the unique target point $(0, 0, 1)$. Then, the vorticity must be zero in all these points of the real space since

$$\begin{aligned} \Omega_i &= \varepsilon_{ijk}\, \varepsilon^{abc}\, n^a\, \left(\partial_j n^b\right)\, \left(\partial_k n^c\right) \\ \Omega_i &\rightarrow 0 \;\; \text{beyond } R \end{aligned}$$

Then the definition of R (called radius of compactification) is actually the reflection of the fact that the vorticity is limited inside the domain of radius R.

Now, when R is large, for the same amount of vorticity squared (the second term of the energy in **Faddeev**) the mapping is unstable and the system takes new configurations where the vorticity is concentrated.

We should show that this has the same nature as finding that the lowest energy of a system of vortices corresponds to collect all of them in a single point, superposed.

This appears to be confirmed by **Jackson Manton Skyrme** S^3.

19.2 Paper Jackson new skyrmion S^3 solutions for baryons

The

$$\begin{aligned} B &= 2 \\ & \text{(baryon number is 2)} \end{aligned}$$

state is interesting because it is here that emerges the *one-pion excahnge* interaction between baryons.

The use of *product ansatz*.

But in the approach based on the symmetry

$$O_L(2) \times O_R(2)$$

it is found that the lowest energy is for states with *coincident* skyrmions.

See **Gibbons Steif** S^3 symmetries, one-forms, vector fields - for gravitation sphaleron (above).

And

at $L \to \infty$ *the skyrmions do not separate*

NOTE

This is interesting, the best solutions correspond to the accumulation of solitons in one single point, like in **Schaposnikov** and others.

END.

The solutions for Skyrme system with *assumed* symmetry $O_L(2) \times O_R(2)$ are searched in the form given by the metric on the *basis* space $S^3(L)$

$$(\mu, \phi_1, \phi_2)$$

with the coordinates

$$\begin{aligned} &L \sin\mu \cos\phi_1 \\ &L \sin\mu \sin\phi_1 \\ &L \cos\mu \cos\phi_2 \\ &L \cos\mu \sin\phi_2 \end{aligned}$$

and the intervals

$$
\begin{aligned}
0 &\leq \mu \leq \frac{\pi}{2} \\
0 &\leq \phi_1 \leq 2\pi \\
0 &\leq \phi_2 \leq 2\pi
\end{aligned}
$$

The metric in the *basis* space is

$$ds^2 = L^2 \left(d\mu^2 + \sin^2 \mu \ d\phi_1^2 + \cos^2 \mu \ d\phi_2^2 \right)$$

and volume element in the *basis* space

$$dV = L^3 \sin \mu \cos \mu \ d\mu \ d\phi_1 d\phi_2$$

This is the origin of the mapping.

The symmetry means that, acting on the system with the transformations belonging to the group

$$O_L (2) \times O_R (2)$$

the system remains invariant.

The equations do not change, the mapping does not change.

The transformations are made in the basis space.

The rotations in the plane

$$(1,2)$$

change ϕ_1 and the rotations in the plane

$$(3,4)$$

change ϕ_2, and the two types of rotations commute.

The *target* of the mapping is the space

$$S^3 (1)$$

(note the distinction: L in the source, 1 in the target).

The fields in target $S^3 (1)$ are

$$\left(\Phi^1, \Phi^2, \Phi^3, \Phi^4 \right)$$

with

$$\Phi^\alpha \Phi^\alpha = 1$$

The idea about the use of this Skyrme theory is the identification between the *target* fields Φ^α and the physical fields

$$\sigma, \ \pi^z, \ \pi^x, \pi^y$$

The ansatz that defines the target fields as a restriction verifying the symmetry

$$O_L(2) \times O_R(2)$$

is

$$\begin{aligned}
\Phi^1 &= \sin[f(\mu)]\ \cos(p\phi_1) \\
\Phi^2 &= \sin[f(\mu)]\ \sin(p\phi_1) \\
\Phi^3 &= \cos[f(\mu)]\ \cos(q\phi_2) \\
\Phi^4 &= \cos[f(\mu)]\ \sin(q\phi_2)
\end{aligned}$$

where the parameters are

$$f(\mu) \equiv \text{function}$$

with limits

$$\begin{aligned}
f(0) &= 0 \\
f\left(\frac{\pi}{2}\right) &= \frac{\pi}{2}
\end{aligned}$$

and

$$(p, q) \equiv \text{integers}$$

and

$$\begin{aligned}
pq &\equiv \text{baryon number} \\
&= B
\end{aligned}$$

We note that the *assumed* form of the fields Φ^α is periodic on the *basis* variable ϕ_1 with period $2\pi/p$ and on variable ϕ_2 with period $2\pi/q$. The expressions are also periodic on the *basis* variable μ via a function $f(\mu)$.

The *baryon* number is

$$B = pq$$

This number results from the calculation of the Hopf invariant.

The symmetry *left*, $O(2)_L$

$$\begin{aligned}
& O(2)_L \\
\phi_1 &\rightarrow \phi_1 + \alpha \\
\begin{pmatrix} \Phi^1 \\ \Phi^2 \end{pmatrix} &\rightarrow \begin{pmatrix} \cos(p\alpha) & \sin(p\alpha) \\ -\sin(p\alpha) & \cos(p\alpha) \end{pmatrix} \begin{pmatrix} \Phi^1 \\ \Phi^2 \end{pmatrix}
\end{aligned}$$

plus the reflection

$$\begin{aligned}
\phi_1 &\rightarrow -\phi_1 \\
\begin{pmatrix} \Phi^1 \\ \Phi^2 \end{pmatrix} &\rightarrow \begin{pmatrix} \Phi^1 \\ -\Phi^2 \end{pmatrix}
\end{aligned}$$

the same for $O_R(2)$, involving this time

$$\begin{pmatrix} \Phi^3 \\ \Phi^4 \end{pmatrix}$$

and their periodicity with q.

NOTE

check **Gibbons Steif** S^3 gravitation.

END

NOTE that the grouping is (σ, π^z) that are acted upon by $O_L(2)$ and (π^x, π^y) that are acted upon $O_R(2)$.

These actions are not mixing.

The energy

$$E = \int_{S^3(L)} dV \ \left[\mathrm{Tr}\,(K) + \frac{1}{2} \left\{ (\mathrm{Tr}K)^2 - \mathrm{Tr}\left(K^2\right) \right\} \right]$$

where

$$K = \begin{pmatrix} \frac{\partial \Phi^a}{\partial \mu} \frac{\partial \Phi^\alpha}{\partial \mu} & \frac{1}{\sin\mu} \frac{\partial \Phi^\alpha}{\partial \mu} \frac{\partial \Phi^\alpha}{\partial \phi_1} & \frac{1}{\cos\mu} \frac{\partial \Phi^\alpha}{\partial \mu} \frac{\partial \Phi^\alpha}{\partial \phi_2} \\ \frac{1}{\sin\mu} \frac{\partial \Phi^\alpha}{\partial \mu} \frac{\partial \Phi^\alpha}{\partial \phi_1} & \frac{1}{\sin^2\mu} \frac{\partial \Phi^\alpha}{\partial \phi_1} \frac{\partial \Phi^\alpha}{\partial \phi_1} & \frac{1}{\sin\mu\cos\mu} \frac{\partial \Phi^\alpha}{\partial \phi_1} \frac{\partial \Phi^\alpha}{\partial \phi_2} \\ \frac{1}{\cos\mu} \frac{\partial \Phi^\alpha}{\partial \mu} \frac{\partial \Phi^\alpha}{\partial \phi_2} & \frac{1}{\sin\mu\cos\mu} \frac{\partial \Phi^\alpha}{\partial \phi_1} \frac{\partial \Phi^\alpha}{\partial \phi_2} & \frac{1}{\cos^2\mu} \frac{\partial \Phi^\alpha}{\partial \phi_2} \frac{\partial \Phi^\alpha}{\partial \phi_2} \end{pmatrix}$$

This is an *elastic energy*, calculated from the departure of the mapping relative to the identity. Adapted to the present symmetry

$$K = \frac{1}{L^2} \begin{pmatrix} \left(\frac{df}{d\mu}\right)^2 & 0 & 0 \\ 0 & p^2 \frac{\sin^2 f}{\sin^2\mu} & 0 \\ 0 & 0 & q^2 \frac{\cos^2 f}{\cos^2\mu} \end{pmatrix}$$

Introducing this expression of K in the energy

$$\begin{aligned} E &= 4\pi^2 L \int d\mu \ \sin\mu \ \cos\mu \ \left[\left(\frac{df}{d\mu}\right)^2 + p^2 \frac{\sin^2 f}{\sin^2\mu} + q^2 \frac{\cos^2 f}{\cos^2\mu} \right] \\ &\quad + 4\pi^2 \frac{1}{L} \int d\mu \ \sin\mu \ \cos\mu \ \left\{ \left(\frac{df}{d\mu}\right)^2 \left[p^2 \frac{\sin^2 f}{\sin^2\mu} + q^2 \frac{\cos^2 f}{\cos^2\mu} \right] + p^2 q^2 \frac{\sin^2 f}{\sin^2\mu} \frac{\cos^2 f}{\cos^2\mu} \right\} \end{aligned}$$

The variational equation for $f(\mu)$ is

$$\begin{aligned}
&\left[-2L\sin\mu\cos\mu-\frac{2}{L}p^2\sin^2 f\frac{\cos\mu}{\sin\mu}-\frac{2}{L}q^2\cos^2 f\frac{\sin\mu}{\cos\mu}\right]\frac{d^2 f}{d\mu^2}\\
&+\left[-2L\left(\cos^2\mu-\sin^2\mu\right)+\frac{2}{L}p^2\frac{\sin^2 f}{\sin^2\mu}-\frac{2}{L}q^2\frac{\cos^2 f}{\cos^2\mu}\right]\frac{df}{d\mu}\\
&-\frac{2}{L}\sin f\cos f\left[p^2\frac{\cos\mu}{\sin\mu}-q^2\frac{\sin\mu}{\cos\mu}\right]\left(\frac{df}{d\mu}\right)^2\\
&+2L\sin f\cos f\left[p^2\frac{\cos\mu}{\sin\mu}-q^2\frac{\sin\mu}{\cos\mu}\right]\\
&+\frac{2}{L}p^2q^2\sin f\cos f\frac{\left[\cos^2 f-\sin^2 f\right]}{\sin\mu\cos\mu}\\
=\;&0
\end{aligned}$$

The simplest solution is

$$\begin{aligned}
p &= 1, q=1\\
\text{or }\; B &= 1
\end{aligned}$$

and the solution is

$$f(\mu)=\mu$$

This leads to

$$\frac{E}{12\pi^2}=\frac{1}{2}\left(L+\frac{1}{L}\right)$$

The topological lower bound is

$$E\geq 12\pi^2 B$$

and is saturated when

$$L=1$$

This is an *isometry*.

It must be related to the *instability* of the mapping

$$\text{base space}\rightarrow\text{target space}$$

which occurs for

$$R\,(\equiv L)\quad>\sqrt{2}$$

For $R>\sqrt{2}$ the identity map becomes unstable and collapses to a single point.

OBS

$$\begin{aligned} L + \frac{1}{L} &= \left(\sqrt{L} + \frac{1}{\sqrt{L}}\right)^2 - 2 > 0 \\ \sqrt{L} + \frac{1}{\sqrt{L}} &> \sqrt{2} \end{aligned}$$

END

NOTE

In the paper **Aratyn Fereira** with Lagrangian ∼power 3/4. The coordinates in the *base* space are toroidal.

Then it is defined a set

$$(\Phi_1, \Phi_2, \Phi_3, \Phi_4)$$

and the function u which is introduced via a stereoscopic projection in the *target* space

$$\begin{aligned} S^2 &\rightarrow R^2 \\ \mathbf{n} &\rightarrow u, u^* \end{aligned}$$

is now

$$\begin{aligned} u &= \frac{Z_1}{Z_0} \\ &= \frac{\Phi_1 + i\Phi_2}{\Phi_3 + i\Phi_4} \end{aligned}$$

where

$$|Z_0|^2 + |Z_1|^2 = 1$$

This is a connection with **Zee Lee** and many others.

But Φ_i may be connected with $(\sigma, \boldsymbol{\pi})$ of **Jackson Manton** S^3.

END

20 The Magnus force and Lorentz force on skyrmions in planar ferromagnetics

The paper by **M. Stone.**

Several papers are in *biblio, classical systems, Skyrme Faddeev.*

skyrmions_Hall_sondhi_kivelson; **Michael Stone – Supersymmetry and the quantum mechanics of spin**. **Kochetov, E. A. – SU(2) coherent-state path integral**. **Klauder, John R. – Path integrals and stationary-phase approximations**.

Also in *research functional.*

20.1 Derivation of the Landau Lifshitz equation for a spin (in a continuous distribution of spins)

The dynamics of a single spin whose direction is

$$\mathbf{n}$$

The classical action for a spin in a magnetic field $\mathbf{B}$ depends on the *history* of the motion of the spin

$$\mathbf{n}(t)$$

and is

$$S = -J \int \dot{\mathbf{n}} \cdot \mathbf{A}(\mathbf{n}) \ dt + \mu \int \mathbf{B} \cdot \mathbf{n} dt$$

where the second term is

$$\begin{aligned} \mu \int \mathbf{B} \cdot \mathbf{n} dt &\equiv -\int H \ dt \\ \text{where } H &= -\mu \mathbf{B} \cdot \mathbf{n} \end{aligned}$$

H is the hamiltonian for a spin of moment $\mu\mathbf{n}$ in the field $\mathbf{B}$.

In the first term, $\mathbf{A}(\mathbf{n})$ is the gauge potential of the magnetic field of a unit *monopole* of 4π *flux*.

The *magnetic monopole* is located in the center of the sphere S^2 on which $\mathbf{n}$ exists.

When the motion of $\mathbf{n}$ is periodic the action can be rewritten

$$S = -J \int \mathbf{n} \cdot \left(\frac{\partial \mathbf{n}}{\partial \tau} \times \frac{\partial \mathbf{n}}{\partial t} \right) d\tau dt + \mu \int \mathbf{B} \cdot \mathbf{n} \ dt$$

The two "time" variables t and τ are introduced in connection with the surface covered by the tip of the spin on the target (internal) sphere S^2 in its evolution.

NOTE

The expression

$$\mathbf{n} \cdot \left(\frac{\partial \mathbf{n}}{\partial x^\mu} \times \frac{\partial \mathbf{n}}{\partial x^\nu} \right) = F_{\mu\nu}$$

is examined in the text *Note Sci, Geometry, Topological* in a chapter on $O(3)$ model, **Skyrme**, **Faddeev**.

We note the replacement of the integrand of the first term in the action S (above) by

$$\dot{\mathbf{n}} \cdot \mathbf{A} \rightarrow \int d\tau \ \mathbf{n} \cdot \left(\frac{\partial \mathbf{n}}{\partial \tau} \times \frac{\partial \mathbf{n}}{\partial t} \right)$$

where $\mathbf{A}$ is the gauge field of the magnetic monopole. The integral in the RHS is of $F_{\mu\nu}$ on one direction, x^μ. If $F_{\mu\nu}$ is the electric field E then the integral along a path $x^\mu \equiv \tau$ from 0 to 1 (where 1 is the value of τ on the circumference of the region Γ) is a difference of potential, like $\delta\varphi$.

END

There is a region Γ on the target sphere S^2 and is bounded by the curve traced by the tip of $\mathbf{n}(t)$ in its *periodic* motion. The variable τ takes value

$$\tau = 1$$

on the curve $\mathbf{n}(t)$ that defines the region Γ. The variable τ extends the physical variable t such as the two (t, τ) cover the region Γ.

A portion of the plane (t, τ) is mapped by $\mathbf{n}$ to the region Γ on the sphere.

The first term of the action S is the *area* of Γ.

This is related to the geometric explanation offered by **Ward** for the **Faddeev** term in the Lagrangian E_2 and E_4.

The *base space* here is a plane (t, τ), two times-like.

NOTE

This is not a trivial replacement.

Initially the term was

$$\frac{\partial \mathbf{n}}{\partial t} \cdot \mathbf{A}$$

where $\mathbf{A}$ is the gauge field of a unit magnetic monopole placed in the center of the sphere.

Now we have

$$\dot{\mathbf{n}} \cdot \mathbf{A} \to \int d\tau \ \mathbf{n} \cdot \left(\frac{\partial \mathbf{n}}{\partial \tau} \times \frac{\partial \mathbf{n}}{\partial t} \right)$$

END

NOTE

The integration by parts using periodicity gives

$$\frac{\partial \mathbf{n}}{\partial t} \cdot \mathbf{A} \quad \to -\mathbf{n} \cdot \quad \frac{\partial \mathbf{A}}{\partial t} = \mathbf{n} \cdot \mathbf{E}$$

precisely on the points of the curve Γ on S^2

it is like a flux of $\mathbf{E}$ but only on Γ

END

The classical equation of motion must be found by varying the action to $\mathbf{n}(t)$.

The variation of the *area* to the variation of its contour

$$\begin{aligned} \delta S &= -J \oint \left[\mathbf{n} \cdot \left(\delta \mathbf{n} \times \dot{\mathbf{n}} \right) \right] dt \\ & \quad + \mu \oint \mathbf{B} \cdot \delta \mathbf{n} \ dt \end{aligned}$$

But we also have to take into account that

$$|\mathbf{n}| = 1$$

which means

$$\begin{aligned} \mathbf{n}^2 - 1 &= 0 \\ 2\mathbf{n}\cdot\delta\mathbf{n} &= 0 \end{aligned}$$

and this can be realized by taking the "variation" $\delta\mathbf{n}$ to lie along the direction perpendicular to the plane formed by $\mathbf{n}$ and another vector, $\delta\mathbf{w}$,

$$\delta\mathbf{n} = \mathbf{n}\times\delta\mathbf{w}$$

Then

$$\delta S = \oint \delta\mathbf{w} \cdot \left[J\dot{\mathbf{n}} - \mu\left(\mathbf{n}\times\mathbf{B}\right)\right] \ dt$$

The extremum of the action gives the equation

$$J\dot{\mathbf{n}} - \mu\left(\mathbf{n}\times\mathbf{B}\right) = 0$$

This equation for the unit vector $\mathbf{n}$ is the *precession* about the direction of the magnetic field $\mathbf{B}$.

One multiplies the equation vectorially by $\mathbf{n}$

$$J\left(\dot{\mathbf{n}}\times\mathbf{n}\right) + \mu\left[\mathbf{B} - \left(\mathbf{B}\cdot\mathbf{n}\right)\mathbf{n}\right] = 0$$

The first term is the Lorentz force acting on the particle with charge J in the field of a monopole, the charge being constrained to move on S^2.

The second term is the component of the field of the monopole, $\mu\mathbf{B}$ which is *tangent* to the sphere S^2. This results from the fact that from $\mathbf{B}$ one subtracts the projection of $\mathbf{B}$ along $\mathbf{n}$ ($\mathbf{n}$ is just a radius in the target sphere S^2). *This is the force attempting to align the vector* $\mathbf{n}$ *which is the spin, with the direction of the field* $\mathbf{B}$.

The continuum version.

one introduces the density ρ of spins of the ferromagnet, per unit area, each of magnitude J.

The first term in the action remains defined by the area on S^2 but now we add all these terms for the ensemble of spins with area density ρ. This is valid for *periodic* motions of the spin $\mathbf{n}(t)$.

In the second term one replaces the field $\mathbf{B}$ with the field-effect of all the neighbor spins that surrounds the current one.

$$S = -J\rho\int \dot{\mathbf{n}}\cdot\left(\frac{\partial\mathbf{n}}{\partial\tau}\times\frac{\partial\mathbf{n}}{\partial t}\right) d\tau dt \ d^2x - \frac{1}{2}K\int\left(\nabla\mathbf{n}\right)^2 \ dtd^2x$$

By this replacement the magnetic field is no more "external, a monopole", but is the intrinsic result of the presence of sources that surround the spin.

The equation of motion is

$$J\rho\dot{\mathbf{n}} - K\mathbf{n}\times\nabla^2\mathbf{n} = 0$$

This is Landau Lifshitz equation and describes the *precession of a spin in the magnetic field created by its neighbors.*

Note see **Saffman Betchov da Rios** in $3D$ fluid and Vorticity Filament Equation - Hasimoto - NSEq **Calini Ivey,** with isoperiodic deformation by **Grinevich**.

END.

The static solutions are those of

$$\begin{aligned} \nabla^2\mathbf{n} &= 0 \\ \mathbf{n}^2 - 1 &= 0 \end{aligned}$$

The solutions are *skyrmions*

$$\mathbf{n} : \mathbf{R}^2 \to S^2$$

The parameterization of the sphere (internal space of $\mathbf{n}$) is

$$(\theta, \varphi) \equiv \text{polar coordinates}$$

and the solution is

$$\exp(i\varphi)\cot\frac{\theta}{2} = \frac{a}{z}$$

where

$$z \equiv \left(x^1, x^2\right) = x^1 + ix^2$$

the spin in the center is UP and progressively with the distance from the center the spins flip. At infinity of the plane, all spins are down.

General solution

$$\exp(i\varphi)\cot\frac{\theta}{2} = f(z)$$

where $f(z)$ has poles and zeros. **Rajaraman**.

[similar solution is given by **Polyakov Belavin** in their first examination of $S^2 \to S^2$ model]

[similar solution is in **Battye Sutcliffe** on the Skyrme model].

20.2 Dynamical aspect of a skyrmion

Assume there is a skyrmion placed in the origin of the base space

$$\mathbf{n}_0(\mathbf{r})$$

Note we would expect that the Skyrmion was a $SU(2)$ matrix. **End.**

It generates a field at a distance $\mathbf{R}$ of it

$$\mathbf{R} = \left(R^1, R^2\right)$$

$$\mathbf{n}(\mathbf{r}, t) = \mathbf{n}_0(\mathbf{r} - \mathbf{R}(t))$$

The skyrmion will move in the base plane and its field at a certain distance will be calculated using this expression.

This expression of the field $\mathbf{n}(\mathbf{r}, t)$ is inserted in the Action functional, $S = -J \int \dot{\mathbf{n}} \cdot \mathbf{A}(\mathbf{n})\ dt + \mu \int \mathbf{B} \cdot \mathbf{n} dt$.

It is then calculated the cost in action for a closed loop of the skyrmion in the base plane.

$$S = -J\rho \int \dot{\mathbf{n}} \cdot \mathbf{A}[\mathbf{n}(\mathbf{r}, t)]\ dtd^2x$$

where we replace $\mathbf{n}(\mathbf{r}, t)$ as above.

NOTE that here the gauge potential $\mathbf{A}$ has a source: before there was the *magnetic monopole* placed in the center of the targett space.

Later it is taken as generated by the magnetic effect of all spins around the one we have chosen.

END

Consider the variation of the action when the path $\mathbf{R}(t)$ is modified

$$\delta S = -J\rho \int \mathbf{n}\cdot \left(\delta\mathbf{n}\times\dot{\mathbf{n}}\right)\ dtd^2x$$

The variations are

$$\dot{\mathbf{n}} = -\frac{\partial}{\partial x^i}\mathbf{n}_0(\mathbf{r} - \mathbf{R})\ \dot{R}^i$$

$$\delta\mathbf{n} = -\frac{\partial}{\partial x^i}\mathbf{n}_0(\mathbf{r} - \mathbf{R})\,\delta R^i$$

and the variation of the action

$$\delta S = -J\rho \int \mathbf{n}_0(\mathbf{r} - \mathbf{R}) \cdot \left[\frac{\partial \mathbf{n}_0(\mathbf{r} - \mathbf{R})}{\partial x^i} \times \frac{\partial \mathbf{n}_0(\mathbf{r} - \mathbf{R})}{\partial x^j}\right] \delta R^i\ \dot{R}^j\ dtd^2x$$

$$\delta S = -J\rho \int \delta R^i\ \dot{R}^j\ \left\{\int \mathbf{n}_0(\mathbf{r} - \mathbf{R}) \cdot \left[\frac{\partial \mathbf{n}_0(\mathbf{r} - \mathbf{R})}{\partial x^i} \times \frac{\partial \mathbf{n}_0(\mathbf{r} - \mathbf{R})}{\partial x^j}\right] d^2x\right\} dt$$

The factor in curly braces is the *winding number*

$$\mathbf{n} : \mathbf{R}^2 \rightarrow S^2$$

$$4\pi\mathcal{N}\varepsilon_{ij}$$

then

$$\delta S = -4\pi\mathcal{N}J\rho\oint\left(\delta R^1 \dot{R^2} - \delta R^2 \dot{R^1}\right)dt$$

which is a variation of the action

$$S = -2\pi\mathcal{N}J\rho\oint\left(R^1 \dot{R^2} - R^2 \dot{R^1}\right)dt$$

In the path integral

$$\mathcal{Z} = \int D[\mathbf{n}]\exp(iS)$$

it results that the skyrmion accumulates a phase of 2π for each spin that is inside its loop.

The part

$$2\pi\mathcal{N}J\rho\oint\left(R^1 \dot{R^2} - R^2 \dot{R^1}\right)dt$$

occurs in the action of a particle of charge $\mathcal{N}$ that moves in the uniform magnetic field of strength $2\pi J\rho$.

The idea of this paper is that **Magnus** force and **Lorentz** force, - the motions imposed by these two sources are actually identical, a single object.

It discusses q-Hall effect using Zhang Kievelson Hansen model.

Intersting remark: the Landau motion of a charge would become gyration if it had a mass.

21 The $O(3)$ system

Since this part is useful for *Beltrami insertions*, it is also found in *fluid.tex*.

And in *field theory notes* from *notes_sci*.

In **Ward** it is discussed the $O(3)$ σ-model or the CP^1 model (the same thing).

The space $\mathbf{R}^{2+1}$ is $2+1$ dimensional

$$x^\mu = (t, x, y)$$

with the metric

$$\eta_{\mu\nu} = (-1, 1, 1)$$

By his definition, the $O(3)$ - σ-model or CP^1 deals with *fields* which are functions defined on the space-time and taking values on the Riemann sphere.

For the time-independent case. There is a *potential* energy E_P

$$E_P = \int_{\mathbf{R}^2} dxdy \frac{1}{\left(1+|W|\right)^2} \delta^{ik} \left(\partial_i W\right) \left(\partial_k W^*\right)$$

After compactification

$$\mathbf{R}^2 \to S^2$$

the fields (W is complex so there are two) are mappings between two spheres

$$S^2 \to CP^1 \simeq S^2$$

and are classified by an integer N. This is, roughly, the number of solitons in the plane.

$$E_P \geq 2\pi N$$

The equality $E_P = 2\pi N$ is attained if and only if the W is a *meromorphic* function of $z = x + iy$. Then W is simply given in terms of *zeros* and *poles*.

The graphic representation of the potential energy E_P shows a number of N lumps, solitons on the xy-plane.

21.1 Duality in the $O(3)$ model Wen Zee

The paper is **Duality 2p1 Wen Zee**.

The base space is $2+1$ dimensions.

The model is the *sigma* model of $O(3)$.

The target is the sphere S^2.

It has solitons.

They can be quantized such as to have *fractional spin and statistics*. The paper shows first how to do that.

The field is that of a **unitary vector n**.

$$\mathbf{n}^2 - 1 = 0$$

The topological current

$$J^\mu = \frac{1}{8\pi} \varepsilon^{\mu\nu\rho} \varepsilon_{abc} n^a \partial_\nu n^b \partial_\rho n^c$$

We note here two products,

- one is a *vector product*, in the real space $\mu\nu\rho$, leading to a vector with components μ, and
- one is a *mixed vector product*, in the *internal* $O(3)$ space, abc, with full contraction of the indices

note that this is the vector of *vorticity* according to **Kuznetsov**.
The current occurs in the **Faddeev Skyrme** model.
It is a simple description of a vector line in space.

A theory with only the unit vector field $\mathbf{n}(x,y,z,t)$ is a topological theory since it maps a sphere (compactified plane $\mathbf{R}^2$) on a sphere, the space of $\mathbf{n}^2-1=0$.

In order to obtain the fractional statistics, one has to couple the field $\mathbf{n}$ with a potential A^μ. The lagrangian which initially was

$$\begin{aligned} L_{O(3)} &= \frac{1}{2}(\partial_\mu \mathbf{n})^2 \\ \text{and the condition } \mathbf{n}^2-1 &= 0 \end{aligned}$$

becomes, after including the field A^μ:

$$\begin{aligned} L_{O(3)+A^\mu} &= \left(\frac{2}{g^2}\right)(\partial_\mu \mathbf{n})^2 \quad \text{(kinetic, model } O(3)\text{)} \\ &\quad +A^\mu J_\mu \quad \text{(interaction)} \\ &\quad +\alpha\, \varepsilon^{\mu\nu\rho} A_\mu F_{\nu\rho} \quad \text{(Chern-Simons)} \end{aligned}$$

The gauge is Landau

$$\partial_\mu A^\mu = 0$$

and the potential A^μ can be integrated out in the partition function. It is found the action

$$\begin{aligned} S_0 &= \int d^3x \left(\frac{2}{g^2}\right)(\partial_\mu \mathbf{n})^2 \\ &\quad +\theta H \end{aligned}$$

where the so-called Hopf term is *non-local*

$$H = \frac{1}{4\pi}\int d^3x \varepsilon^{\mu\nu\rho} J_\mu \partial_\nu \frac{1}{\partial^2} J_\rho$$

the factor θ is defined as

$$\theta = -\frac{1}{8\alpha}$$

NOTE see **Haldane** about the term θH. **END.**

Note The expression $\varepsilon^{\mu\nu\rho} J_\mu \partial_\nu \frac{1}{\partial^2} J_\rho$ can be understood as follows. The current, in electromagnetism is the Laplacian of the gauge potential,

$$j = \Delta A$$

then

$$\frac{1}{\partial^2} J_\rho \sim A_\rho$$

and then the expression is

$$\varepsilon^{\mu\nu\rho} J_\mu \partial_\nu \frac{1}{\partial^2} J_\rho \sim \varepsilon^{\mu\nu\rho}\ J_\mu\ \partial_\nu A_\rho \sim J \cdot (\boldsymbol{\nabla} \times \mathbf{A}) = \mathbf{J} \cdot \mathbf{B}$$

End.

The boundary condition is

$$\mathbf{n} \rightarrow \mathbf{n}_0 \quad \text{as} \quad (\mathbf{x}, t) \rightarrow \infty$$

which means that the vector **n** has an unique orientation on the boundary, the same in every point of a large circle (or sphere in $(\mathbf{x}, t)$?). This is equivalent to a map

$$S^3 \rightarrow S^2$$

for which we know that

$$\pi_3 \left(S^2\right) = \mathbf{Z}$$

We conclude that H is **integer** and that *classically* the quantity θ has *NO* effect. But in quantum mechanics it has. Each space-time history is associated in the path integral with a factor:

$$\exp\left(in\theta\right)$$

where n labels the homotopy class of the history.

The origin of the fractional statistics is: the equation of motion

$$2\alpha\varepsilon^{\mu\nu\rho} F_{\nu\rho} = -J^\mu$$

which in particular is

$$F_{12} = \left(-\frac{1}{4\alpha}\right) J_0$$

A soliton located at the origin and carrying q_0 charge

$$q_0 = \int d^2 x J_0$$

Far away of the center of the soliton we have

$$F_{12} \rightarrow 0 \quad \text{at} \quad \mathbf{x} \rightarrow \infty$$

and this implies that A^μ is a pure gauge.

But, it is not topologically trivial.

Going around a closed contour we have

$$\begin{aligned}
\oint_C dx_i A^i &= \\
&= \iint d^2x F_{12} \quad \text{(flux)} \\
&= \iint d^2x \left(-\frac{1}{4\alpha}\right) J_0 \quad \text{(topological charge)} \\
&= -\frac{q_0}{4\alpha}
\end{aligned}$$

Taking a soliton around a closed curve the change of the wavefunction consists of multiplying by a phase factor,

$$\begin{aligned}
&\sim \exp\left(\oint_C dx_i A^i\right) \\
&= \exp\left(-\frac{q_0}{4\alpha}\right)
\end{aligned}$$

This phase can be interpreted as fractional statistics.

NOTE the paper by **Niemi Semenoff** on the difference between the number of fermion zero modes

$$\begin{aligned}
N_R - N_L &\sim \Phi \\
&= \text{flux of magnetic field}
\end{aligned}$$

and this is fractional.

END

Further, in **membranes hopf wu zee** paper.

The topological current is

$$J^\mu = \varepsilon^{\mu\nu\rho}\varepsilon_{abc}\, n^a\ \partial_\nu n^b\ \partial_\rho n^c$$

as in **Fadeev Skyrme** , the vorticity line of **Kuznetsov**.

[It can also be expressed in terms of a gauge potential A_μ after changing to complex matrix Z].

The Chern Simons term $\int AF$ can be expressed in terms of the gauge field A_μ which is defined in terms of $\mathbf{n}$. It then represents the Hopf mapping

$$S^3 \to S^2$$

Define

$$\mathbf{n} = z^\dagger \boldsymbol{\sigma} z$$

with

$$z \equiv \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

and

$$z^\dagger z = 1$$

The action contains

- the interaction jA (Hopf term) between the current (which is topological, Faddeev Skyrme, $O(3)$), and
- the Chern Simons term

It is

$$S = \int d^3x \left[J^\mu A_\mu + g\varepsilon^{\mu\nu\rho} A_\mu F_{\nu\rho} \right]$$

with equation

$$2g\varepsilon^{\mu\nu\lambda} F_{\nu\lambda} = -J^\mu$$

When the sizes of solitons are small compared with the separation, the current becomes

$$J^\mu = \int d\tau \; \delta^{(3)} \left(x - y(\tau) \right) \; \frac{dy^\mu(\tau)}{d\tau}$$

The current-gauge field interaction part of the action (the Hopf term)

$$\exp\left(i \int d^3x \; J^\mu A_\mu \right)$$

This is a phase factor that can be calculated

$$\exp\left(-\frac{i}{4g} \right)$$

The calculation is possible because

- we dispose of the expression of the current J in terms of the unit vector $\mathbf{n}$;
- one can solve the equation of motion derived from the action and find A in terms of the current J. The equation of motion is $B \sim J$ and further $\mathbf{\nabla} \times \mathbf{A} = \mathbf{B}$ can be solved using the Biot Savart integral.

In the point particle limit the Hopf term reduces to Gauss linking integral

In physical terms, the Gauss linking integral is the *work* done on a magnetic monopole for it to move around a closed loop under the action of the magnetic field produced by a current flowing along another closed loop.

The Hopf term is

$$\int \mathbf{J} \cdot \mathbf{B} \, d^3x$$

is this work.

This work is zero except if the monopole moves along a closed *flux* line produced by the current $\mathbf{J}$. This condition means that the two loops are linked.

In the generating functional which consists of the functional integral over the fields that appear in the action functional $S = \int d^3x \left[J^\mu A_\mu + g\varepsilon^{\mu\nu\rho} A_\mu F_{\nu\rho}\right]$ one can integrate over the field A_μ. The result is the Hopf integral which appears as an interaction

$$\int d^3x \; d^3y \; \varepsilon^{\mu\nu\rho} J_\mu(x) \frac{x-y}{|x-y|^3} J_\nu(x)$$

This integral is not singular for

$$|x-y| \to 0$$

The problem that arises is the renormalization. The Hopf soliton is an extended object from a point-like object. The current has a particular simpler expression when the soliton can be considered a point.

Nice extension from the Hopf mapping $S^3 \to S^2$ where the objects are linked lines to objects in higher dimension, like a membrane. The membrane is the limit form of a thick fuzzy object, just like the point is the limit form of the finite-extension soliton.

21.2 Soliton instability in the CP^1 or σ model

From **Ward**.

The basic model is topological.

This is discussed in **Scattering Nonlinear Sigma**.

For the model $O(N)$ there is the lagrangian density

$$\mathcal{L} = (\partial_\mu \mathbf{n})^2$$

and the condition

$$\mathbf{n}^2(x) - 1 = 0$$

The action

$$S = -\frac{1}{2g_0}\int (\partial_\mu \mathbf{n})^2 \, d^2x$$

is supplemented with the term arising from the condition

$$\prod_x \delta\left(\mathbf{n}^2(x) - 1\right)$$

with a Lagrange multiplier $\lambda(x)$. The partition function obtains a term of the form

$$\exp\left(-\frac{N}{2}\ln\det\left\|-\partial^2 - \lambda^{1/2}(x)\right\|\right)$$

From this expression we conclude that $\lambda^{1/2}(x)$, square root of the Lagrange multiplier, appears as a mass term, by analogy with the Klein-Gordon operator.

In general, the Lagarange multiplier has the effect of **the mass**. **This may suggest that the mass is possibly connected with the presence of a constraint in a free theory**.

More generally, the screening arising in the expression of a Green operator (propagator) can be associated with a constraint.

The quantity $\lambda^{1/2}(x)$ is also **the inverse correlation length** $\langle n_i(x)\, n_j(y)\rangle$. Conversely, the finite correlation length can be associatted with a constaint present in the theory through a Lagrangian multiplier.

21.3 Solitons $O(3)$ by Lee

In the paper **9510141 Solitons** $O(3)$ model by **Lee** it is discussed a model

$$\begin{array}{l} \text{gauged } (A_\mu) \\ O(3) \\ \text{sigma model} \end{array}$$

in the base space

$$2+1 \text{ dimensions}$$

Particularly, the gauge field A_μ is coupled to the scalar field via a current that is $U(1)$ and NOT the topological current. For the specific potential which is of 6^{th} order, there is a Bogomol 'nyi bound.

The field is $\boldsymbol{\phi}$

$$\begin{aligned} \boldsymbol{\phi} &: \mathbf{R}^2 \overset{\phi}{\rightarrow} S^2 \\ |\boldsymbol{\phi}|^2 &= 1 \end{aligned}$$

There is a topological current

$$\begin{aligned} k_\alpha &= \frac{1}{8\pi}\varepsilon_{\alpha\beta\rho}\boldsymbol{\phi}\cdot\left(\partial^\beta\boldsymbol{\phi}\times\partial^\rho\boldsymbol{\phi}\right) \\ &= \frac{1}{8\pi}\varepsilon_{\alpha\beta\rho}\varepsilon^{abc}\phi_a\left(\partial^\beta\phi_b\right)\left(\partial^\rho\phi_c\right) \end{aligned}$$

is conserved. If $\boldsymbol{\phi}$ approaches at spatial infinity a *constant* unitary vector, then $\boldsymbol{\phi}$ realizes a mapping $\boldsymbol{\phi} : \mathbf{R}^2 \xrightarrow{\phi} S^2$ and the topological charge is the integral of the 0-component of the topological current

$$Q = \int d^2 x k_0$$

Now the model will be extended with a gauge field.

Later for the gauge field it will be adopted the *Chern Simons* Lagrangian density.

The derivative operator becomes a covariant derivative operator defined as

$$D_\mu \boldsymbol{\phi} = \partial_\alpha \boldsymbol{\phi} + A_\mu \widehat{\mathbf{n}} \times \boldsymbol{\phi}$$

where $\widehat{\mathbf{n}}$ is a fixed versor in the space of $\boldsymbol{\phi}$, taken

$$\widehat{\mathbf{n}} = (0,0,1)$$

The gauge-invariant generalization of the topological current is

$$\begin{aligned} K_\alpha &= \frac{1}{8\pi} \varepsilon_{\alpha\beta\rho} \boldsymbol{\phi} \cdot \left(D^\beta \boldsymbol{\phi} \times D^\rho \boldsymbol{\phi} \right) \\ &\quad + \frac{1}{8\pi} \varepsilon_{\alpha\beta\rho} F^{\beta\rho} \left(v - \widehat{\mathbf{n}} \cdot \boldsymbol{\phi} \right) \end{aligned}$$

where v is a real parameter.

$$\begin{aligned} K_a &= k_\alpha \\ &\quad + \frac{1}{4\pi} \varepsilon_{\alpha\beta\rho} \partial^\beta \left[\left(v - \widehat{\mathbf{n}} \cdot \boldsymbol{\phi} \right) A^\rho \right] \end{aligned}$$

The second term is also a *current* (the only indice is α) and is the rotational of a vector, which is a scalar $\times$ gauge field A^ρ.

NOTE

The paper by **Ward** explains the meaning of the expression of the current k_α in terms of geometry of area.

To determine the area one has to use vectors directed along the tangents at the surface in a point, $\partial_\mu \mathbf{n}$ whose vector product is *projected* along the vector $\mathbf{n}$.

But, what is the vector that we obtain when the derivative is *covariant* ?

$$\partial_\mu \rightarrow D_\mu = \partial_\mu + A...$$

END

According to the condition at the infinite limit, there are

1. two symmetric phases
$$\lim_{|x| \rightarrow \infty} \boldsymbol{\phi}(t, \mathbf{x}) = \pm \widehat{\mathbf{n}}$$

2. one asymmetric phase, for $v < 1$,

$$\lim_{|x|\to\infty} \widehat{\mathbf{n}} \cdot \boldsymbol{\phi}(t, \mathbf{x}) = v$$

The Lagrangian

$$\begin{aligned} L &= \frac{\kappa}{2}\varepsilon^{\alpha\beta\rho} A_\alpha \partial_\beta A_\rho \\ &\quad + \frac{1}{2}(D_\alpha \boldsymbol{\phi})^2 \\ &\quad - \frac{1}{2\kappa^2}(v - \widehat{\mathbf{n}} \cdot \boldsymbol{\phi})^2 (\widehat{\mathbf{n}} \times \boldsymbol{\phi})^2 \end{aligned}$$

22 The Skyrme model

In the paper **9506099 Skyrme Piette** it is introduced the model.

It is $(3+1)$ dimensional.

Has soliton solutions which after quantization are models for *baryons*.

It is invariant to $SO(3)_{iso}$ of iso-rotations.

The field can be coupled with a $U(1)$ gauge field.

That paper analyses a $(2+1)$ dimensional *baby Skyrmion* model, with dynamical $U(1)$ (*i.e.* Abelian) gauge field A^μ.

It has soliton solutions that are stable for topological reasons. They carry magnetic flux.

The gauge field symmetry is NOT broken. Therefore the solitons are different of the flux tubes or *ANO* vortices of the Abelian-Higgs model.

The magnetic flux of the solitons of the $(2+1)$ baby Skyrme is NOT quantized.

The content of the model:

$$\begin{aligned} \boldsymbol{\phi} &\equiv (\phi_1, \phi_2, \phi_3) \quad \text{with} \quad \phi_1^2 + \phi_2^2 + \phi_3^2 = 1 \text{ (scalar matter field)} \\ A_\mu \ \ , \ \mu &= 0, 1, 2 \quad \text{(gauge field)} \end{aligned}$$

in the *base* space

$$x^\alpha \quad , \quad \alpha = 0, 1, 2$$

with the metric

$$(-1, +1, +1)$$

NOTE

The gauge field $A_{0,1,2}$ has components that are considered relative to the coordinates of the *base* space (t, x, y).

END

The Scalar matter field $\boldsymbol{\phi}$ is a unitary vector with the tip lying on the 2-sphere

$$S^2_\phi$$

which is the *space of internal symmetry.*

The Lagrangian will be assumed *invariant* to certain rotations in the internal space. The rotations are chosen to preserve a certain direction in the internal space. Denoting this direction with the vector

$$\mathbf{n} \equiv (0,0,1)$$

in internal space, the rotations are in the plane perpendicular to $\mathbf{n}$

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \to \begin{pmatrix} \cos\chi & \sin\chi & 0 \\ -\sin\chi & \cos\chi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}$$

These rotations are called $SO(2)_{iso}$ being *plane* rotations, in *isospace.*

The term used in this paper for coupling the matter with a gauge field is : **We couple electromagnetism to the baby Skyrme model by gauging the rotations of $SO(2)_{iso}$ symmetry. Thus we require that the theory is invariant to rotations of $SO(2)_{iso}$ which are now** *dependent of point*

$$\phi \to O(x)\,\phi$$

where

$$O(x) \in SO(2)_{iso} \quad \text{is a matrix that depends on } x$$

Take an infinitesimal angle ε of such a rotation (whose axis is $\mathbf{n}$), then

$$\boldsymbol{\phi} \to \boldsymbol{\phi} + \varepsilon\mathbf{n} \times \boldsymbol{\phi}$$

NOTE

In the work of **Lee previous notes**, the change from partial derivatives to *covariant* derivatives is

$$D_\mu\boldsymbol{\phi} = \partial_\alpha\boldsymbol{\phi} + A_\mu\widehat{\mathbf{n}} \times \boldsymbol{\phi}$$

END

The Abelian gauge field A_μ transforms as

$$A_\mu \to A_\mu - \partial_\mu\varepsilon$$

From here it is defined the *covariant derivative*

$$D_\alpha\boldsymbol{\phi} = \partial_\alpha\boldsymbol{\phi} + A_\alpha\mathbf{n} \times \boldsymbol{\phi}$$

and the invariance is ensured by

$$D_\alpha \left(O\left(x\right) \boldsymbol{\phi}\right) = O\left(x\right) D_\alpha \boldsymbol{\phi}$$

There is a curvature field

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$$

The Lagrangian is

$$\begin{aligned} L &= \int d^2x \left[\frac{1}{2} \left(D_\alpha \boldsymbol{\phi}\right)^2 \quad \text{the gauged } O\left(3\right) \text{ sigma model} \right. \\ & + \frac{\lambda^2}{4} \left(D_\alpha \boldsymbol{\phi} \times D_\beta \boldsymbol{\phi}\right)^2 \quad \text{gauged Skyrme term} \\ & + \mu^2 \left(1 - \mathbf{n} \cdot \boldsymbol{\phi}\right) \quad \text{the pion mass} \\ & \left. + \frac{1}{4g^2} \left(F_{\alpha\beta}\right)^2 \right] \quad \text{the Maxwell term} \end{aligned}$$

NOTE in the Internet site of Hopfions it is recalled that the *area* $\mathbf{n} \cdot \left(\frac{\partial \mathbf{n}}{\partial x^\mu} \times \frac{\partial \mathbf{n}}{\partial x^\nu}\right)$ squared as it should in a Lagrangian can take a simpler form $\left(\partial_\mu \mathbf{n} \times \partial_\nu \mathbf{n}\right)^2$ as above. **END.**

The energy of a

$$\left(\boldsymbol{\phi}, A_\alpha \right) \quad \text{configuration}$$

is composed of two parts

kinetic energy (contains time-derivations plus the energy of the electric field, which itself is $\partial A_i / \partial t$)

$$T = \int d^2x \left[\frac{1}{2} \left(D_0 \boldsymbol{\phi}\right)^2 + \frac{\lambda^2}{4} \left(D_0 \boldsymbol{\phi} \times D_i \boldsymbol{\phi}\right)^2 + \frac{1}{2g^2} E_i^2 \right]$$

potential energy (contains space derivations plus the energy of the magnetic field)

$$V = \int d^2x \left[\frac{1}{2} \left(D_1 \boldsymbol{\phi}\right)^2 + \frac{1}{2} \left(D_2 \boldsymbol{\phi}\right)^2 + \frac{\lambda^2}{2} \left(D_1 \boldsymbol{\phi} \times D_2 \boldsymbol{\phi}\right)^2 + \mu^2 \left(1 - \mathbf{n} \cdot \boldsymbol{\phi}\right) + \frac{1}{2g^2} B^2 \right]$$

NOTE

In the paper **9812103 Low energy vortex dyn Abelian Fuertes Guilarte** it is done a similar separation

$$S \text{ (the action)} = \int dt \left(T - V \right)$$

where

$$\begin{aligned} T &= \int d^2x \left[\frac{1}{2} \left(\frac{\partial \rho}{\partial t} \right)^2 \right. \\ & \left. + \frac{\kappa}{2} \varepsilon_{kl} \left(\frac{\partial A_k}{\partial t} \right) A_l - \frac{\kappa}{2e} \left(\frac{\partial \chi}{\partial t} \right) F_{12} \right] \end{aligned}$$

and

$$\begin{aligned} V &= \int d^2x \left[\frac{1}{2} \left(D_k \phi\right) \left(D^k \phi\right) \right. \\ &\quad + \frac{\lambda}{8} \rho^2 \left(\rho^2 - 1\right)^2 \\ &\quad \left. + \frac{1}{2} \frac{\kappa^2}{e\rho^2} \left(F_{12}\right)^2 \right] \end{aligned}$$

and the Hamiltonian is

$$H = \int dt \left[\frac{1}{2} \int d^2x \left(\frac{\partial \rho}{\partial t} \right)^2 + V \right]$$

Here the following *unusual* notations have been used

$$\phi = \rho \exp\left(i \frac{\chi}{2}\right)$$

END

It is restricted the theory to *finite energy* configurations, which imposes

$$\lim_{r \to \infty} \boldsymbol{\phi}(x) = \mathbf{n}$$

(which means that the field $\boldsymbol{\phi}$ takes a unique direction in isospace, at spatial infinity).

The plane $\mathbf{R}^2$ can be compactified to a sphere $S^2_{\mathbf{x}}$, due to this choice of asymptotic condition for $\boldsymbol{\phi}$.

The space of internal symmetry (isospin) is also a sphere $S^2_{\boldsymbol{\phi}}$.

Any configuration with that asymptotic behavior will be a mapping between the two spheres

$$S^2_{\mathbf{x}} \to S^2_{\boldsymbol{\phi}}$$

and the **topological** quantity is conserved

$$Q = \frac{1}{4\pi} \int_{S^2_{\mathbf{x}}} d^2x \; \boldsymbol{\phi} \cdot \left(\partial_1 \boldsymbol{\phi} \times \partial_2 \boldsymbol{\phi}\right)$$

The Hopf index.

22.1 The Magnus force and the Lorenz force on a Skyrmion Stone

Few papers in *biblio, classical systems, Skyrme Faddeev.*

23 BPS skyrmions

Review.

24 Hopf map in Yang Mills theory (Guo)

The paper is **Phys Lett B 739 (2014) 83**.

The space $\mathbf{R}^4$ is described by two complex variables

$$\begin{aligned} z_1 &= x_1 + i x_2 \\ z_2 &= x_3 + i x_4 \end{aligned}$$

The *compactified* version of the space $\mathbf{R}^4$ is now S^3:

$$S^3 : \; |z_1|^2 + |z_2|^2 = 1$$

The Hopf mapping

$$f : S^3 \to S^2$$

has homotopy group $\mathbf{Z}$.

See **Zee**.

24.1 The connection Hopf-map and the sphaleron

Also in **Helicity notes 2017 corfu**.

The basis to construct the gauge field with topological content (winding number) is the Hopf mapping.

The Hopf mapping

$$S^3 \to S^2$$

has a nontrivial homotopy group

$$\pi_3\left(S^2\right) = \mathbf{Z}$$

The Hopf map is

$$\chi : \mathbf{R}^3 \to \mathbf{C}$$

with the condition

$$\lim_{|\mathbf{x}| \to \infty} \chi = \chi_0 = \text{const} \in \mathbf{C}$$

The pre-images of a fixed point on the target sphere, $\chi =$const $\in S^2$ is a closed curve in $\mathbf{R}^3$. Two closed curves are linked N times, where $N \equiv$Hopf index, $\in \pi_3\left(S^2\right)$.

Good reference is **Kuznetsov** or **Kundu Rybakov**.

The pullback of the *area* two-form of the target space S^2 is the *curvature* defined in $\mathbf{R}^3$. Take the point on S^2 defined by two variables of the real space (ρ, σ).

χ is a point on S^2 but (ρ, σ) is a set of points in the real (base) space, a *curve.*

$$\chi = \rho \exp(i\sigma)$$

the expression

$$\mathbf{B} = \frac{2}{i} \frac{\boldsymbol{\nabla}\chi^* \times \boldsymbol{\nabla}\chi}{(1+\rho^2)}$$

Defining the gauge field by

$$F_{ij} = \varepsilon_{ijk} B_k$$

and the curvature by

$$\frac{1}{2} F_{ij} dx_i dx_j$$

The field $\mathbf{B}$ is tangent to the closed curves that correspond in $\mathbf{R}^3$ to χ =const on S^2. To $\mathbf{B}$ it corresponds the gauge potential $\mathbf{A}$.

The Hopf index appears as the integral over space of the density of topological charge (Chern-Simons) of the gauge field $\mathbf{B}$, the *helicity*

$$N = \frac{1}{16\pi^2} \int d^3x \ \mathbf{A} \cdot \mathbf{B}$$

The simplest Hopf map, with $N = 1$ is

$$\chi = \frac{2(x_1 + ix_2)}{2x_3 - i(1 - r^2)}$$

Here χ is a point on S^2 in the target space and (x_1, x_2, x_3) are points in $\mathbf{R}^3$ that represent a *string* (a line in space).

This is the simplest *hypermagnetic* knot.

With this Hopf map, according to above formulas, we calculate explicit expressions for the potential and magnetic field

$$A_k(t, \mathbf{x}) = \frac{N_k(\mathbf{x})}{1 + r^2}$$

$$B_k(t, \mathbf{x}) = \varepsilon_{kij} \partial_i A_j = 4 \frac{N_k(\mathbf{x})}{(1 + r^2)^2}$$

where

$$N_k(\mathbf{x}) = \frac{1}{(1 + r^2)} \begin{pmatrix} 2xz - 2y \\ 2yz + 2x \\ 1 - x^2 - y^2 + z^2 \end{pmatrix}$$

Here is the model that must be used for application:

- there is *one fermion species.*

- its current is chiral and retaining only the *left* component, J_μ^L, its divergence is equal with the density of topological charge of the gauge field

$$\partial_\mu J_L^\mu = \frac{1}{16\pi^2} \varepsilon^{\mu\nu\rho\lambda} F_{\nu\mu} F_{\rho\lambda}$$

- the dimension of the BASE space is four $(1,-1,-1,-1)$. There is no Chern Simons term possible. But $F\widetilde{F}$ is a scalar, the density of the topological charge

- the gauge field is ABELIAN.

- it is taken in the gauge

$$A_0 = 0$$

- the integral over time between the initial and final states is taken after both terms are integrated over space.

- the number of particles created between

$$[t_i, t_f]$$

is

$$\int d^3x \left[J_0^L(t_f,\mathbf{x}) - J_0^L(t_i,\mathbf{x})\right] = -\frac{1}{4\pi^2}\int d^3x \left[(\mathbf{A}\cdot\mathbf{B})_{(t_f,\mathbf{x})} - (\mathbf{A}\cdot\mathbf{B})_{(t_i,\mathbf{x})}\right]$$

- the fermion field obeys the Dirac equation with gauge A_μ field and NO mass. [in **Boyanovsky** there is mass]

- the Dirac equation is reduced to work on Weil spinors

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

this may leave the false impression that we have a NON-ABELIAN theory. It is NOT. It is a Dirac equation.

The zero modes.

The fermion field verifies the massless Dirac equation

$$\sigma_k \left(\frac{\partial}{\partial x^k} - iA_k\right)\Psi = 0$$

where Ψ is a Weil spinor (only the chiral aspect is important), *i.e.* it is a two-component column matrix

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$$

The condition of zero mode: divergence of the spin current is zero

$$\frac{\partial}{\partial x^k}\Sigma_k = 0$$

where the density of spin is

$$\Sigma^k = \Psi^\dagger \sigma^k \Psi$$

Conversely, if one has a solution which verifies the equation of zero-modes, then together with the Dirac equation, one can find the gauge field

$$\begin{aligned} A_k &= \frac{1}{|\mathbf{\Sigma}|}\left(\frac{1}{2}\varepsilon_{klm}\partial_l \Sigma_m + \operatorname{Im}\Psi^\dagger \partial_k \Psi\right) \\ & \frac{1}{2}\varepsilon_{klm}\frac{\partial}{\partial x^l}\left(\ln|\mathbf{\Sigma}|\right)N_m + \frac{1}{2}\varepsilon_{klm}\partial_l N_m \\ & +\widehat{\Psi}^\dagger \Sigma_k \widehat{\Psi} \end{aligned}$$

where

$$N_k = \frac{\Sigma_k}{|\mathbf{\Sigma}|}$$

$$\widehat{\Psi} = \frac{\Psi}{\left[\Psi^\dagger \Psi\right]^{1/2}}$$

Solution found by **Loss Yau**.

Take

$$-i\sigma_k \frac{\partial}{\partial x^k}\Psi = h\Psi$$

together with the Dirac equation, offers an explicit expression for the gauge field that verifies Dirac equation for this Ψ ($h(\mathbf{x})$ is an arbitrary function)

$$A_k = h\frac{\Psi^\dagger \sigma_k \Psi}{\Psi^\dagger \Psi}$$

Then Loss Yau give the expression for $\Psi(\mathbf{x})$,

For $c = 3$,

$$\Psi = \frac{4}{\left(1+r^2\right)^{3/2}}\left(1 + i\mathbf{x}\cdot\boldsymbol{\sigma}\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

For this Dirac zero mode the density of spin is

$$\Sigma_k = \Psi^\dagger \sigma_k \Psi = \frac{16}{\left(1+r^2\right)^2} N_k$$

and the gauge field is

$$A_k = \frac{3}{\left(1+r^2\right)} N_k$$

- VERY strange hypothesis: *the gauge field of the sphaleron solution is adopted with invariant shape but the amplitude is factorized as a time dependent coefficient*

$$A_k(t,\mathbf{x}) = c(t) A_k(\mathbf{x})$$

then a natural question is : how the gauge field can have different winding numbers at t_i and t_f ? The possible answer: the gauge field does not have different winding numbers at the initial and final state, it has a single winding number, fixed by its spatial shape $A_k(\mathbf{x})$. This fixed amount of winding will be transferred to the fermionic field which will undergo a change *of the number of particles.*

- therefore the appearence in time at t_i of the gauge field and its extinction at time t_f just injects all the content of winding number to the fermion field. The fermion field benefits of this winding number, takes it and transform it into a set of NEW particles.

- the number of new fermions that are generated between t_i and t_f is given by the gauge field.

- the *sphaleron* is known and its expression is given, for the field A_k. It consists of two factors. The first factor is the time-variable factor

$$c(t)$$

and it goes to zero at both ends: the gauge field must disappear. The second factor is a space construction whose structure gives the winding number.

$$A_k(t,\mathbf{x}) = c(t)\frac{N_k(\mathbf{x})}{1+r^2}$$

$$B_k(t,\mathbf{x}) = \varepsilon_{kij}\partial_i A_j = 4c(t)\frac{N_k(\mathbf{x})}{(1+r^2)^2}$$

The Chern-Simons number of this gauge field is

$$\frac{1}{4\pi^2}\int d^3x \ A_k(t,\mathbf{x}) B_k(t,\mathbf{x}) = \frac{1}{4}c^2(t)$$

This is the total amount of *helicity* in the volume.

For certain values

$$c = 1+2K \ , \ K = 1,2,3,...$$

the Dirac equation has K zero modes.

Consider the relaxation of the gauge configuration from an initial value of

$$c(t_i) = 1+2K$$

with K very large to a small final value. Then the number of particles that are created is equal to the variation of the Chern-Simons number

$$\begin{aligned} N &= (CS)^{fin} - (CS)^{ini} = \frac{c^2(t_f)}{4} - \frac{c^2(t_i)}{4} \\ &= K(K+1) + \frac{1}{4}\left(1-c^2(t_f)\right) \end{aligned}$$

About the sphaleron shape.

Take

$$x_3 = \text{const}$$

We then have the vectors in the plane x_3 =const.

We calculate

$$\begin{aligned} & [N_x(x,y)]^2 + [N_y(x,y)]^2 \\ = & \ (2xz-2y)^2 + (2yz+2x)^2 \\ = & \ 4x^2z^2 + 4y^2 - 8xyz + 4y^2z^2 + 4x^2 + 8xyz \\ = & \ 4z^2\left(x^2+y^2\right) + 4\left(x^2+y^2\right) \\ = & \ \left(x^2+y^2\right)\left(4z^2+4\right) \end{aligned}$$

25 Anyion statistics: Chern Simons, Linking, Hopf mapping

The paper **Tze, I.J. Mod Phys A3 (1988) 1959 manifold splitting regularization, self-linking, twisting, writhing**

It is a detailed comment on the work of **Polyakov** where the calculation of the average of a Wilson exponential, with CS action, leads to something similar to the Gauss linking integral but for a *UNIQUE* path in space time. Then one has to apply regularization method of calculation, as Calugareanu did.

The effect is that the Self Linking which is an integer, appears to be composed of two terms

write

twist

and they can have continuous values, NOT integers. The statistics will then change from 1/2 spin to *ANY* statistics.

The θ angle.

Dynamics of anti-ferromagnetic magnons in $D=3$,

$$S = \frac{1}{\gamma_0} \int d^3x \left[\sum_{k=1,2} |D_\mu Z_k|^2 + \frac{\theta}{16\pi^2} \varepsilon_{\mu\nu\rho} A^\mu \left(\partial^\nu A^\rho\right) \right]$$

where

$$\begin{aligned} Z &= \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \\ |Z|^2 &= 1 \\ Z &\in S^3 \end{aligned}$$

These variables are connected with $\mathbf{n}$ of a $O(3)$ model

$$\begin{aligned} \mathbf{n} &= Z^\dagger \boldsymbol{\sigma} Z \\ &\in S^2 \end{aligned}$$

$$D_\mu = \frac{\partial}{\partial x^\mu} - iA_\mu$$

The interpretations

- the model is a nonlinear chiral

$$CP^1 \sim SU(2)/U(1)$$

with an additional topological term, the Hopf invariant

the Hopf invariant

=

the gauge Chern-Simons term

$$A_\mu = iZ^\dagger \frac{\partial Z}{\partial x^\mu} \qquad \text{(def)}$$
classically nonpropagating

$$\begin{aligned} J^\mu &= -\frac{i}{16\pi^2}\varepsilon^{\mu\nu\lambda}\left(D_\nu Z\right)^\dagger\left(D_\lambda Z\right) \\ &= \text{conserved topological current} \end{aligned}$$

and the interaction is

$$\frac{\theta}{16\pi^2}\varepsilon_{\mu\nu\rho}A^\mu\partial^\nu A^\rho = \frac{\theta}{16\pi^2}A_\mu J^\mu$$

- the second interpretation, is connected with the

  T_c superconductivity

  - at small distances and high momenta,

    $$\begin{aligned} Z &\equiv \text{spin } 0 \\ A_\mu &\equiv \text{spin } 1 \end{aligned}$$

    are independent. Like in asymptotic freedom.
  - at large distances small momenta

    $$A_\mu = iZ^\dagger\frac{\partial}{\partial x^\mu}Z$$

    a condition that is obtained imposing that at large distances the covariant derivative is zero, equivalent with the boundary condition that changes $\mathbf{R}^3$ into S^3.

In the second interpretation we have a Heisenberg anti-ferromagnet, where the angle θ must be determined by some underlying physical *fermion* theory

Study of the long range interaction between the z quanta, mediated by the gauge field with <u>nontrivial</u> <u>topology</u>

The partition function for the Z quanta

$$\begin{aligned} \mathcal{Z} &= \sum_{\{Path\}}^{all\ closed\ paths} \exp\left[-mL\left(path\right)\right]\left\langle \exp\left(i\int_{path} dx^\mu A_\mu\right)\right\rangle \\ L\left(path\right) &\equiv \text{length of the path} \\ m &\equiv \text{mass of the } z \text{ quanta} \end{aligned}$$

$$L\left(path\right) \rightarrow \text{a curve in the Euclidean } R^3$$

The average $\langle\rangle$ is done with the measure of long range A_μ action, which is Chern Simons

$$\begin{aligned} & i\int_x^y dx_\mu A^\mu(x) \\ = \quad & i\int d^3x\ \eta_\mu(x)\ \ A^\mu(x) \end{aligned}$$

Note that first we replace the line integration of the gauge vector field $A^\mu(x)$ with a *volume* integration, of the projection of $A^\mu(x)$ on the field $\eta_\mu(x)$.

Then

$$\begin{aligned} & \left\langle \exp\left(i\int_{path} dx^\mu A_\mu\right)\right\rangle \\ & \text{average with action Chern Simons} \\ = \quad & \exp\left\{-\frac{1}{2}\int_{path}\int_{path} \eta^\mu(x)\,D_{\mu\nu}(x-y)\,\eta^\nu(y)\ \ d^3x d^3y\right\} \\ = \quad & \exp\left\{-2\pi i\left[\frac{1}{4\pi}\oint_{path}\oint_{path} dx^\mu dy^\nu\ \varepsilon_{\mu\nu\lambda}\frac{\left(x^\lambda-y^\lambda\right)}{|x-y|^3}\right]\right\} \\ = \quad & \exp\{2\pi i\ I(path)\} \end{aligned}$$

Here

$$\begin{aligned} D_{\mu\nu} \quad &= \quad i\varepsilon_{\mu\nu\lambda}\frac{\left(x^\lambda-y^\lambda\right)}{|x-y|^3} \\ &\equiv \quad \text{propagator of } A^\mu \text{ field} \\ & \quad\quad \text{governed by Chern-Simons action} \end{aligned}$$

The integral

$$I(path)$$

is the Gauss integral but performed over the same space curve.

Average of a space-time loop, Wilson.
Remember

$$\begin{aligned} & \mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A} \\ \iint \mathbf{dS}\cdot\mathbf{B} \quad &= \quad \iint \mathbf{dS}\cdot(\nabla\times\mathbf{A}) \\ &= \quad \oint \mathbf{dl}\cdot\mathbf{A} \end{aligned}$$

then to integrate $\mathbf{A}$ along a closed contour means to integrate the magnetic field $\mathbf{B}$ flux through the surface bounded by the contour. We must make the average of the exponential of the magnetic flux,once the *path* has been chosen, adopting a measure in the statistical ensemble.

The magnetic flux $\varepsilon^{\mu\nu}\partial_\mu A_\nu$ is a Chern Simons *simplified* by the low dimensionality.

Then we actually make the average of the exponential of a topological quantity.

The Biot Savart inversion of the *rotational* gives the propagator.

26 Dynamic transition of skyrmion driven by current

The skyrmion is topological with degree

$$N_{sk} = \frac{1}{4\pi}\iint dx\ dy\ \mathbf{n}\cdot\left(\frac{\partial \mathbf{n}}{\partial x} \times \frac{\partial \mathbf{n}}{\partial y}\right)$$

where

$$\begin{aligned} \mathbf{n} &= \frac{\mathbf{M}}{M} \\ \mathbf{M} &\equiv \text{magnetization} \end{aligned}$$

The skyrmions can be in motion under the action of low currents.

The skyrmions can be deformed but they keep the topological degree. In this way they avoid obstacles like impurities. Other formations: helices and domain walls *cannot* avoid impurities with preservation of their topology.

The mechanism of motion of skyrmions under excitation by electric field:
spin-transfer torque
Papers
- current_driven_skyrmion_in_magnets
- colossal_spin_transfer_torque_edge_skyrmion
- chirality-magnetism-and-magnetoelectricity-2021.pdf
- univ_current_velocity_skyrmion_magnetic

according to the Thiele equation

$$\begin{aligned} & \mathbf{G}\times(\mathbf{v}_s - \mathbf{v}_d) \\ +\mathcal{D} \quad : \quad & (\beta\ \mathbf{v}_s - \alpha\ \mathbf{v}_d) \\ & +\mathbf{F}_{pin} \\ = \quad & 0 \end{aligned}$$

where

$$\begin{aligned} v_d &= \sqrt{v_x^2 + v_y^2} \\ &\equiv \text{skyrmion drift velocity} \end{aligned}$$

$$\mathbf{v}_s \equiv \text{ conduction electron velocity}$$

$$\begin{aligned} & \text{Magnus vector} \\ \mathbf{G} \quad = \quad & (0, 0, 4\pi N_{sx}) \end{aligned}$$

then

$$\mathbf{G}\times(\mathbf{v}_s - \mathbf{v}_d) = \text{Magnus force}$$

And

$$\alpha \equiv \text{ Gilbert damping factor}$$

$$\beta \equiv \text{ nonadiabatic coefficient}$$

then

$$\mathcal{D} : (\beta\ \mathbf{v}_s - \alpha\ \mathbf{v}_d) \equiv \text{ dissipative term}$$

And

$$\begin{aligned} \mathbf{F}_{pin} \quad = \quad & \text{pinning force} \\ & \text{due to impurities} \end{aligned}$$

Neel type skyrmionic bubbles are micrometric. Large currents are necessary to overcome the pinning force that keep them imobile.

Nanometric skyrmions require far less current density to move.