# The asymptotic quasi-stationary states of the two-dimensional magnetically confined plasma and of the planetary atmosphere

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**Abstract**

We derive the differential equation governing the asymptotic quasi-stationary states of the two dimensional plasma immersed in a strong confining magnetic field and of the planetary atmosphere. These two systems are related by the property that there is an intrinsic constant length: the Larmor radius and respectively the Rossby radius and a condensate of the vorticity field in the unperturbed state related to the cyclotronic gyration and respectively to the Coriolis frequency. Although the closest physical model is the Charney-Hasegawa-Mima (CHM) equation, our model is more general and is related to the system consisting of a discrete set of point-like vortices interacting in plane by a short range potential. A field-theoretical formalism is developed for describing the continuous version of this system. The action functional can be written in the Bogomolnyi form (emphasizing the role of Self-Duality of the asymptotic states) but the minimum energy is no more topological and the asymptotic structures appear to be non-stationary, which is a major difference with respect to traditional topological vortex solutions. Versions of this field theory are discussed and we find arguments in favor of a particular form of the equation. We comment upon the significant difference between the CHM fluid/plasma and the Euler fluid and respectively the Abelian-Higgs vortex models.

# Contents

# 1 Introduction

The instabilities of a plasma embedded in a confining magnetic field (*e.g.* experimental fusion devices, like tokamak) evolve in a geometry that is strongly anisotropic. The motion of the electrons along the magnetic field lines is in many cases sufficiently fast to produce a density perturbation that has the Boltzmann distribution in the electric potential. In these cases a two dimensional approximation (the dynamical equations are written in the plane that is transversal to the magnetic field) may be satisfactory. The Boltzmann electron distribution suppresses the convective nonlinearity, leaving the ion polarization drift as the essential nonlinearity. This is of high differential degree (therefore it is enhanced at small spatial scales) and basically describes the advection of the fluctuating vorticity by the velocity fluctuations. The differential equation for the electrostatic potential has been derived by Hasegawa and Mima [1]. A similar situation appears in the physics of the atmosphere and the differential equation for the streamfunction of the velocity field has been derived by Charney [2].

Many plasma instabilities (in particular in tokamak) depend on this nonlinear term and will likely show some common aspects. In numerical simulations of the Charney-Hasegawa-Mima (CHM) equation it has been proved that the plasma is evolving at large times (in the absence of external driving forces and starting from an irregular flow pattern) to states that are characterised by a regular form of the potential. The evolution toward a very regular pattern of vortical flow is also characteristic to the incompressible ideal fluid, described by the Euler equation [3], [4], [5], [6], [7], [8], [9]. In this case the time-asymptotic states consists of few vortices, of regular shape, with very slow motion. The streamfunction obeys, in these states, the *sinh*-Poisson equation. In the case of CHM equation, the cuasi-stationary states also consist of structures but it has not been possible to derive an equation for the streamfunction [10]. There are seveal studies of the CHM (or very similar) equations showing that at large time the flow is strongly organized and dominated by structures [11], [12], [13], [14] (and references therein). At the oposite limit the turbulent regime can be treated with renormalization group methods [15].

The formulation of the problem exclusively in terms of experimentally-accessible quantities, velocity and vorticity seems to not allow too much freedom in elaborating a theoretical model from which the stationary states to be determined. In all our considerations we will be guided by the analogous experience in the case of the Euler fluid. In that case the existence of a parallel formulation (although the mathematical equivalence is still not fully proven) has been decissive. That model consists of the discrete set of point-like vortices evolving in plane due to a potential given as the natural logarithm of the relative distance. Comparing with the differential equation of the Euler fluid this alternative formulation provides something fundamentally new: it splits the dynamics into two objects of distinct nature: point-like vortices and potential, or, in other words, matter and field carrying the interaction. Going along this model we are led to consider the standard treatment in these cases. We must assume that each of these objects can evolve freely (*i.e.* independent of the other) and in addition there is an interaction between them. This formulation is standard in electrodynamics (classical or quantum). Going to continuum, the discrete set of point-like vortices becomes a field of "matter" (which must be assumed in general complex), the potential will be a free field (called "gauge", similar with the free electromagnetic field) and, in addition, there is the interaction between the matter and the gauge fields (similar to the classical $j_\mu A_\mu$ charge-field interaction). Basic properties of the original, *i.e.* physical, model impose constraints to this field-theoretical formulation. The presence of vorticity requires a particular form of the potential between discrete vortices: it is the *curl* of a sum of natural logarithms. This can only be derived from a gauge-field Lagrangean density of Chern-Simons type, instead of Maxwell type. Therefore the field-theoretical model will contain the Chern-Simons Lagrangean. The matter field has a nonlinear self-interaction that reflects the stationary structure of the free matter field. The coupling of the two fields (the interaction) is minimal, via the covariant derivatives. In a previous work we have formulated this model and have obtained, in this way, a purely analytic derivation of the *sinh*-Poisson equation for the Euler fluid [16].

This paper will develop a model for the Charney-Hasegawa-Mima equation along the same lines. However several differences will impose new features of the model.

In a previous work [17] we have investigated a model based on the similarity with the superfluid field theory, the Abelian-Higgs model. This is able to describe *positive* fluid vortices. We have also proposed, without details, a more extended model, which seemed able to describe the physical vortices of plasma and atmosphere within the regime of the CHM equation.

The full development of this model is the main objective of the present work.

## 2 The physical problem and the Charney-Hasegawa-Mima equation

The analytical model we develop in the present work is intended to describe a system characterized by the following elements:

1. the existence of an intrinsic length; this is the Larmor radius $\rho_s$ for the two-dimensional plasma immersed in a strong, confining, transversal magnetic field; and the Rossby radius for the two-dimensional quasi-geostrophical approximation of the planetary atmosphere;

2. the existence of a condensate of vorticity as a background in the system, in the absence of any perturbation. This background consists of the cyclotronic gyration of ions, for the plasma; and of the Coriolis effect resulting from the planetary rotation, for the atmosphere.

These basic elements are very general and other systems belong to the class that is defined by them. In particular the non-neutral plasma produced in laboratory experiments and the vortices produced on a fluid in a rotating tank (see Schecter, Nezlin, Hopfinger and Van Heijst, etc.)

To make the discussion more specific we will refer in the following to the physical model developed for the two-dimensional plasma and atmosphere, which leads to the equation of Charney-Hasegawa-Mima (CHM):

$$\left(1-\nabla_{\perp}^{2}\right)\frac{\partial\phi}{\partial t}-\kappa\frac{\partial\phi}{\partial y}-\left[\left(-\nabla_{\perp}\phi\times\widehat{\mathbf{n}}\right)\cdot\nabla_{\perp}\right]\nabla_{\perp}^{2}\phi=0 \tag{1}$$

where $\kappa\widehat{\mathbf{e}}_y=-\widehat{\mathbf{n}}\times\nabla_{\perp}\ln n_0$. For simplicity we will refer to the problem of plasma physics, the adaptation to the problem in the physics of atmosphere being easily done (see [14]). The quantities appearing in the Eq.(1) are the *physical* ones after having been normalized :

$$\begin{aligned}\phi &= \frac{|e|\,\phi^{phys}}{T_e} \\ (x,y) &= \left(x^{phys}/\rho_s, y^{phys}/\rho_s\right) \\ t &= t^{phys}\Omega_{ci}\end{aligned} \tag{2}$$

where $\Omega_{ci} = |e| B/m_i$, $\rho_s = c_s/\Omega_{ci}$, $c_s^2 = T_e/m_i$. The derivation of the equation, in the drift instability in tokamaks is done in the Appendix A.

For comparison, the Euler equation is

$$\begin{aligned} \frac{d\omega}{dt} &= 0 \text{ or} \\ \frac{\partial}{\partial t}\left(\nabla_\perp^2 \phi\right) + \left[\left(-\nabla_\perp \phi \times \widehat{\mathbf{n}}\right) \cdot \nabla_\perp\right] \nabla_\perp^2 \phi &= 0 \end{aligned} \tag{3}$$

where $\phi$ is the *streamfunction*.

The similarity between the two equation, Eq.(1) and Eq.(3) is apparent. However, with regard to the form and properties of the stationary states of the two equations we should not attempt of simply taking any previous conclusion derived from the Euler fluid context into the CHM context. This is because the naive stationarity ($\partial/\partial t = 0$) imposed in the two equations leads to an equation with a vast degree of generality and does not provide, by itself, a clear identification of the final vortex shapes. We may represent the family of all possible solutions of the naive stationary limit as a subset in a function space. In the time evolution the two equations produces two series of configurations representing functions belonging to two distinct paths, ending in this set at distinct points (*i.e.* configurations).

Compared with the equation for the ideal fluid (Euler equation) the CHM eq. is not scale invariant [18]. To see this we make the rescaling of the space variables

$$(x, y) \rightarrow (x', y') = (\lambda x, \lambda y)$$

Expressing the Euler equation in the new variables, we have

$$\frac{\partial}{\partial t}\left(\nabla_\perp'^2 \phi\right) + \lambda^2 \left[\left(-\nabla_\perp' \phi \times \widehat{\mathbf{n}}\right) \cdot \nabla_\perp'\right] \nabla_\perp'^2 \phi = 0$$

The factor $\lambda^2$ can be absorbed in a rescaling of the time variable and the equation preserves its form. By contrast the equation CHM becomes (for simplicity we take $\kappa = 0$)

$$\left(1 - \lambda^2 \nabla_\perp'^2\right) \frac{\partial \phi}{\partial t} - \lambda^4 \left[\left(-\nabla_\perp' \phi \times \widehat{\mathbf{n}}\right) \cdot \nabla_\perp'\right] \nabla_\perp'^2 \phi = 0$$

While the factor $\lambda^4$ can be absorbed by time rescaling, the factor $\lambda^2$ in the first paranthesis cannot be absorbed. The form of the equation is invariant only for $\lambda = 1$ which means the space is measured in units of $\rho_s$. The equation CHM exhibits an intrinsic spatial scale, which is $\rho_s$.

## 3 An equivalent discrete model

In the case of the Euler fluid, there is an equivalent model whose dynamical evolution is considered to be identical with that of the physical system (for a list of references, see [16]). It consists of a collection of discrete point-like vortices in two-dimensions evolving from mutual interaction defined in terms of a potential. The potential is the natural logarithm of the relative distance of two vortices. This model has been proposed and used long ago (Kirchoff, Onsager, etc.) but the rigurous proof of the equivalence between it and the physical Euler description is a difficult mathematical problem [19].

For the CHM equation there is a similar model: a collection of point-like vortices interacting by a potential that has a short range. This model has been proposed in meteorology by Morikawa [20] and Stewart [21] (see Horton and Hasegawa [14]). For a set of $N$ vortices with strength $\omega_j$, $j = 1, N$ with instantaneous positions $\mathbf{r}_j$ the streamfunction $\psi(x, y)$ has the following expression

$$\psi(\mathbf{r}) = \sum_j \psi_j(\mathbf{r}) = \sum_j \omega_j K_0(m|\mathbf{r} - \mathbf{r}_j|) \tag{4}$$

where $m$ is a constant. The differential equation from which the contributions to the streamfunction $\psi$ in Eq.(4) are derived, is

$$\left(\Delta - m^2\right)\psi_j(\mathbf{r}) = -2\pi\omega_j\delta(\mathbf{r} - \mathbf{r}_j) \tag{5}$$

in two dimensions. For a single vortex of strength $\omega$ placed in the origin, the azimuthal component of the velocity can be derived from the streamfunction $\psi(\mathbf{r})$

$$v_\theta = \frac{\partial\psi}{\partial r} = -\omega m K_1(mr) \tag{6}$$

At small distances

$$v_\theta \sim \frac{1}{r} \text{ for } r \to 0 \tag{7}$$

which is the same as for the Euler case, where $\psi(\mathbf{r})$ is given by the natural logarithm. The streamfunction decays fast at large $r$ since the modified Bessel function of the second kind $K_0$ decays exponentially at large argument

$$\psi \sim \frac{1}{\sqrt{mr}} \exp(-mr) \text{ for } r \to \infty \tag{8}$$

This means that the vortices are shielded. The elementary vortex of the Charney-Hasegawa-Mima equation is localised by $m^{-1}$ and one can associate a finite spatial extension to it, $\rho_s$ in plasma physics, $\rho_g$ in the physics of atmosphere [14].

The equations of motion for the vortex $\omega_k$ at $(x_k, y_k)$ under the effect of the others are [20]

$$\begin{aligned} -2\pi\omega_k \frac{dx_k}{dt} &= \frac{\partial W}{\partial y_k} \\ -2\pi\omega_k \frac{dy_k}{dt} &= -\frac{\partial W}{\partial x_k} \end{aligned} \tag{9}$$

where

$$W = \pi \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} \omega_i \omega_j K_0 \left( m \left| \mathbf{r}_i - \mathbf{r}_j \right| \right) \tag{10}$$

This is the Kirchhoff function for the system of interacting point-like vortices in plane. It is the Hamiltonian for the system of $N$ vortices. If we introduce the versor of the normal to the plane, $\widehat{\mathbf{n}}$, the equations can be expressed

$$\frac{d\mathbf{r}}{dt} = -\nabla\psi \times \widehat{\mathbf{n}} \tag{11}$$

We will develop a formalism for this discrete system which in the continuum becomes a field theory.

# 4 The main components and the main steps of construction of the continuum model

We will develop a continuum model whose equations of motion could reproduce, in the discrete approximation, the Eqs.(4), (9) and the energy (10). The model must be defined in terms of a Lagrangean density for two interacting fields:

- the field associated with the density of point-like vortices $\phi(x, y)$; we will call it the *matter* (or *scalar*, or *Higgs*) field; and
- the field associated with the potential carrying the interaction between the vortices; we will call it the *gauge* field.

## 4.1 The gauge field

We note that the interaction potential $K_0(|\mathbf{r} - \mathbf{r}_j|)$ (assume that $m$ is normalised as $m = 1$) appearing in the discrete model proposed in meteorology is similar to the potential appearing in the Euler problem, $\ln(|\mathbf{r} - \mathbf{r}_j|)$ in the following sense: they both have topological properties, in sense to be explained. For the Euler equation, the potential can be represented using the angle made by the line connecting the reference ($\mathbf{r}_j$) and the current ($\mathbf{r}$) points with a fixed line, and in order to remove the multivaluedness one has to make a cut in the plane from the center (where is singular) to infinity [22].

$$\begin{aligned} \text{Euler fluid} \quad : \quad \frac{dr_i^\alpha}{dt} &= \left(-\nabla\phi \times \widehat{\mathbf{n}}\right)^\alpha \\ &= \varepsilon^{\alpha\beta} \sum_{j\neq i}^{N} \omega_j \frac{r^\beta - r_j^\beta}{\left|\mathbf{r}-\mathbf{r}_j\right|^2} \end{aligned} \tag{12}$$

(Here $i$, $j$ label the point-like vortices and $\alpha$ , $\beta$ label the coordinates of the position vectors, $r_i^\alpha$, $\alpha = 1, 2$). Since

$$\varepsilon^{\alpha\beta} \frac{r^\beta}{r^2} = \varepsilon^{\alpha\beta} \partial_\beta \ln r \tag{13}$$

we see that the potential in Eq.(12) is expressed through the Green function of the $2D$ Laplacian, defined by the equation

$$\nabla^2 \ln r = 2\pi \delta^2 \left(r\right) \tag{14}$$

The potential is obtained by applying the rotational operator $\varepsilon^{\alpha\beta}\partial_\beta$ on this Green function.

The CHM case is similar, with the difference that the $K_0\left(mr\right)$ is the Green function of the Helmholtz operator, as results from Eq.(5). The series representation of the function $K_0$ is

$$K_0\left(z\right) = -I_0\left(z\right) \ln \frac{z}{2} + \sum_{k=0}^{\infty} \frac{z^{2k}}{2^{2k}\left(k!\right)^2} \psi\left(k+1\right)$$

we see that close to the origin the two potentials are similar

$$K_0\left(r \rightarrow 0\right) = -\ln \frac{r}{2} + ... \tag{15}$$

We note that the potential in the Euler fluid case may be presented as a singular *pure gauge* [22]:

$$\begin{aligned} \frac{1}{2\pi} \varepsilon^{\alpha\beta} \frac{r^\beta}{r^2} &= -\frac{1}{2\pi} \frac{\partial}{\partial r^\alpha} \arctan \frac{y}{x} \\ &= -\frac{1}{2\pi} \frac{\partial}{\partial r^\alpha} \theta \end{aligned} \tag{16}$$

The individual contributions to the potential are the derivatives of the angle $\theta$ made by the particle position vector with an arbitrary fixed direction in plane. In the case of the CHM equation we have from Eqs.(4), (9), (5)

$$\begin{aligned}\text{CHM plasma} \quad : \quad & \frac{dr_i^\alpha}{dt} = \left(-\nabla\phi \times \widehat{\mathbf{n}}\right)^\alpha \\ = \quad & \varepsilon^{\alpha\beta} \sum_{j \neq i}^{N} \omega_j \frac{r^\beta - r_j^\beta}{\left|\mathbf{r} - \mathbf{r}_j\right|^2} \left[m \left|\mathbf{r} - \mathbf{r}_j\right| K_1 \left(m \left|\mathbf{r} - \mathbf{r}_j\right|\right)\right]\end{aligned} \tag{17}$$

Using the small argument expansion (see formula 8.446 in [23] or, alternatively, the Eq.(15))

$$m \left|\mathbf{r} - \mathbf{r}_j\right| K_1 \left(m \left|\mathbf{r} - \mathbf{r}_j\right|\right) \to 1 \text{ for } \left|\mathbf{r} - \mathbf{r}_j\right| \to 0$$

we note that Eq.(17) can be written like Eq.(16). The function in the right paranthesis only changes the spatial decay.

The angle $\theta(x, y)$ is a (multivalued) scalar function and we have in both cases a typical situation of the type

$$\left(-\nabla\phi \times \widehat{\mathbf{n}}\right)^\alpha \sim \partial_\alpha \theta \tag{18}$$

so that the potential can be considered at large distances a pure gauge

$$\left(-\nabla\phi \times \widehat{\mathbf{n}}\right)^\alpha \sim g^{-1} dg \tag{19}$$

with $g \in U(1)$ *i.e.* $g = \exp(i\theta)$. In other words, for every point $(x, y)$ on a large circle on the plane, we have a value of the angle $\theta$. These potentials have therefore a *topological* nature, since they map the circle at infinity in $2D$, $(r \to \infty)$ onto the set of values of the angle $\theta$, which is also a circle. This is a typical homotopic classification of states and the potentials $\phi$ are classified into distinct sets characterised by an integer, representing how many times the circle in the plane is covered by the circle representing the values of $\theta$.

The main difference between the Euler and the CHM cases is the short range of the potential in the latter case. If we use formally the concept of photon (a "particle" that mediates the gauge interaction as in electrodynamics), one can say that in the Euler case we have the usual (two dimensional) *massless photon*, whereas in the CHM case we have a *massive photon*. The fact that the photon is massive is another way to express the fast spatial decay of the potential function, *i.e.* the short range and we will often use this formulation. From physical reasons we know that this short spacial range must be of the order of $\rho_s$, the intrinsic length in the CHM equation. The need that the equations of motion lead to a short-range potential (finite-mass photon) and that the potential has a topological nature represent constraints for the part of the Lagrangean density coming from the gauge field.

As we have shown in the case of Euler fluid [16], the *topological nature* of the potential is provided by the Chern-Simons (CS) Lagrangean

$$\mathcal{L}_{CS} = \frac{\kappa}{2} \varepsilon^{\alpha\beta\gamma} A_\alpha \partial_\beta A_\gamma \tag{20}$$

where $\varepsilon^{\alpha\beta\gamma}$ is the totally antisymmetric tensor in $2+1$ dimensions ($\alpha$, $\beta$ and $\gamma$ can take three values: $0, 1, 2$, corresponding to the time and the two coordinates $(x, y)$) and $\kappa$ is a constant. This Lagrangean is essentially the density of "magnetic" helicity. It is known that this Lagrangean does not lead by itself to dynamical equations for the potential $A_\mu$ since it is first order in the time derivatives; it only represents a constraint on the dynamics, analogous to the Lorentz force in an external magnetic field given by the combination of $\kappa$ with the other constants of the model. We can heuristically say that the Chern-Simons Lagrangean induce vortical effects on the dynamics. This will become more clear later.

The gauge field dynamics can be introduced either by coupling the Chern-Simons potential with the matter field, or by including the Maxwell Lagrangean density

$$\mathcal{L}_M = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{21}$$

or both. We note that any of these combinations provide a finite mass for the photon but the way they do that is different.

For the combination Maxwell and Chern-Simons, the vortical effect of the Chern-Simons part induces decay of the field on structures of small spatial scale, with an extension governed by the "external magnetic field", $\kappa$. This corresponds to the gyration motion.

For the combination Maxwell and matter field the generation of the photon mass is due to a classical Higgs mechanism. The matter field has a nonlinear self-interaction which vanishes at certain non-zero values of the field. Therefore the extremum of the action implies the minimum of this self-interaction potential and the matter field will take one of these values (called: the vacuum value) at infinity. This is the symmetry breaking leading to the Higgs mechanism. The motion of the photon in a polarisable medium consisting of this background matter density induces a finite mass effect for the photon. Then the value of the mass for the photon (the short range of the spatial decay) is determined by this vacuum value of the matter field and by the coefficient of the Maxwell contribution in the Lagrangean (the electric charge).

The combination Maxwell, Chern-Simons and matter has therefore two possible ways to obtain a finite mass for the photon: the gyration (due to Chern-Simons) and the Higgs mechanism, due to the finite background of matter field corresponding to one of its "vacuum" values. There is a mixing of these two ways and there are two possible masses, or short ranges for the gauge potential. At the limit where the Maxwell term is suppressed from the Lagrangean, the short range of Chern-Simons with matter is recovered.

We argue that including the Maxwell term is not necessary in our model describing the flow governed by the CHM equation. This would only provide for the gauge field an independent dynamical evolution since, even when the matter field is absent, the Maxwell Lagrangean leads to plane waves, *i.e.* a propagating field, without any meaning or justification in our case.

## 4.2 The matter field

The matter field $\phi$ is associated to (without being identical with) the density of point-like vortices in the discrete model. The matter field must be complex since the vorticity carried by any point-like vortex appears as a sort of electrical charge (only complex fields can represent charged particles). The kinematical part of the matter field in the Lagrangean consists as usual in the squared momentum but with the covariant derivatives, to reflect the so-called *minimal coupling* with the gauge field

$$\mathcal{L}_{kin} = -\frac{1}{2} \left( D^{\mu}\phi \right)^{\dagger} \left( D_{\mu}\phi \right) \tag{22}$$

where

$$D_{\mu} = \partial_{\mu} + A_{\mu} \tag{23}$$

In the Hamiltonian formulation of the discrete vortices model for the Euler equation it has been derived an equation connecting the gauge field with the "density" of the point-like vortices. Going to the continuum (*i.e.* field-theoretical) version it appeared that the only way to keep this constraint was to assume a self-interaction of the matter field. The same reasons act in our present case, but now the problem is more complicated. The self-interaction potential $V(\phi)$ must have a minimum at a nonzero value of the matter field such as to ensure the background that will induce (together with the CS term) the short-range of the gauge field. Comparing with the Euler case (we neglect the various constant factors)

$$V_{Euler}(\phi) \sim |\phi|^4 \tag{24}$$

the simplest form would be

$$V_{CHM}(\phi) \sim \left( |\phi|^2 - v^2 \right)^2 \tag{25}$$

where $v$ is the vacuum value of the matter field. However, it will be shown below that this form cannot provide, for the Lagrangean density consisting of Chern-Simons and matter part, the most symmetrical extremum of the action functional for the system. This particular state is called self-duality and we adopt the point of view that this is a fundamental requirement on the model. In particular in our previous paper [16] it was shown that the stationary states of the ideal fluid obeying the *sinh*-Poisson equation correspond to the self-duality.

In a different context [26], [27] the form of the self-interaction potential able to support self-duality has been found as

$$V_{CHM}(\phi) \sim |\phi|^2 \left(|\phi|^2 - v^2\right)^2 \tag{26}$$

and we will work with this form.

### 4.3 The necessity to consider "pairing" of the fields

The main physical content of the states governed by the CHM equation is the vorticity, which is organizing in large scale vortical structures. The vorticity and the velocity fields generate the kinematic helicity, which is a topological invariant for a dissipationless fluid/plasma. It has been shown [29] that the helicity is determined from the boundary condition (this becomes evident in the Clebsch representation of the velocity): the values of helicity on the boundary of the volume are sufficient to determine the value at any internal point. We understand that besides the fields defined in the internal points, we need to consider fields that carry the information from the boundary toward the interior, on equal foot with fields that carry informations from the points of the internal volume to the boundary. This *pairing* of functions suggests that all quantities involved in our model will be matrices. The model becomes non-Abelian and the quantities are elements of the algebra of the group $SU(2)$.

A more formal explanation of the necessity to adopt a non-Abelian algebraic structure of the theory results from the consideration of the spinorial nature of the elementary point-like vortices and from the Parity, Charge Conjugation and Time inversion invariances of the theory.

## 5 The field theoretical formalism

The continuum limit of the system of discrete point-like vortices is a field theory. From the discussion of the previous Section, we have the field theoretical model: covariant, $SU(2)$, Chern-Simons for the gauge field and $6^{th}$ order self-interaction for the matter field.

- gauge field, with "potential" $A^\mu$, ($\mu = 0, 1, 2$ for $(t, x, y)$) described by the Chern-Simons Lagrangean;
- matter ("Higgs" or "scalar") field $\phi$ described by the covariant kinematic Lagrangean (*i.e.* covariant derivatives, implementing the minimal coupling of the gauge and matter fields)
- matter-field self-interaction given by a potential $V\left(\phi, \phi^\dagger\right)$ with $6^{th}$ power of $\phi$;
- the matter and gauge fields belong to the *adjoint* representation of the algebra $SU(2)$

The Lagrangean density for such a model has been used in $(2+1)$ field theories and reads

$$\begin{aligned}\mathcal{L} &= -\kappa\varepsilon^{\mu\nu\rho}\mathrm{tr}\left(\partial_\mu A_\nu A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho\right) \\ &\quad -\mathrm{tr}\left[(D^\mu\phi)^\dagger (D_\mu\phi)\right] \\ &\quad -V\left(\phi, \phi^\dagger\right)\end{aligned} \tag{27}$$

What follows is already exposed in field-theoretical literature, in particular in **Dunne** [30], [35], [34]. For the Abelian version, see [28].

The transformations of the space and time variables must be connected through the condition of the general covariance of the theory. The metric of the space-time is

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{28}$$

This means that we have to take account of the covariant and contravariant coordinates of vectors, tensors and operators. We can use both notations $(A_1, A_2)$ and $(A_x, A_y)$ since no confusion is possible. We have

$$\begin{aligned} x^\mu &\equiv (t, x, y) \\ x_\mu &= g_{\mu\nu}x^\nu = (-t, x, y) \end{aligned} \tag{29}$$

$$\begin{aligned} A^\mu &\equiv \left(A^0, A^1, A^2\right) \\ A_\mu &\equiv (A_0, A_1, A_2) = g_{\mu\nu}A^\nu = \left(-A^0, A^1, A^2\right) \end{aligned} \tag{30}$$

the derivation operator is

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \tag{31}$$

and

$$\begin{aligned} \partial^\mu &= g^{\mu\nu}\partial_\nu \\ &= \left(-\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \end{aligned} \tag{32}$$

The covariant derivatives are

$$D_\mu = \partial_\mu + [A_\mu, ] \tag{33}$$

(note that we need not introduce an electric charge, $e$). We write the detailed expression

$$\begin{aligned} D^\mu\phi &= g^{\mu\nu}D_\nu\phi \\ &= g^{\mu\nu}\partial_\nu\phi + g^{\mu\nu}\left[A_\nu, \phi\right] \\ &= \partial^\mu\phi + A^\mu\phi - \phi A^\mu \end{aligned} \tag{34}$$

For comparison we write them in detail (see also Eq.(30) )

$$D_\mu\phi = \left\{\frac{\partial\phi}{\partial t} + A_0\phi - \phi A_0, \frac{\partial\phi}{\partial x} + A_1\phi - \phi A_1, \frac{\partial\phi}{\partial y} + A_2\phi - \phi A_2\right\}$$

$$D^\mu\phi = \left\{-\frac{\partial\phi}{\partial t} + A^0\phi - \phi A^0, \frac{\partial\phi}{\partial x} + A^1\phi - \phi A^1, \frac{\partial\phi}{\partial y} + A^2\phi - \phi A^2\right\} \tag{35}$$

The *Hermitean conjugate* of a matrix is the transpose matrix with complex conjugated entries. For Eq.(34) the Hermitian conjugate is

$$\begin{aligned} \left(D^\mu\phi\right)^\dagger &= \left(\partial^\mu\phi\right)^\dagger + \left[A^\mu, \phi\right]^\dagger \\ &= \partial_\mu\phi^\dagger + \left[\phi^\dagger, A^{\mu\dagger}\right] \\ &= \partial_\mu\phi^\dagger + \phi^\dagger A^{\mu\dagger} - A^{\mu\dagger}\phi^\dagger \end{aligned} \tag{36}$$

or, in detail

$$\begin{aligned} \left(D^\mu\phi\right)^\dagger = & \left\{-\frac{\partial\phi^\dagger}{\partial t} + \phi^\dagger A^{0\dagger} - A^{0\dagger}\phi^\dagger,\right. \\ & +\frac{\partial\phi^\dagger}{\partial x} + \phi^\dagger A^{1\dagger} - A^{1\dagger}\phi^\dagger, \\ & \left. +\frac{\partial\phi^\dagger}{\partial y} + \phi^\dagger A^{2\dagger} - A^{2\dagger}\phi^\dagger\right\} \end{aligned} \tag{37}$$

keeping the following rules ($*$ is complex conjugate and $T$ is transpose)

$$\begin{aligned} A^{\mu\dagger} &= (A_\mu)^{*T} \\ A_\mu^\dagger &= (A^\mu)^{*T} \end{aligned} \tag{38}$$

This means

$$\begin{aligned} A^{0\dagger} &= A^{0*T} = (-A_0)^{*T} \\ A^{k\dagger} &= A^{k*T} = (A_k)^{*T} \ , \ k = 1, 2 \end{aligned} \tag{39}$$

It has been found that the only possiblity this model has to reach self-dual states is to choose a matter field nonlinear self-interaction given by a sixth order potential [26]

$$V\left(\phi, \phi^\dagger\right) = \frac{1}{4\kappa^2} \mathrm{tr}\left[\left(\left[\left[\phi, \phi^\dagger\right], \phi\right] - v^2\phi\right)^\dagger \left(\left[\left[\phi, \phi^\dagger\right], \phi\right] - v^2\phi\right)\right]. \tag{40}$$

The trace is taken in a finite dimensional representation of the compact simple Lie algebra $\mathcal{G}$ to which the gauge field $A_\mu$ and the charged matter field $\phi$ and $\phi^\dagger$ belong.

The Euler Lagrange equations are

$$D_\mu D^\mu \phi = \frac{\partial V}{\partial \phi^\dagger} \tag{41}$$

$$-\kappa \varepsilon^{\nu\mu\rho} F_{\mu\rho} = i J^\nu \tag{42}$$

where the field strength is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \tag{43}$$

(We may note that the second equation of motion shows a typical Chern-Simons' action property, *i.e.* the proportionality between the magnetic field and the current density, which in physical systems would be called **force-free**).

The current is

$$J^\mu = -i\left(\left[\phi^\dagger, D^\mu\phi\right] - \left[\left(D^\mu\phi\right)^\dagger, \phi\right]\right) \tag{44}$$

with the conservation (covariant)

$$D^\mu J_\mu = 0 \tag{45}$$

The Gauss law constraint is the 0-component of the second equation of motion

$$\begin{aligned} -\kappa\left(\varepsilon^{012}F_{12}+\varepsilon^{021}F_{21}\right) &= iJ^0 \\ -2\kappa F_{12} &= iJ^0 \\ &= -iJ_0 \end{aligned} \tag{46}$$

since

$$J^0 = g^{0\mu}J_\mu = g^{00}J_0 = -J_0 \tag{47}$$

Using the Eq.(44) we have the Gauss law

$$2\kappa F_{12} = \left[\phi^\dagger, D_0\phi\right] - \left[\left(D_0\phi\right)^\dagger, \phi\right] \tag{48}$$

in the nonabelian form. The equation written above is the 0-th component ($\varepsilon^{012} = 1$, $\varepsilon^{021} = -1$) of the equation of motions connecting the field tensor with the current of matter, Eq.(42). We identify in the right hand side the magnetic field, since

$$\begin{aligned} F_{\mu\nu} &\equiv \partial_\mu A_\nu - \partial_\nu A_\mu + \left[A_\mu, A_\nu\right] \\ &= \begin{pmatrix} 0 & -E_y & -E_x \\ E_y & 0 & -B \\ E_x & B & 0 \end{pmatrix} \end{aligned} \tag{49}$$

$(\mu, \nu = 0, 1, 2)$ *i.e.*

$$F_{12} = -B \tag{50}$$

The energy density of the system is

$$\begin{aligned} \mathcal{E} &= \mathrm{tr}\left(\left(D_0\phi\right)^\dagger\left(D_0\phi\right)\right) + \\ &\quad +\mathrm{tr}\left(\left(D_k\phi\right)^\dagger\left(D_k\phi\right)\right) \\ &\quad +V\left(\phi,\phi^\dagger\right) \end{aligned} \tag{51}$$

The energy density can be rewritten in the Bogomolnyi form, *i.e.* as a sum of squares plus a quantity that integrated over the plane becomes a lower bound for the energy. The space part of the Lagrangean containing covariant derivatives can be written [37], [35]

$$\mathrm{tr}\left(\left(D_k\phi\right)^\dagger\left(D_k\phi\right)\right) = \mathrm{tr}\left(\left(D_-\phi\right)^\dagger\left(D_-\phi\right)\right) - i\mathrm{tr}\left(\phi^\dagger\left[F_{12},\phi\right]\right) \tag{52}$$

with the notation

$$D_\pm = D_1 \pm iD_2 \tag{53}$$

In the following we will verify the equality

$$-i\mathrm{tr}\left(\phi^\dagger\left[F_{12},\phi\right]\right)=\frac{i}{2\kappa}\mathrm{tr}\left(\left[\left[\phi,\phi^\dagger\right],\phi\right]^\dagger D_0\phi-\left[\left[\phi,\phi^\dagger\right],\phi\right]\left(D_0\phi\right)^\dagger\right) \tag{54}$$

Using Eq.(48) we replace $F_{12}$ and obtain

$$\begin{aligned}
& -i\mathrm{tr}\left(\phi^\dagger\left[F_{12},\phi\right]\right) \\
= & -i\mathrm{tr}\left(\phi^\dagger\left[\frac{1}{2\kappa}\left(\left[\phi^\dagger,D_0\phi\right]-\left[\left(D_0\phi\right)^\dagger,\phi\right]\right),\phi\right]\right) \\
= & -\frac{i}{2\kappa}\mathrm{tr}\left\{\phi^\dagger\left\{\left(\left[\phi^\dagger,D_0\phi\right]-\left[\left(D_0\phi\right)^\dagger,\phi\right]\right)\phi-\phi\left(\left[\phi^\dagger,D_0\phi\right]-\left[\left(D_0\phi\right)^\dagger,\phi\right]\right)\right\}\right\} \\
= & -\frac{i}{2\kappa}\mathrm{tr}\left\{\phi^\dagger\left\{\left[\phi^\dagger,D_0\phi\right]\phi-\left[\left(D_0\phi\right)^\dagger,\phi\right]\phi\right.\right. \\
& \left.\left.-\phi\left[\phi^\dagger,D_0\phi\right]+\phi\left[\left(D_0\phi\right)^\dagger,\phi\right]\right\}\right\} \\
= & -\frac{i}{2\kappa}\mathrm{tr}\left\{\phi^\dagger\left[\phi^\dagger,D_0\phi\right]\phi\right. \\
& -\phi^\dagger\left[\left(D_0\phi\right)^\dagger,\phi\right]\phi \\
& -\phi^\dagger\phi\left[\phi^\dagger,D_0\phi\right] \\
& \left.+\phi^\dagger\phi\left[\left(D_0\phi\right)^\dagger,\phi\right]\right\}
\end{aligned} \tag{55}$$

We expand the commutators in order to collect together the factors of $D_0\phi$ and respectively $(D_0\phi)^\dagger$.

$$\begin{aligned}
& -i\mathrm{tr}\left(\phi^\dagger\left[F_{12},\phi\right]\right) \\
= & -\frac{i}{2\kappa}\mathrm{tr}\left\{\phi^\dagger\left(\phi^\dagger\left(D_0\phi\right)-\left(D_0\phi\right)\phi^\dagger\right)\phi\right. \\
& -\phi^\dagger\left(\left(D_0\phi\right)^\dagger\phi-\phi\left(D_0\phi\right)^\dagger\right)\phi \\
& -\phi^\dagger\phi\left(\phi^\dagger\left(D_0\phi\right)-\left(D_0\phi\right)\phi^\dagger\right) \\
& \left.+\phi^\dagger\phi\left(\left(D_0\phi\right)^\dagger\phi-\phi\left(D_0\phi\right)^\dagger\right)\right\}
\end{aligned} \tag{56}$$

$$
\begin{aligned}
& -i\mathrm{tr}\left(\phi^{\dagger}\left[F_{12},\phi\right]\right) \qquad (57)\\
= & -\frac{i}{2\kappa}\mathrm{tr}\left\{\phi^{\dagger}\phi^{\dagger}\left(D_0\phi\right)\phi\right.\\
& -\phi^{\dagger}\left(D_0\phi\right)\phi^{\dagger}\phi\\
& -\phi^{\dagger}\left(D_0\phi\right)^{\dagger}\phi\phi\\
& +\phi^{\dagger}\phi\left(D_0\phi\right)^{\dagger}\phi\\
& -\phi^{\dagger}\phi\phi^{\dagger}\left(D_0\phi\right)\\
& +\phi^{\dagger}\phi\left(D_0\phi\right)\phi^{\dagger}\\
& +\phi^{\dagger}\phi\left(D_0\phi\right)^{\dagger}\phi\\
& \left.-\phi^{\dagger}\phi\phi\left(D_0\phi\right)^{\dagger}\right\}
\end{aligned}
$$

We take separately the terms containing $D_0\phi$ and use the cyclic symmetry of the Trace operator

$$
\begin{aligned}
& \phi^{\dagger}\phi^{\dagger}\left(D_0\phi\right)\phi \qquad (58)\\
& -\phi^{\dagger}\left(D_0\phi\right)\phi^{\dagger}\phi\\
& -\phi^{\dagger}\phi\phi^{\dagger}\left(D_0\phi\right)\\
& +\phi^{\dagger}\phi\left(D_0\phi\right)\phi^{\dagger}\\
\rightarrow & \ \phi\phi^{\dagger}\phi^{\dagger}\left(D_0\phi\right)\\
& -\phi^{\dagger}\phi\phi^{\dagger}\left(D_0\phi\right)\\
& -\phi^{\dagger}\phi\phi^{\dagger}\left(D_0\phi\right)\\
& +\phi^{\dagger}\phi^{\dagger}\phi\left(D_0\phi\right)\\
= & \left\{\left(\phi\phi^{\dagger}-\phi^{\dagger}\phi\right)\phi^{\dagger}-\phi^{\dagger}\left(\phi\phi^{\dagger}-\phi^{\dagger}\phi\right)\right\}\left(D_0\phi\right)\\
= & \left[\left[\phi,\phi^{\dagger}\right],\phi^{\dagger}\right]\left(D_0\phi\right)
\end{aligned}
$$

and analogously for the factors of $(D_0\phi)^\dagger$.

$$\begin{aligned} & -\phi^\dagger (D_0\phi)^\dagger \phi\phi \\ & +\phi^\dagger\phi (D_0\phi)^\dagger \phi \\ & +\phi^\dagger\phi (D_0\phi)^\dagger \phi \\ & -\phi^\dagger\phi\phi (D_0\phi)^\dagger \\ \rightarrow & -\phi\phi\phi^\dagger (D_0\phi)^\dagger \\ & +\phi\phi^\dagger\phi (D_0\phi)^\dagger \\ & +\phi\phi^\dagger\phi (D_0\phi)^\dagger \\ & -\phi^\dagger\phi\phi (D_0\phi)^\dagger \\ = & \left\{ \left(\phi\phi^\dagger - \phi^\dagger\phi\right)\phi - \phi\left(\phi\phi^\dagger - \phi^\dagger\phi\right)\right\} (D_0\phi)^\dagger \\ & \left\{ \left[\phi, \phi^\dagger\right]\phi - \phi\left[\phi, \phi^\dagger\right]\right\} (D_0\phi)^\dagger \\ = & \left[\left[\phi, \phi^\dagger\right], \phi\right] (D_0\phi)^\dagger \end{aligned} \tag{59}$$

It results

$$\begin{aligned} & -i\mathrm{tr}\left(\phi^\dagger \left[F_{12}, \phi\right]\right) \\ = & -\frac{i}{2\kappa}\mathrm{tr}\left\{ \left[\left[\phi, \phi^\dagger\right], \phi^\dagger\right] (D_0\phi) + \left[\left[\phi, \phi^\dagger\right], \phi\right] (D_0\phi)^\dagger \right\} \\ = & \frac{i}{2\kappa}\mathrm{tr}\left\{ - \left[\left[\phi, \phi^\dagger\right], \phi^\dagger\right] (D_0\phi) - \left[\left[\phi, \phi^\dagger\right], \phi\right] (D_0\phi)^\dagger \right\} \end{aligned} \tag{60}$$

and we will prove that

$$\mathrm{tr}\left\{ - \left[\left[\phi, \phi^\dagger\right], \phi^\dagger\right]\right\} = \mathrm{tr}\left\{ \left[\left[\phi, \phi^\dagger\right], \phi\right]^\dagger \right\} \tag{61}$$

The right hand side is

$$\begin{aligned} & \left[\left[\phi, \phi^\dagger\right], \phi\right]^\dagger \\ = & \left\{ \left(\phi\phi^\dagger - \phi^\dagger\phi\right)\phi - \phi\left(\phi\phi^\dagger - \phi^\dagger\phi\right)\right\}^\dagger \\ = & \left\{ \phi\phi^\dagger\phi - \phi^\dagger\phi\phi - \phi\phi\phi^\dagger + \phi\phi^\dagger\phi \right\}^\dagger \\ = & \phi^\dagger\phi\phi^\dagger - \phi^\dagger\phi^\dagger\phi - \phi\phi^\dagger\phi^\dagger + \phi^\dagger\phi\phi^\dagger \end{aligned} \tag{62}$$

and the left hand side

$$\begin{aligned} & -\left[\left[\phi, \phi^\dagger\right], \phi^\dagger\right] \\ = & -\left\{ \left(\phi\phi^\dagger - \phi^\dagger\phi\right)\phi^\dagger - \phi^\dagger\left(\phi\phi^\dagger - \phi^\dagger\phi\right)\right\} \\ = & -\left\{ \phi\phi^\dagger\phi^\dagger - \phi^\dagger\phi\phi^\dagger - \phi^\dagger\phi\phi^\dagger + \phi^\dagger\phi^\dagger\phi \right\} \\ = & -\phi\phi^\dagger\phi^\dagger + \phi^\dagger\phi\phi^\dagger + \phi^\dagger\phi\phi^\dagger - \phi^\dagger\phi^\dagger\phi \end{aligned} \tag{63}$$

and one can see the identity of the two expressions. Then

$$\begin{aligned} & -i\mathrm{tr}\left(\phi^{\dagger}\left[F_{12},\phi\right]\right) \\ = & \frac{i}{2\kappa}\mathrm{tr}\left\{\left[\left[\phi,\phi^{\dagger}\right],\phi\right]^{\dagger}\left(D_{0}\phi\right)-\left[\left[\phi,\phi^{\dagger}\right],\phi\right]\left(D_{0}\phi\right)^{\dagger}\right\} \end{aligned} \tag{64}$$

This is the expression that is used in Eq.(52). We have

$$\begin{aligned} \mathrm{tr}\left(\left(D_{k}\phi\right)^{\dagger}\left(D_{k}\phi\right)\right) &= \mathrm{tr}\left(\left(D_{-}\phi\right)^{\dagger}\left(D_{-}\phi\right)\right)-i\mathrm{tr}\left(\phi^{\dagger}\left[F_{12},\phi\right]\right) \\ &= \mathrm{tr}\left(\left(D_{-}\phi\right)^{\dagger}\left(D_{-}\phi\right)\right) \\ &\quad +\frac{i}{2\kappa}\mathrm{tr}\left\{\left[\left[\phi,\phi^{\dagger}\right],\phi\right]^{\dagger}\left(D_{0}\phi\right)-\left[\left[\phi,\phi^{\dagger}\right],\phi\right]\left(D_{0}\phi\right)^{\dagger}\right\} \end{aligned} \tag{65}$$

# 6 The term containing the time-derivatives

We remind that the objective of this calculation is to find the configurations for which the energy Eq.(51) is minimum. According to the usual approach to this problem, we will try to reexpress Eq.(51) as a sum of squared terms plus a "residual" term. The squared contributions, being always positive, are minimum when the corresponding expressions are zero, while the additional term in general has a topological meaning. In the particular case of the CHM fluids, the residual energy cannot be associated to a topological quantity, for reasons that will be discussed later. Since we are no more guided by the physical significance of the additional energy as resulting from a topological property of the system we must accept that there is no unique way of separating in Eq.(51) the square terms and the additional term. We present in the following two such formulations and discuss them comparatively.

## 6.1 First mode of separating the squared terms in the energy expression

Now we have to write in detail the term from the Lagrangian containing the zero-th covariant derivatives. This is done by including the constant $v$

representing the asymptotic value of the charged field.

$$\begin{aligned}
& \operatorname{tr}\left((D_0\phi)^\dagger (D_0\phi)\right) \qquad (66)\\
= & \operatorname{tr}\left(\left(D_0\phi - \frac{i}{2\kappa}\left(\left[\left[\phi,\phi^\dagger\right],\phi\right] - v^2\phi\right)\right)^\dagger\right.\\
& \qquad \left.\times\left(D_0\phi - \frac{i}{2\kappa}\left(\left[\left[\phi,\phi^\dagger\right],\phi\right] - v^2\phi\right)\right)\right)\\
& -\frac{i}{2\kappa}\operatorname{tr}\left(\left(\left[\left[\phi,\phi^\dagger\right],\phi\right] - v^2\phi\right)^\dagger D_0\phi - \left(\left[\left[\phi,\phi^\dagger\right],\phi\right] - v^2\phi\right)(D_0\phi)^\dagger\right)\\
& -\frac{1}{4\kappa^2}\operatorname{tr}\left(\left(\left[\left[\phi,\phi^\dagger\right],\phi\right] - v^2\phi\right)^\dagger \left(\left[\left[\phi,\phi^\dagger\right],\phi\right] - v^2\phi\right)\right)
\end{aligned}$$

This expression is obtained after several cicliv translations of the factors, which is allowed under the Trace operator. The last term, when we return to Eq.(51), cancels the potential $V\left(\phi,\phi^\dagger\right)$ given in (40). We now understand that only this choice allows to write the energy in the Bogomolnyi form.

The total energy in the system, Eq.(51), written in Bogomolnyi form, results from the expressions (65) and (66)

$$\begin{aligned}
\mathcal{E} = & \operatorname{tr}\left(\left(D_0\phi - \frac{i}{2\kappa}\left(\left[\left[\phi,\phi^\dagger\right],\phi\right] - v^2\phi\right)\right)^\dagger\right. \qquad (67)\\
& \qquad \left.\times\left(D_0\phi - \frac{i}{2\kappa}\left(\left[\left[\phi,\phi^\dagger\right],\phi\right] - v^2\phi\right)\right)\right)\\
& +\operatorname{tr}\left((D_-\phi)^\dagger (D_-\phi)\right)\\
& +\frac{iv^2}{2\kappa}\operatorname{tr}\left(\phi^\dagger (D_0\phi) - (D_0\phi)^\dagger \phi\right)
\end{aligned}$$

The energy contains a sum of positive quantities (squares) and is minimised by those states where these terms are vanishing. The last term shows that there is a lower bound the energy

$$\mathcal{E} \geqslant \frac{iv^2}{2\kappa}\operatorname{tr}\left(\phi^\dagger (D_0\phi) - (D_0\phi)^\dagger \phi\right) \qquad (68)$$

#### 6.1.1 The first form of the self-duality equations

The vanishing of the squared terms in the energy leads to the *self-duality equations*

$$\begin{aligned}
D_-\phi &= 0 \qquad (69)\\
D_0\phi &= \frac{i}{2\kappa}\left(\left[\left[\phi,\phi^\dagger\right],\phi\right] - v^2\phi\right)
\end{aligned}$$

Combining these two equations such as to put in evidence the gauge field $F_{+-}$, whose expression is in Eq.(C.17), (the calculation is presented in detail in *Appendix C*)

$$
\begin{aligned}
D_{-}\phi &= 0 \qquad (70)\\
F_{+-} &= \frac{1}{\kappa^2}\left[v^2\phi - \left[\left[\phi,\phi^\dagger\right],\phi\right],\phi^\dagger\right]
\end{aligned}
$$

Using Eqs.(69) we can derive a new expression for the energy in the self-dual state

$$
\mathcal{E}_{SD} = \frac{v^2}{2\kappa^2}\mathrm{tr}\left(\phi^\dagger\left(v^2\phi - \left[\left[\phi,\phi^\dagger\right],\phi\right]\right)\right) \qquad (71)
$$

which is the saturated lower bound shown above.

## 6.2 Second mode of separating squared terms in the expression of the energy

We look for an expression of a square term that differs from the previous one by a change of sign within the first two terms of Eq.(66)

$$
\mathrm{tr}\left\{\left(D_0\phi + \frac{i}{2\kappa}\left(\left[\left[\phi,\phi^\dagger\right],\phi\right] - v^2\phi\right)\right)^\dagger\left(D_0\phi + \frac{i}{2\kappa}\left(\left[\left[\phi,\phi^\dagger\right],\phi\right] - v^2\phi\right)\right)\right\} \qquad (72)
$$

$$
\begin{aligned}
&= \mathrm{tr}\left\{\left(\left(D_0\phi\right)^\dagger - \frac{i}{2\kappa}\left(\left[\left[\phi,\phi^\dagger\right],\phi\right] - v^2\phi\right)^\dagger\right)\left(D_0\phi + \frac{i}{2\kappa}\left(\left[\left[\phi,\phi^\dagger\right],\phi\right] - v^2\phi\right)\right)\right\}\\
&= \mathrm{tr}\left\{\left(D_0\phi\right)^\dagger\left(D_0\phi\right)\right.\\
&\quad -\frac{i}{2\kappa}\left(\left[\left[\phi,\phi^\dagger\right],\phi\right] - v^2\phi\right)^\dagger\left(D_0\phi\right)\\
&\quad +\left(D_0\phi\right)^\dagger\frac{i}{2\kappa}\left(\left[\left[\phi,\phi^\dagger\right],\phi\right] - v^2\phi\right)\\
&\quad \left.+\frac{1}{4\kappa^2}\left(\left[\left[\phi,\phi^\dagger\right],\phi\right] - v^2\phi\right)^\dagger\left(\left[\left[\phi,\phi^\dagger\right],\phi\right] - v^2\phi\right)\right\}
\end{aligned}
$$

The two median lines are expanded and the full expression is rewritten

$$
\begin{aligned}
&\mathrm{tr}\left\{\left(D_0\phi+\frac{i}{2\kappa}\left(\left[\left[\phi,\phi^\dagger\right],\phi\right]-v^2\phi\right)\right)^\dagger\left(D_0\phi+\frac{i}{2\kappa}\left(\left[\left[\phi,\phi^\dagger\right],\phi\right]-v^2\phi\right)\right)\right\}\\
&= \mathrm{tr}\left\{\left(D_0\phi\right)^\dagger\left(D_0\phi\right)\right.\\
&\quad -\frac{i}{2\kappa}\left(\left[\left[\phi,\phi^\dagger\right],\phi\right]\right)^\dagger\left(D_0\phi\right)+\frac{i}{2\kappa}v^2\phi^\dagger\left(D_0\phi\right)\\
&\quad +\frac{i}{2\kappa}\left(D_0\phi\right)^\dagger\left(\left[\left[\phi,\phi^\dagger\right],\phi\right]\right)-\frac{i}{2\kappa}\left(D_0\phi\right)^\dagger v^2\phi\\
&\quad \left.+\frac{1}{4\kappa^2}\left(\left[\left[\phi,\phi^\dagger\right],\phi\right]-v^2\phi\right)^\dagger\left(\left[\left[\phi,\phi^\dagger\right],\phi\right]-v^2\phi\right)\right\} \qquad (73)
\end{aligned}
$$

and we can now get an expression for the first contribution to the energy

$$
\begin{aligned}
&\mathrm{tr}\left\{\left(D_0\phi\right)^\dagger\left(D_0\phi\right)\right\} \qquad (74)\\
&= \mathrm{tr}\left\{\left(D_0\phi+\frac{i}{2\kappa}\left(\left[\left[\phi,\phi^\dagger\right],\phi\right]-v^2\phi\right)\right)^\dagger\left(D_0\phi+\frac{i}{2\kappa}\left(\left[\left[\phi,\phi^\dagger\right],\phi\right]-v^2\phi\right)\right)\right\}\\
&\quad -\mathrm{tr}\left\{-\frac{i}{2\kappa}\left(\left[\left[\phi,\phi^\dagger\right],\phi\right]\right)^\dagger\left(D_0\phi\right)+\frac{i}{2\kappa}v^2\phi^\dagger\left(D_0\phi\right)\right.\\
&\quad \left.+\frac{i}{2\kappa}\left(D_0\phi\right)^\dagger\left(\left[\left[\phi,\phi^\dagger\right],\phi\right]\right)-\frac{i}{2\kappa}\left(D_0\phi\right)^\dagger v^2\phi\right\}\\
&\quad -\mathrm{tr}\left\{\frac{1}{4\kappa^2}\left(\left[\left[\phi,\phi^\dagger\right],\phi\right]-v^2\phi\right)^\dagger\left(\left[\left[\phi,\phi^\dagger\right],\phi\right]-v^2\phi\right)\right\}
\end{aligned}
$$

With the two expressions detailed above, we can rewrite the energy

Eq.(51)

$$\begin{aligned}
\mathcal{E} &= \operatorname{tr}\left((D_0\phi)^\dagger (D_0\phi)\right) + \operatorname{tr}\left((D_k\phi)^\dagger (D_k\phi)\right) + V\left(\phi,\phi^\dagger\right) \qquad (75)\\
&= \operatorname{tr}\left\{\left(D_0\phi + \frac{i}{2\kappa}\left(\left[\left[\phi,\phi^\dagger\right],\phi\right] - v^2\phi\right)\right)^\dagger \left(D_0\phi + \frac{i}{2\kappa}\left(\left[\left[\phi,\phi^\dagger\right],\phi\right] - v^2\phi\right)\right)\right\}\\
&\quad -\operatorname{tr}\left\{-\frac{i}{2\kappa}\left(\left[\left[\phi,\phi^\dagger\right],\phi\right]\right)^\dagger (D_0\phi) + \frac{i}{2\kappa} v^2\phi^\dagger (D_0\phi)\right.\\
&\quad \left. + \frac{i}{2\kappa}(D_0\phi)^\dagger\left(\left[\left[\phi,\phi^\dagger\right],\phi\right]\right) - \frac{i}{2\kappa}(D_0\phi)^\dagger v^2\phi\right\}\\
&\quad -\operatorname{tr}\left\{\frac{1}{4\kappa^2}\left(\left[\left[\phi,\phi^\dagger\right],\phi\right] - v^2\phi\right)^\dagger\left(\left[\left[\phi,\phi^\dagger\right],\phi\right] - v^2\phi\right)\right\}\\
&\quad +\operatorname{tr}\left((D_-\phi)^\dagger (D_-\phi)\right) + \frac{i}{2\kappa}\operatorname{tr}\left\{\left[\left[\phi,\phi^\dagger\right],\phi\right]^\dagger (D_0\phi) - \left[\left[\phi,\phi^\dagger\right],\phi\right](D_0\phi)^\dagger\right\}\\
&\quad +\frac{1}{4\kappa^2}\operatorname{tr}\left\{\left(\left[\left[\phi,\phi^\dagger\right],\phi\right] - v^2\phi\right)^\dagger\left(\left[\left[\phi,\phi^\dagger\right],\phi\right] - v^2\phi\right)\right\}
\end{aligned}$$

We note the cancellation of the term which is the potential $V\left(\phi^\dagger,\phi\right)$ and we obtain

$$\begin{aligned}
\mathcal{E} &= \operatorname{tr}\left\{\left(D_0\phi + \frac{i}{2\kappa}\left(\left[\left[\phi,\phi^\dagger\right],\phi\right] - v^2\phi\right)\right)^\dagger \left(D_0\phi + \frac{i}{2\kappa}\left(\left[\left[\phi,\phi^\dagger\right],\phi\right] - v^2\phi\right)\right)\right\}\\
&\quad +\operatorname{tr}\left((D_-\phi)^\dagger (D_-\phi)\right)\\
&\quad +\mathcal{E}^a \qquad (76)
\end{aligned}$$

where the additional term in the energy expression is

$$\begin{aligned}
\mathcal{E}^a &= -\operatorname{tr}\left\{-\frac{i}{2\kappa}\left(\left[\left[\phi,\phi^\dagger\right],\phi\right]\right)^\dagger (D_0\phi) + \frac{i}{2\kappa}v^2\phi^\dagger (D_0\phi)\right. \qquad (77)\\
&\quad \left. + \frac{i}{2\kappa}(D_0\phi)^\dagger \left[\left[\phi,\phi^\dagger\right],\phi\right] - \frac{i}{2\kappa}(D_0\phi)^\dagger v^2\phi\right\}\\
&\quad + \frac{i}{2\kappa}\operatorname{tr}\left\{\left[\left[\phi,\phi^\dagger\right],\phi\right]^\dagger (D_0\phi) - \left[\left[\phi,\phi^\dagger\right],\phi\right](D_0\phi)^\dagger\right\}\\
&= \frac{i}{\kappa}\operatorname{tr}\left\{\left[\left[\phi,\phi^\dagger\right],\phi\right]^\dagger (D_0\phi)\right\}\\
&\quad -\frac{i}{\kappa}\operatorname{tr}\left\{(D_0\phi)^\dagger\left[\left[\phi,\phi^\dagger\right],\phi\right]\right\} \quad \text{(using cyclic permutation in Trace)}\\
&\quad -\frac{iv^2}{2\kappa}\operatorname{tr}\left\{\phi^\dagger (D_0\phi) - (D_0\phi)^\dagger \phi\right\}
\end{aligned}$$

We write

$$\mathcal{E}^a = \mathcal{E}^{a(1)} + \mathcal{E}^{a(2)} + \mathcal{E}^{a(3)} \tag{78}$$

for the last three lines of the equation above and these will be calculated below. At this point it is more useful to focus on the set of equations at self-duality that are derived from this choice adopted in Eq.(72).

### 6.2.1 Second form of the Self-Duality equations

The NEW equations as they result from the alternative Bogomolny form of the action are

$$\begin{aligned} D_-\phi &= 0 \\ D_0\phi &= -\frac{i}{2\kappa}\left(\left[\left[\phi, \phi^\dagger\right], \phi\right] - v^2\phi\right) \end{aligned} \tag{79}$$

We introduce the field tensor $F_{+-}$ and it is shown in Appendix C that

$$\begin{aligned} F_{+-} &= -\frac{J^0}{\kappa} = \frac{J_0}{\kappa} \\ &= \frac{1}{\kappa}\left\{-i\left(\left[\phi^\dagger, D_0\phi\right] - \left[\left(D_0\phi\right)^\dagger, \phi\right]\right)\right\} \end{aligned} \tag{80}$$

Now, we use the NEW equations

$$\begin{aligned} D_0\phi &= -\frac{i}{2\kappa}\left(\left[\left[\phi, \phi^\dagger\right], \phi\right] - v^2\phi\right) \\ \left(D_0\phi\right)^\dagger &= \frac{i}{2\kappa}\left(\left(\left[\left[\phi, \phi^\dagger\right], \phi\right]\right)^\dagger - v^2\phi^\dagger\right) \end{aligned} \tag{81}$$

Inserting the two operators from the equations above, and after finding that the two commutators in the Eq.(C.6) are equal and opposite, we get an expression for $F_{+-}$ as

$$\begin{aligned} F_{+-} &= -\frac{i}{\kappa}\left\{\left(\left[\phi^\dagger, D_0\phi\right] - \left[\left(D_0\phi\right)^\dagger, \phi\right]\right)\right\} \\ &= -\frac{2i}{\kappa}\left[\phi^\dagger, D_0\phi\right] \end{aligned} \tag{82}$$

where we replace the NEW expression of $D_0\phi$ , *i.e.* Eq.(81), obtaining

$$\begin{aligned} F_{+-} &= -\frac{2i}{\kappa}\left[\phi^\dagger, -\frac{i}{2\kappa}\left(\left[\left[\phi, \phi^\dagger\right], \phi\right] - v^2\phi\right)\right] \\ &= -\frac{1}{\kappa^2}\left[\phi^\dagger, \left[\left[\phi, \phi^\dagger\right], \phi\right] - v^2\phi\right] \\ &= -\frac{1}{\kappa^2}\left[v^2\phi - \left[\left[\phi, \phi^\dagger\right], \phi\right], \phi^\dagger\right] \end{aligned} \tag{83}$$

Then the NEW equations at self-duality are

$$\begin{aligned} D_{-}\phi &= 0 \\ F_{+-} &= -\frac{1}{\kappa^2}\left[v^2\phi - \left[\left[\phi,\phi^\dagger\right],\phi\right],\phi^\dagger\right] \end{aligned} \tag{84}$$

We note that this form of the self-duality equations differs from the previous one by the opposite sign of the right-hand side term of the second equation.

Using the definition of Eq.(53) the left hand side of the first equation of motion (41) can be written

$$D_\mu D^\mu \phi = -D_0 D_0 \phi + D_+ D_- \phi + i\left[F_{12},\phi\right] \tag{85}$$

# 7 The group theoretical ansatz

## 7.1 Elements of the $SU(2)$ algebra structure

In order to solve the self-duality equations it is considered, as in the case of the Euler equation, the Lie algebra of the group $SU(2)$. Then the Chevalley basis is [32]

$$\begin{aligned} \left[E_+,E_-\right] &= H \\ \left[H,E_\pm\right] &= \pm 2E_\pm \\ \mathrm{tr}\left(E_+E_-\right) &= 1 \\ \mathrm{tr}\left(H^2\right) &= 2 \end{aligned} \tag{86}$$

where

$H$ is the Cartan subalgebra generator

Since the *rank* of $SU(2)$ is $r=1$ the generator $H$ is unique.

$E_\pm$ are step (ladder) operators

The $2\times 2$ representation is

$$E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{87}$$

$$E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{88}$$

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{89}$$

The Hemitian conjugates of the generators are the transposed complex conjugated matrices

$$\begin{aligned} E_+^\dagger &= E_- \\ E_-^\dagger &= E_+ \\ H^\dagger &= H \end{aligned} \tag{90}$$

These adjoint generators will be used to express the adjoint fields in the calculations based on a particular *ansatz*.

## 7.2 The fields within the algebraic *ansatz*

According to Dunne [31], [33], the following *ansatz* can be adopted

$$\begin{aligned} \phi &= \sum_{a=1}^{r} \phi_a E_a + \phi_{-M} E_{-M} \\ &= \phi_1 E_+ + \phi_2 E_- \end{aligned} \tag{91}$$

since the rank of $SU(2)$ is $r = 1$. We take the Hermitian conjugate, which is

$$\begin{aligned} \phi^\dagger &= \phi_1^* E_+^\dagger + \phi_2^* E_-^\dagger \\ &= \phi_1^* E_- + \phi_2^* E_+ \end{aligned} \tag{92}$$

In this ansatz the matter Higgs field is represented by a linear combination of the ladder generators plus the generator associated with minus the maximal root.

The gauge potential is taken as

$$\begin{aligned} A_+ &= aH \\ A_- &= -a^* H \end{aligned} \tag{93}$$

The notations with $+$ and $-$ correspond to the combinations of the $x$ and $y$ components, with the coefficient $i$ for the $y$ component.

### 7.2.1 The explicit form of the equations with the *ansatz*

The gauge field tensor

$$\begin{aligned} F_{+-} &= \partial_+ A_- - \partial_- A_+ + [A_+, A_-] \\ &= \partial_+ (-a^* H) - \partial_- (aH) + [aH, -a^* H] \\ &= (-\partial_+ a^* - \partial_- a) H \end{aligned} \tag{94}$$

where

$$\begin{aligned}\partial_+ &= \partial_x + i\partial_y = 2\frac{\partial}{\partial z^*} \\ \partial_- &= \partial_x - i\partial_y = 2\frac{\partial}{\partial z}\end{aligned} \tag{95}$$

We will have to calculate, with this ansatz, the terms of the equations. The right hand side of the second equation

$$\left[v^2\phi - \left[\left[\phi, \phi^\dagger\right], \phi\right], \phi^\dagger\right] \tag{96}$$

will be calculated using the commutator

$$\begin{aligned}\left[\phi, \phi^\dagger\right] &= \left[\phi_1 E_+ + \phi_2 E_-, \phi_1^* E_- + \phi_2^* E_+\right] \\ &= \phi_1 \phi_1^* \left[E_+, E_-\right] + \phi_2 \phi_1^* \left[E_-, E_-\right] \\ &\quad + \phi_1 \phi_2^* \left[E_+, E_+\right] + \phi_2 \phi_2^* \left[E_-, E_+\right]\end{aligned} \tag{97}$$

$$\begin{aligned}\left[\phi, \phi^\dagger\right] &= \phi_1^* \phi_1 \left[E_+, E_-\right] \\ &\quad + \phi_2^* \phi_2 \left[E_-, E_+\right] \\ &= \phi_1^* \phi_1 H - \phi_2^* \phi_2 H\end{aligned} \tag{98}$$

$$\begin{aligned}\left[\phi, \phi^\dagger\right] &= \left(\phi_1^* \phi_1 - \phi_2^* \phi_2\right) H \\ &= \left(\rho_1 - \rho_2\right) H\end{aligned} \tag{99}$$

where we have introduced the notations

$$\begin{aligned}\rho_1 &\equiv \left|\phi_1\right|^2 \\ \rho_2 &\equiv \left|\phi_2\right|^2\end{aligned} \tag{100}$$

The next step is to calculate

$$\left[\left[\phi, \phi^\dagger\right], \phi\right] = \left[\left(\rho_1 - \rho_2\right) H, \phi_1 E_+ + \phi_2 E_-\right] \tag{101}$$

This is

$$\begin{aligned}\left[\left[\phi, \phi^\dagger\right], \phi\right] &= \left(\rho_1 - \rho_2\right) \phi_1 \left[H, E_+\right] + \left(\rho_1 - \rho_2\right) \phi_2 \left[H, E_-\right] \\ &= 2\left(\rho_1 - \rho_2\right)\left(\phi_1 E_+ - \phi_2 E_-\right)\end{aligned} \tag{102}$$

The next level in the commutator is

$$\begin{aligned}v^2\phi - \left[\left[\phi, \phi^\dagger\right], \phi\right] &= v^2 \phi_1 E_+ + v^2 \phi_2 E_- \\ &\quad - 2\left(\rho_1 - \rho_2\right) \phi_1 E_+ + 2\left(\rho_1 - \rho_2\right) \phi_2 E_- \\ &\equiv P E_+ + Q E_-\end{aligned} \tag{103}$$

where

$$\begin{aligned} P &\equiv v^2\phi_1 - 2(\rho_1 - \rho_2)\phi_1 \\ Q &\equiv v^2\phi_2 + 2(\rho_1 - \rho_2)\phi_2 \end{aligned} \tag{104}$$

Returning to the Eq.(70), the full right hand side term is

$$\begin{aligned} \left[v^2\phi - \left[\left[\phi, \phi^\dagger\right], \phi\right], \phi^\dagger\right] &= \left[PE_+ + QE_-, \phi_1^* E_- + \phi_2^* E_+\right] \\ &= P\phi_1^* \left[E_+, E_-\right] + Q\phi_2^* \left[E_-, E_+\right] \\ &= \left(P\phi_1^* - Q\phi_2^*\right)\left[E_+, E_-\right] \\ &= \left(P\phi_1^* - Q\phi_2^*\right) H \end{aligned} \tag{105}$$

or

$$\begin{aligned} & \left[v^2\phi - \left[\left[\phi, \phi^\dagger\right], \phi\right], \phi^\dagger\right] \\ = & \left\{\left(v^2\phi_1 - 2(\rho_1 - \rho_2)\phi_1\right)\phi_1^* - \left(v^2\phi_2 + 2(\rho_1 - \rho_2)\phi_2\right)\phi_2^*\right\} H \\ = & \left\{\left(v^2 - 2(\rho_1 - \rho_2)\right)\rho_1 - \left(v^2 + 2(\rho_1 - \rho_2)\right)\rho_2\right\} H \\ = & \left(v^2 - 2(\rho_1 + \rho_2)\right)(\rho_1 - \rho_2) H \end{aligned} \tag{106}$$

## 7.3 Using the algebraic ansatz in the first version of the SD equations

The second self-duality equation Eq.(70) becomes, using Eqs.(94) and (106)

$$-\frac{\partial a^*}{\partial x_+} - \frac{\partial a}{\partial x_-} = \frac{1}{k^2}(\rho_1 - \rho_2)\left[v^2 - 2(\rho_1 + \rho_2)\right] \tag{107}$$

Now we turn to the first self-duality equation

$$D_-\phi = 0 \tag{108}$$

and its adjoint form. It has been defined

$$D_- \equiv D_1 - iD_2 \tag{109}$$

then

$$D_-\phi = \frac{\partial \phi}{\partial x} + \left[A_x, \phi\right] - i\frac{\partial \phi}{\partial y} - i\left[A_y, \phi\right] \tag{110}$$

To proceed further we express the components of the potential

$$\begin{aligned} A_+ &= A_x + iA_y = aH \\ A_- &= A_x - iA_y = -a^* H \end{aligned} \tag{111}$$

Then

$$
\begin{aligned}
A_x &= \frac{1}{2}\left(a-a^*\right)H \\
A_y &= \frac{1}{2i}\left(a+a^*\right)H
\end{aligned}
\tag{112}
$$

Then

$$
\begin{aligned}
D_-\phi &= \left(\frac{\partial\phi_1}{\partial x} - i\frac{\partial\phi_1}{\partial y}\right)E_+ + \left(\frac{\partial\phi_2}{\partial x} - i\frac{\partial\phi_2}{\partial y}\right)E_- \\
&\quad +\frac{1}{2}\left(a-a^*\right)\phi_1\left[H,E_+\right] \\
&\quad +\frac{1}{2}\left(a-a^*\right)\phi_2\left[H,E_-\right] \\
&\quad -i\frac{1}{2i}\left(a+a^*\right)\phi_1\left[H,E_+\right] \\
&\quad -i\frac{1}{2i}\left(a+a^*\right)\phi_2\left[H,E_-\right]
\end{aligned}
\tag{113}
$$

$$
\begin{aligned}
D_-\phi &= \left(\frac{\partial\phi_1}{\partial x} - i\frac{\partial\phi_1}{\partial y} + 2\frac{1}{2}\left(a-a^*\right)\phi_1 - 2\frac{1}{2}\left(a+a^*\right)\phi_1\right)E_+ \\
&\quad + \left(\frac{\partial\phi_2}{\partial x} - i\frac{\partial\phi_2}{\partial y} - 2\frac{1}{2}\left(a-a^*\right)\phi_2 + 2\frac{1}{2}\left(a+a^*\right)\phi_2\right)E_- \\
&= 0
\end{aligned}
\tag{114}
$$

From the explicit form of the ladder generators we obtain the equations derived from the first self-duality equation

$$
\frac{\partial\phi_1}{\partial x} - i\frac{\partial\phi_1}{\partial y} - 2\phi_1 a^* = 0 \tag{115}
$$

$$
\frac{\partial\phi_2}{\partial x} - i\frac{\partial\phi_2}{\partial y} + 2\phi_2 a^* = 0 \tag{116}
$$

### 7.3.1 The explicit form of the *adjoint* equations with the algebraic *ansatz*

Now we consider the *adjoint* equation (also derived from the extremum of the corresponding part of the action expressed in the Bogomolnyi form)

$$
\left(D_-\phi\right)^\dagger = 0 \tag{117}
$$

We have

$$D_-^\dagger = \frac{\partial}{\partial x} + \left[ , A_x^\dagger \right] + i\frac{\partial}{\partial y} + i \left[ , A_y^\dagger \right] \tag{118}$$

where the adjoint is taken for any matrix as the transpose complex conjugated. The change of the order in the commutators is due to the property that for any two matrices $R_1$ and $R_2$ the Hermitian conjugate of their commutator is

$$\begin{aligned} [R_1, R_2]^\dagger &= (R_1 R_2 - R_2 R_1)^\dagger \\ &= \left( R_2^T R_1^T - R_1^T R_2^T \right)^* \\ &= R_2^\dagger R_1^\dagger - R_1^\dagger R_2^\dagger \\ &= \left[ R_2^\dagger, R_1^\dagger \right] \end{aligned} \tag{119}$$

($^*$ is complex conjugate and $^T$ is the transpose operators) and we take into account that in the expression of $\phi^\dagger$ we have already used the Hermitian conjugated matrices of $E_\pm$.

The Hermitian conjugates of the gauge field matrices are

$$\begin{aligned} A_x^\dagger &= \frac{1}{2}(a^* - a) H^\dagger = \frac{1}{2}(a^* - a) H \\ A_y^\dagger &= -\frac{1}{2i}(a^* + a) H^\dagger = -\frac{1}{2i}(a^* + a) H \end{aligned} \tag{120}$$

Then

$$D_-^\dagger \equiv \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} + \frac{1}{2}(a^* - a)[, H] - \frac{1}{2}(a^* + a)[, H] \tag{121}$$

We recall that

$$\phi^\dagger = \phi_1^* E_- + \phi_2^* E_+ \tag{122}$$

The we have

$$\begin{aligned} (D_- \phi)^\dagger &= \left\{ \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} + \frac{1}{2}(a^* - a)[, H] - \frac{1}{2}(a^* + a)[, H] \right\} \\ &\quad \times (\phi_1^* E_- + \phi_2^* E_+) \\ &= \left( \frac{\partial \phi_1^*}{\partial x} + i\frac{\partial \phi_1^*}{\partial y} \right) E_- + \left( \frac{\partial \phi_2^*}{\partial x} + i\frac{\partial \phi_2^*}{\partial y} \right) E_+ \\ &\quad + \frac{1}{2}(a^* - a)\phi_1^* [E_-, H] \\ &\quad + \frac{1}{2}(a^* - a)\phi_2^* [E_+, H] \\ &\quad - \frac{1}{2}(a^* + a)\phi_1^* [E_-, H] \\ &\quad - \frac{1}{2}(a^* + a)\phi_2^* [E_+, H] \end{aligned} \tag{123}$$

or

$$
\begin{aligned}
(D_-\phi)^\dagger &= 2\frac{\partial\phi_1^*}{\partial z^*}E_- + 2\frac{\partial\phi_2^*}{\partial z^*}E_+ \\
&+\frac{1}{2}\left(a^*-a\right)\phi_1^*\left(2E_-\right) \\
&+\frac{1}{2}\left(a^*-a\right)\phi_2^*\left(-2E_+\right) \\
&-\frac{1}{2}\left(a^*+a\right)\phi_1^*\left(2E_-\right) \\
&-\frac{1}{2}\left(a^*+a\right)\phi_2^*\left(-2E_+\right)
\end{aligned} \tag{124}
$$

The equation becomes

$$
\begin{aligned}
(D_-\phi)^\dagger &= \left(2\frac{\partial\phi_1^*}{\partial z^*} + \left(a^*-a\right)\phi_1^* - \left(a^*+a\right)\phi_1^*\right)E_- \\
&+\left(2\frac{\partial\phi_2^*}{\partial z^*} - \left(a^*-a\right)\phi_2^* + \left(a^*+a\right)\phi_2^*\right)E_+ \\
&= 0
\end{aligned} \tag{125}
$$

Here we have made use of the identifications

$$
\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \equiv 2\frac{\partial}{\partial z^*} \tag{126}
$$

and

$$
\frac{\partial}{\partial x} - i\frac{\partial}{\partial y} \equiv 2\frac{\partial}{\partial z} \tag{127}
$$

The resulting equations are

$$
2\frac{\partial\phi_1^*}{\partial z^*} - 2a\phi_1^* = 0 \tag{128}
$$

and

$$
2\frac{\partial\phi_2}{\partial z^*} + 2a\phi_2^* = 0 \tag{129}
$$

which represent the adjoints of the first set, Eqs(115), as expected.

### 7.3.2 Using the two sets of equations

Now we consider the first equations (*i.e.* those refering to $\phi_1$) in the two sets, Eqs.(115) and (128)

$$
\begin{aligned}
2\frac{\partial\phi_1}{\partial z} - 2a^*\phi_1 &= 0 \\
2\frac{\partial\phi_1^*}{\partial z^*} - 2a\phi_1^* &= 0
\end{aligned} \tag{130}
$$

From here we obtain the expressions of $a$ and $a^*$

$$a = \frac{\partial}{\partial z^*} \ln \left(\phi_1^*\right) \tag{131}$$

$$a^* = \frac{\partial}{\partial z} \ln \left(\phi_1\right) \tag{132}$$

The left hand side of the second self-duality equation (107) is

$$\begin{aligned} -2\frac{\partial a^*}{\partial z^*} - 2\frac{\partial a}{\partial z} &= -2\frac{\partial}{\partial z^*}\frac{\partial}{\partial z} \ln \left(\phi_1\right) - 2\frac{\partial}{\partial z}\frac{\partial}{\partial z^*} \ln \left(\phi_1^*\right) \\ &= -2\frac{\partial^2}{\partial z \partial z^*} \left[\ln \left(\phi_1\right) + \ln \left(\phi_1^*\right)\right] \\ &= -2\frac{\partial^2}{\partial z \partial z^*} \ln \left(\left|\phi_1\right|^2\right) \end{aligned} \tag{133}$$

In the differential operator we recognize the Laplacean,

$$\Delta = 4\frac{\partial^2}{\partial z \partial z^*}$$

Equating the expressions that we have obtained for the left hand side and respectively for right hand side of the second self-duality equation (107) we obtain

$$-\frac{1}{2}\Delta \ln \rho_1 = -\frac{1}{\kappa^2} \left(\rho_1 - \rho_2\right) \left[2\left(\rho_1 + \rho_2\right) - v^2\right] \tag{134}$$

The second equations (those refering to $\phi_2$) in the two sets Eqs.(116) and (129) give the result

$$a^* = -\frac{\partial}{\partial z} \ln \phi_2 \tag{135}$$

and

$$a = -\frac{\partial}{\partial z^*} \ln \phi_2^* \tag{136}$$

from where we obtain the form of the right hand side in the second self-duality equation, (107)

$$\begin{aligned} -2\frac{\partial a^*}{\partial z^*} - 2\frac{\partial a}{\partial z} &= 2\frac{\partial}{\partial z^*}\frac{\partial}{\partial z} \ln \left(\phi_2\right) + 2\frac{\partial}{\partial z}\frac{\partial}{\partial z^*} \ln \left(\phi_2^*\right) \\ &= 2\frac{\partial^2}{\partial z \partial z^*} \left[\ln \left(\phi_2\right) + \ln \left(\phi_2^*\right)\right] \\ &= 2\frac{\partial^2}{\partial z \partial z^*} \ln \left(\left|\phi_2\right|^2\right) \end{aligned} \tag{137}$$

The final form is

$$\frac{1}{2}\Delta\ln\rho_2 = -\frac{1}{\kappa^2}\left(\rho_1-\rho_2\right)\left[2\left(\rho_1+\rho_2\right)-v^2\right] \tag{138}$$

The right hand side in Eqs.(134) and (138) is the same and if we substract the equations we obtain

$$\begin{aligned}\Delta\ln\rho_1+\Delta\ln\rho_2 &= 0 \\ \Delta\ln\left(\rho_1\rho_2\right) &= 0\end{aligned} \tag{139}$$

The function $\ln\left(\rho_1\rho_2\right)$ is an arbitrary harmonic function and this aspect will be discussed later. For the moment we simply take a constant, convenient for normalization,

$$\rho_1\rho_2 = v^4/16 \tag{140}$$

With this relation we return to the equation for $\rho_1$, (134)

$$-\frac{1}{2}\Delta\ln\rho_1 = -\frac{1}{\kappa^2}\left(\rho_1-\frac{v^4/16}{\rho_1}\right)\left[2\left(\rho_1+\frac{v^4/16}{\rho_1}\right)-v^2\right] \tag{141}$$

We add the zero-valued Laplacean of a constant to the left side and factorise in the right hand side

$$\frac{1}{2}\Delta\ln\rho_1-\frac{1}{2}\Delta\ln\left(v^2/4\right) = 4\frac{\left(v^2/4\right)^2}{\kappa^2}\left(\frac{\rho_1}{v^2/4}-\frac{v^2/4}{\rho_1}\right)\left[\frac{1}{2}\left(\frac{\rho_1}{v^2/4}+\frac{v^2/4}{\rho_1}\right)-1\right] \tag{142}$$

Now we introduce a single variable

$$\rho\equiv\frac{\rho_1}{v^2/4}=\frac{v^2/4}{\rho_2} \tag{143}$$

and obtain

$$\frac{1}{2}\Delta\ln\rho = \frac{1}{4}\left(\frac{v^2}{\kappa}\right)^2\left(\rho-\frac{1}{\rho}\right)\left[\frac{1}{2}\left(\rho+\frac{1}{\rho}\right)-1\right] \tag{144}$$

We make the substitution

$$\psi\equiv\ln\rho \tag{145}$$

and we obtain

$$\begin{aligned}\frac{1}{2}\Delta\psi &= \frac{1}{4}\left(\frac{v^2}{\kappa}\right)^2\left[\exp\left(\psi\right)-\exp\left(-\psi\right)\right] \\ &\quad\times\left\{\frac{1}{2}\left[\exp\left(\psi\right)+\exp\left(-\psi\right)\right]-1\right\} \\ &= \frac{1}{2}\left(\frac{v^2}{\kappa}\right)^2\sinh\psi\left(\cosh\psi-1\right)\end{aligned} \tag{146}$$

$$\left(\frac{\kappa}{v^2}\right)^2 \Delta\psi - \sinh\psi\left(\cosh\psi - 1\right) = 0 \tag{147}$$

Exactly the same equation would have been obtained starting from the one for $\rho_2$, (138) after a change of the unknown function, $\psi \to -\psi$.

After normalizing the coordinates by the length $\kappa/v^2$, we obtain

$$\Delta\psi - \sinh\psi\left(\cosh\psi - 1\right) = 0 \tag{148}$$

This is the equation governing the stationary states of the CHM equation, resulting from the first form of the SD equations.

#### 7.3.3 Calculation of the additional energy for the first version of the SD equations

We start from the energy as integral of the density of the Hamiltonian Eq.(51). Since all other terms in the expression of the energy are positive (they vanish after adopting the self-duality and the particular $6^{th}$ order potential), the energy is bounded from below

$$\mathcal{E} \geqslant \frac{iv^2}{2\kappa}\mathrm{tr}\left(\phi^\dagger\left(D_0\phi\right) - \left(D_0\phi\right)^\dagger\phi\right) \tag{149}$$

where the second of the equations at self-duality Eq.(69) is

$$D_0\phi = \frac{i}{2\kappa}\left(\left[\left[\phi,\phi^\dagger\right],\phi\right] - v^2\phi\right) \tag{150}$$

and

$$\left(D_0\phi\right)^\dagger = -\frac{i}{2\kappa}\left(\left[\left[\phi,\phi^\dagger\right],\phi\right] - v^2\phi\right)^\dagger$$

Then we have

$$\mathcal{E} \geqslant -\frac{v^2}{4\kappa^2}\mathrm{tr}\left\{\phi^\dagger\left(\left[\left[\phi,\phi^\dagger\right],\phi\right] - v^2\phi\right) + \left(\left[\left[\phi,\phi^\dagger\right],\phi\right] - v^2\phi\right)^\dagger\phi\right\} \tag{151}$$

and we will prove that the second term in the curly brackets is equal with the first.

$$\begin{aligned}
\left(\left[\left[\phi,\phi^\dagger\right],\phi\right] - v^2\phi\right)^\dagger\phi &= \left[\left[\phi,\phi^\dagger\right],\phi\right]^\dagger\phi - v^2\phi^\dagger\phi \\
&= \left[\phi^\dagger,\left[\phi,\phi^\dagger\right]^\dagger\right]\phi - v^2\phi^\dagger\phi \\
&= \left[\phi^\dagger,\left[\phi,\phi^\dagger\right]\right]\phi - v^2\phi^\dagger\phi \\
&= \left(\phi^\dagger\left[\phi,\phi^\dagger\right] - \left[\phi,\phi^\dagger\right]\phi^\dagger\right)\phi - v^2\phi^\dagger\phi \\
&= \phi^\dagger\left[\phi,\phi^\dagger\right]\phi - \underline{\left[\phi,\phi^\dagger\right]\phi^\dagger\phi} - v^2\phi^\dagger\phi
\end{aligned} \tag{152}$$

We can apply in the second term (underlined), the cyclic symmetry of the tr operator, moving successively the factors $\phi$ and $\phi^{\dagger}$ in the first position and have

$$\begin{aligned} & \mathrm{tr}\left\{\left(\left[\left[\phi,\phi^{\dagger}\right],\phi\right]-v^{2}\phi\right)^{\dagger}\phi\right\} \\ = & \mathrm{tr}\left\{\phi^{\dagger}\left[\phi,\phi^{\dagger}\right]\phi-\phi^{\dagger}\phi\left[\phi,\phi^{\dagger}\right]-v^{2}\phi^{\dagger}\phi\right\} \\ = & \mathrm{tr}\left\{\phi^{\dagger}\left[\left[\phi,\phi^{\dagger}\right],\phi\right]-v^{2}\phi^{\dagger}\phi\right\} \\ = & \mathrm{tr}\left\{\phi^{\dagger}\left(\left[\left[\phi,\phi^{\dagger}\right],\phi\right]-v^{2}\phi\right)\right\} \end{aligned} \tag{153}$$

and the equality with the first term in Eq.(151) is proved. It results

$$\mathcal{E}\geqslant-\frac{v^{2}}{2\kappa^{2}}\mathrm{tr}\left\{\phi^{\dagger}\left(\left[\left[\phi,\phi^{\dagger}\right],\phi\right]-v^{2}\phi\right)\right\} \tag{154}$$

but at self-duality (since we have already used the equations derived from self-duality) the limit is saturated

$$\mathcal{E}_{SD}=\frac{v^{2}}{2\kappa^{2}}\mathrm{tr}\left\{\phi^{\dagger}\left(v^{2}\phi-\left[\left[\phi,\phi^{\dagger}\right],\phi\right]\right)\right\} \tag{155}$$

We can obtain the explicit formula using the algebraic representation of the fields

$$\begin{aligned} \phi &= \phi_{1}E_{+}+\phi_{2}E_{-} \\ \phi^{\dagger} &= \phi_{1}^{*}E_{-}+\phi_{2}^{*}E_{+} \end{aligned} \tag{156}$$

and recall the previous result

$$v^{2}\phi-\left[\left[\phi,\phi^{\dagger}\right],\phi\right]=PE_{+}+QE_{-} \tag{157}$$

where

$$\begin{aligned} P &\equiv v^{2}\phi_{1}-2\left(\rho_{1}-\rho_{2}\right)\phi_{1} \\ Q &\equiv v^{2}\phi_{2}+2\left(\rho_{1}-\rho_{2}\right)\phi_{2} \end{aligned} \tag{158}$$

A detailed calculation, starting from Eq.(155):

$$\begin{aligned} \mathcal{E}_{SD} &= \frac{v^{2}}{2\kappa^{2}}\mathrm{tr}\left(\phi^{\dagger}\left(v^{2}\phi-\left[\left[\phi,\phi^{\dagger}\right],\phi\right]\right)\right) \\ &= \frac{v^{2}}{2\kappa^{2}}\mathrm{tr}\left\{\left(\phi_{1}^{*}E_{-}+\phi_{2}^{*}E_{+}\right)\left(PE_{+}+QE_{-}\right)\right\} \\ &= \frac{v^{2}}{2\kappa^{2}}\mathrm{tr}\left\{\phi_{1}^{*}PE_{-}E_{+}+\phi_{1}^{*}QE_{-}E_{-}+\phi_{2}^{*}PE_{+}E_{+}+\phi_{2}^{*}QE_{+}E_{-}\right\} \end{aligned} \tag{159}$$

We calculate separately

$$\begin{aligned}\operatorname{tr}\left(\phi_1^* P E_- E_+\right) &= \operatorname{tr}\left[\phi_1^*\left(v^2\phi_1 - 2\left(\rho_1-\rho_2\right)\phi_1\right)E_-E_+\right] \\ &= \left[v^2\rho_1 - 2\left(\rho_1-\rho_2\right)\rho_1\right]\operatorname{tr}\begin{pmatrix} 0 & 0 \\ 0 & 1\end{pmatrix} \\ &= v^2\rho_1 - 2\left(\rho_1-\rho_2\right)\rho_1\end{aligned} \tag{160}$$

$$\begin{aligned}\operatorname{tr}\left(\phi_1^* Q E_- E_-\right) &= \operatorname{tr}\left[\phi_1^*\left(v^2\phi_2 + 2\left(\rho_1-\rho_2\right)\phi_2\right)E_-E_-\right] \\ &= \phi_1^*\left(v^2\phi_2 + 2\left(\rho_1-\rho_2\right)\phi_2\right)\operatorname{tr}\begin{pmatrix} 0 & 0 \\ 0 & 0\end{pmatrix} \\ &= 0\end{aligned} \tag{161}$$

$$\begin{aligned}\operatorname{tr}\left(\phi_2^* P E_+ E_+\right) &= \operatorname{tr}\left[\phi_2^*\left(v^2\phi_1 - 2\left(\rho_1-\rho_2\right)\phi_1\right)E_+E_+\right] \\ &= \phi_2^*\left(v^2\phi_1 - 2\left(\rho_1-\rho_2\right)\phi_1\right)\operatorname{tr}\begin{pmatrix} 0 & 0 \\ 0 & 0\end{pmatrix} \\ &= 0\end{aligned} \tag{162}$$

$$\begin{aligned}\operatorname{tr}\left(\phi_2^* Q E_+ E_-\right) &= \operatorname{tr}\left[\phi_2^*\left(v^2\phi_2 + 2\left(\rho_1-\rho_2\right)\phi_2\right)E_+E_-\right] \\ &= \left[v^2\rho_2 + 2\left(\rho_1-\rho_2\right)\rho_2\right]\operatorname{tr}\begin{pmatrix} 1 & 0 \\ 0 & 0\end{pmatrix} \\ &= v^2\rho_2 + 2\left(\rho_1-\rho_2\right)\rho_2\end{aligned} \tag{163}$$

Summing up the contributions

$$\begin{aligned}\mathcal{E}_{SD} &= \frac{v^2}{2\kappa^2}\left[v^2\rho_1 - 2\left(\rho_1-\rho_2\right)\rho_1 + v^2\rho_2 + 2\left(\rho_1-\rho_2\right)\rho_2\right] \\ &= \frac{v^2}{2\kappa^2}\left[v^2\left(\rho_1+\rho_2\right) - 2\left(\rho_1-\rho_2\right)^2\right]\end{aligned} \tag{164}$$

Expressed in the normalised variable, the energy is

$$\begin{aligned}\mathcal{E}_{SD} &= \frac{v^2}{2\kappa^2}\left[v^2\left(\rho_1+\rho_2\right) - 2\left(\rho_1-\rho_2\right)^2\right] \\ &= \frac{v^2}{2\kappa^2}\left[v^2\left(\rho\frac{v^2}{4} + \frac{v^2}{4}\frac{1}{\rho}\right) - 2\left(\rho\frac{v^2}{4} - \frac{v^2}{4}\frac{1}{\rho}\right)^2\right] \\ &= \frac{v^2}{2\kappa^2}4\left(\frac{v^2}{4}\right)^2\left[\rho + \frac{1}{\rho} - \frac{1}{2}\left(\rho - \frac{1}{\rho}\right)^2\right] \\ &= -\frac{v^2}{8}\frac{1}{\rho_s^2}\left[\frac{1}{2}\left(\rho - \frac{1}{\rho}\right)^2 - \left(\rho + \frac{1}{\rho}\right)\right]\end{aligned} \tag{165}$$

Introducing the streamfunction $\rho \equiv \exp(\psi)$ we get

$$\begin{aligned}
\mathcal{E}_{SD} &= -\frac{v^2}{8}\frac{1}{\rho_s^2}\left[\frac{1}{2}\left(\rho-\frac{1}{\rho}\right)^2-\left(\rho+\frac{1}{\rho}\right)\right] \\
&= -\frac{v^2}{8}\frac{1}{\rho_s^2}\left[2(\sinh\psi)^2-2\cosh\psi\right] \\
&= -\frac{v^2}{4}\frac{1}{\rho_s^2}\left[(\cosh\psi)^2-\cosh\psi-1\right]
\end{aligned} \tag{166}$$

or

$$\mathcal{E}_{SD} = v^2\frac{1}{\rho_s^2}\frac{1}{4}\left[-(\cosh\psi)^2+\cosh\psi+1\right] \tag{167}$$

We note that this expression must be integrated over the plane (the factor $1/\rho_s^2$ will ensure the correct dimension) and the dimension of the *energy* is actually given by $v^2 \equiv \Omega_{ci}$.

## 7.4 Using the algebraic ansatz in the second version of the SD equations

We now turn to the second version of the SD equations (84) and introduce the algebraic ansatz. Then the second equation of the second version of the Self-Duality becomes

$$\begin{aligned}
F_{+-} &= -\frac{1}{\kappa^2}\left[v^2\phi-\left[\left[\phi,\phi^\dagger\right],\phi\right],\phi^\dagger\right] \\
&= -\frac{1}{\kappa^2}\left[v^2-2(\rho_1+\rho_2)\right](\rho_1-\rho_2)H
\end{aligned} \tag{168}$$

Using Eq.(94)

$$-\frac{\partial a^*}{\partial x_+}-\frac{\partial a}{\partial x_-} = -\frac{1}{\kappa^2}(\rho_1-\rho_2)\left[v^2-2(\rho_1+\rho_2)\right] \tag{169}$$

From the first equation of self duality, which is common to the two choices

$$a = \frac{\partial}{\partial z^*}\ln(\phi_1^*) \tag{170}$$

$$a^* = \frac{\partial}{\partial z}\ln(\phi_1) \tag{171}$$

The left hand side of the second self-duality equation (107) is

$$\begin{aligned} -2\frac{\partial a^*}{\partial z^*} - 2\frac{\partial a}{\partial z} &= -2\frac{\partial}{\partial z^*}\frac{\partial}{\partial z}\ln(\phi_1) - 2\frac{\partial}{\partial z}\frac{\partial}{\partial z^*}\ln(\phi_1^*) \qquad (172) \\ &= -2\frac{\partial^2}{\partial z\partial z^*}\left[\ln(\phi_1) + \ln(\phi_1^*)\right] \\ &= -2\frac{\partial^2}{\partial z\partial z^*}\ln\left(|\phi_1|^2\right) \end{aligned}$$

Since the Laplace operator is defined as

$$\Delta = 4\frac{\partial^2}{\partial z\partial z^*} \qquad (173)$$

we get

$$-\frac{1}{2}\Delta\ln\left(|\phi_1|^2\right) = -\frac{1}{\kappa^2}(\rho_1 - \rho_2)\left[v^2 - 2(\rho_1 + \rho_2)\right] \qquad (174)$$

or

$$\frac{1}{2}\Delta\ln\rho_1 = \frac{1}{\kappa^2}(\rho_1 - \rho_2)\left[v^2 - 2(\rho_1 + \rho_2)\right] \qquad (175)$$

and we can now replace

$$\rho \equiv \frac{\rho_1}{v^2/4} = \frac{v^2/4}{\rho_2} \qquad (176)$$

$$\begin{aligned} \frac{1}{2}\Delta\ln\rho &= \frac{1}{\kappa^2}\left(\frac{v^2}{4}\right)^2\left(\rho - \frac{1}{\rho}\right)\left[4 - 2\left(\rho + \frac{1}{\rho}\right)\right] \qquad (177) \\ \frac{1}{2}\Delta\ln\rho &= \frac{v^4}{2\kappa^2}\left(\frac{1}{2}\right)\left(\rho - \frac{1}{\rho}\right)\left[1 - \frac{1}{2}\left(\rho + \frac{1}{\rho}\right)\right] \end{aligned}$$

and introduce the streamfunction $\psi$

$$\rho = \exp(\psi) \qquad (178)$$

$$\frac{1}{2}\Delta\psi = \frac{1}{2}\left(\frac{v^2}{\kappa}\right)^2\sinh\psi\,(1 - \cosh\psi) \qquad (179)$$

or

$$\Delta\psi + \left(\frac{v^2}{\kappa}\right)^2\sinh\psi\,(\cosh\psi - 1) = 0 \qquad (180)$$

The unit of space is

$$\frac{1}{\rho_s} = \frac{v^2}{\kappa} \qquad (181)$$

and the equation results

$$\Delta\psi + \sinh\psi\,(\cosh\psi - 1) = 0 \qquad (182)$$

All the other calculations, in particular those implying the function $\phi_2$ and the complex conjugated, $\phi_1^*$ and $\phi_2^*$ are similar to the calculations made for the first version of the SD equations.

### 7.4.1 Calculation of the additional energy for the second version of the self-duality

The additional term in the Bogomolnyi form of the energy, in the second version, Eq.(78) consists of three contributions. The first contribution is

$$\mathcal{E}^{a(1)} \equiv \frac{i}{\kappa}\mathrm{tr}\left\{\left[\left[\phi,\phi^{\dagger}\right],\phi\right]^{\dagger}\left(D_0\phi\right)\right\} \tag{183}$$

and we use the previously derived expression

$$\left[\left[\phi,\phi^{\dagger}\right],\phi\right] = 2\left(\rho_1-\rho_2\right)\left(\phi_1 E_+ - \phi_2 E_-\right) \tag{184}$$

and

$$\left[\left[\phi,\phi^{\dagger}\right],\phi\right]^{\dagger} = 2\left(\rho_1-\rho_2\right)\left(\phi_1^* E_- - \phi_2^* E_+\right) \tag{185}$$

Also we use the following relation

$$v^2\phi - \left[\left[\phi,\phi^{\dagger}\right],\phi\right] = PE_+ + QE_- \tag{186}$$

where

$$\begin{aligned} P &\equiv v^2\phi_1 - 2\left(\rho_1-\rho_2\right)\phi_1 \\ Q &\equiv v^2\phi_2 + 2\left(\rho_1-\rho_2\right)\phi_2 \end{aligned} \tag{187}$$

Using the second (new) equation of self-duality we have

$$\begin{aligned} \mathcal{E}^{a(1)} &\equiv \frac{i}{\kappa}\mathrm{tr}\left\{\left[\left[\phi,\phi^{\dagger}\right],\phi\right]^{\dagger}\left(D_0\phi\right)\right\} \\ &= \frac{i}{\kappa}\mathrm{tr}\left\{2\left(\rho_1-\rho_2\right)\left(\phi_1^* E_- - \phi_2^* E_+\right)\left(-\frac{i}{2\kappa}\left(\left[\left[\phi,\phi^{\dagger}\right],\phi\right]-v^2\phi\right)\right)\right\} \\ &= \frac{1}{2\kappa^2}2\left(\rho_1-\rho_2\right)\mathrm{tr}\left\{\left(\phi_1^* E_- - \phi_2^* E_+\right)\left(-PE_+ - QE_-\right)\right\} \end{aligned} \tag{188}$$

The trace is

$$\begin{aligned} & \mathrm{tr}\left\{\left(\phi_1^* E_- - \phi_2^* E_+\right)\left(-PE_+ - QE_-\right)\right\} \\ =\; & \phi_1^*\left(-P\right)\mathrm{tr}\left(E_-E_+\right) \quad \text{(trace is 1)} \\ & +\left(-\phi_2^*\right)\left(-P\right)\mathrm{tr}\left(E_+E_+\right) \quad \text{(trace is 0)} \\ & +\phi_1^*\left(-Q\right)\mathrm{tr}\left(E_-E_-\right) \quad \text{(trace is 0)} \\ & +\left(-\phi_2^*\right)\left(-Q\right)\mathrm{tr}\left(E_+E_-\right) \quad \text{(trace is 1)} \\ =\; & \phi_1^*\left(-P\right)+\left(-\phi_2^*\right)\left(-Q\right) \\ =\; & -\phi_1^*\left[v^2\phi_1 - 2\left(\rho_1-\rho_2\right)\phi_1\right] + \phi_2^*\left[v^2\phi_2 + 2\left(\rho_1-\rho_2\right)\phi_2\right] \\ =\; & -\rho_1\left[v^2 - 2\left(\rho_1-\rho_2\right)\right] + \rho_2\left[v^2 + 2\left(\rho_1-\rho_2\right)\right] \\ =\; & -v^2\left(\rho_1-\rho_2\right) + 2\left(\rho_1-\rho_2\right)\left(\rho_1+\rho_2\right) \\ =\; & -\left(\rho_1-\rho_2\right)\left[v^2 - 2\left(\rho_1+\rho_2\right)\right] \end{aligned} \tag{189}$$

and the first contribution to the residual energy becomes

$$\begin{aligned}\mathcal{E}^{a(1)} &= \frac{1}{\kappa^2}(\rho_1-\rho_2)\left\{-(\rho_1-\rho_2)\left[v^2-2(\rho_1+\rho_2)\right]\right\} \\ &= -\frac{1}{\kappa^2}(\rho_1-\rho_2)^2\left[v^2-2(\rho_1+\rho_2)\right]\end{aligned} \tag{190}$$

Now we calculate the second contribution to the residual energy

$$\begin{aligned}\mathcal{E}^{a(2)} &= -\frac{i}{\kappa}\mathrm{tr}\left\{(D_0\phi)^\dagger\left[\left[\phi,\phi^\dagger\right],\phi\right]\right\} \\ &= -\frac{i}{\kappa}\mathrm{tr}\left\{(D_0\phi)^\dagger 2(\rho_1-\rho_2)(\phi_1E_+-\phi_2E_-)\right\}\end{aligned} \tag{191}$$

and we replace, according to the new second SD equation

$$\begin{aligned}(D_0\phi)^\dagger &= \left(-\frac{i}{2\kappa}\right)^*\left(\left[\left[\phi,\phi^\dagger\right],\phi\right]-v^2\phi\right)^\dagger \\ &= \frac{i}{2\kappa}(-PE_+-QE_-)^\dagger \\ &= -\frac{i}{2\kappa}(P^*E_-+Q^*E_+)\end{aligned} \tag{192}$$

and replacing in the previous equation

$$\begin{aligned}\mathcal{E}^{a(2)} &= -\frac{i}{\kappa}\mathrm{tr}\left\{(D_0\phi)^\dagger 2(\rho_1-\rho_2)(\phi_1E_+-\phi_2E_-)\right\} \\ &= -\frac{i}{\kappa}\mathrm{tr}\left\{\left(-\frac{i}{2\kappa}(P^*E_-+Q^*E_+)\right)2(\rho_1-\rho_2)(\phi_1E_+-\phi_2E_-)\right\} \\ &= -\frac{1}{2\kappa^2}2(\rho_1-\rho_2)\,\mathrm{tr}\left\{(P^*E_-+Q^*E_+)(\phi_1E_+-\phi_2E_-)\right\}\end{aligned} \tag{193}$$

The trace is calculated separately

$$\begin{aligned}& \mathrm{tr}\left\{(P^*E_-+Q^*E_+)(\phi_1E_+-\phi_2E_-)\right\} \\ =\ & P^*\phi_1\mathrm{tr}\left\{E_-E_+\right\}\quad \text{(trace is 1)} \\ & +Q^*\phi_1\mathrm{tr}\left\{E_+E_+\right\}\quad \text{(trace is 0)} \\ & +P^*(-\phi_2)\,\mathrm{tr}\left\{E_-E_-\right\}\quad \text{(trace is 0)} \\ & +Q^*(-\phi_2)\,\mathrm{tr}\left\{E_+E_-\right\}\quad \text{(trace is 1)} \\ =\ & P^*\phi_1+Q^*(-\phi_2) \\ =\ & \left[v^2\phi_1-2(\rho_1-\rho_2)\phi_1\right]^*\phi_1-\left[v^2\phi_2+2(\rho_1-\rho_2)\phi_2\right]^*\phi_2 \\ =\ & \rho_1\left[v^2-2(\rho_1-\rho_2)\right]-\rho_2\left[v^2+2(\rho_1-\rho_2)\right] \\ =\ & v^2(\rho_1-\rho_2)-2(\rho_1-\rho_2)(\rho_1+\rho_2) \\ =\ & (\rho_1-\rho_2)\left[v^2-2(\rho_1+\rho_2)\right]\end{aligned} \tag{194}$$

and the second contribution is

$$\begin{aligned}\mathcal{E}^{a(2)} &= -\frac{1}{\kappa^2}\left(\rho_1-\rho_2\right)\left(\rho_1-\rho_2\right)\left[v^2-2\left(\rho_1+\rho_2\right)\right] \\ &= -\frac{1}{\kappa^2}\left(\rho_1-\rho_2\right)^2\left[v^2-2\left(\rho_1+\rho_2\right)\right]\end{aligned} \tag{195}$$

We note that the two contributions $\mathcal{E}^{a(1)}$ and $\mathcal{E}^{a(2)}$ are equal.

The third contribution is

$$\mathcal{E}^{a(3)} = -\frac{iv^2}{2\kappa}\mathrm{tr}\left\{\phi^\dagger\left(D_0\phi\right)-\left(D_0\phi\right)^\dagger\phi\right\} \tag{196}$$

We can use the new second SD equation

$$D_0\phi = -\frac{i}{2\kappa}\left(\left[\left[\phi,\phi^\dagger\right],\phi\right]-v^2\phi\right) \tag{197}$$

together with

$$v^2\phi-\left[\left[\phi,\phi^\dagger\right],\phi\right] = PE_+ + QE_- \tag{198}$$

We find, as in previous cases,

$$\begin{aligned}D_0\phi &= -\frac{i}{2\kappa}\left(-PE_+ - QE_-\right) \\ &= \frac{i}{2\kappa}\left(PE_+ + QE_-\right)\end{aligned} \tag{199}$$

and

$$\begin{aligned}\left(D_0\phi\right)^\dagger &= \frac{-i}{2\kappa}\left(PE_+ + QE_-\right)^\dagger \\ &= \frac{-i}{2\kappa}\left(P^*E_- + Q^*E_+\right)\end{aligned} \tag{200}$$

Then

$$
\begin{aligned}
\mathcal{E}^{a(3)} &= -\frac{iv^2}{2\kappa}\mathrm{tr}\left\{\phi^\dagger\left(D_0\phi\right)-\left(D_0\phi\right)^\dagger\phi\right\} \qquad (201)\\
&= -\frac{iv^2}{2\kappa}\mathrm{tr}\left\{\phi^\dagger\frac{i}{2\kappa}\left(PE_+ + QE_-\right)-\frac{-i}{2\kappa}\left(P^*E_- + Q^*E_+\right)\phi\right\}\\
&= -\frac{iv^2}{2\kappa}\left(\frac{i}{2\kappa}\right)\mathrm{tr}\left\{\left(\phi_1^*E_- + \phi_2^*E_+\right)\left(PE_+ + QE_-\right)+\left(P^*E_- + Q^*E_+\right)\left(\phi_1E_+ + \phi_2E_-\right)\right\}\\
&= \frac{v^2}{4\kappa^2}\left\{\phi_1^*P\mathrm{tr}\left(E_-E_+\right)\quad \text{(trace is 1)}\right.\\
&\quad +\phi_1^*Q\mathrm{tr}\left(E_-E_-\right)\quad \text{(trace is 0)}\\
&\quad +\phi_2^*P\mathrm{tr}\left(E_+E_+\right)\quad \text{(trace is 0)}\\
&\quad +\phi_2^*Q\mathrm{tr}\left(E_+E_-\right)\quad \text{(trace is 1)}\\
&\quad +P^*\phi_1\mathrm{tr}\left(E_-E_+\right)\quad \text{(trace is 1)}\\
&\quad +Q^*\phi_1\mathrm{tr}\left(E_+E_+\right)\quad \text{(trace is 0)}\\
&\quad +P^*\phi_2\mathrm{tr}\left(E_-E_-\right)\quad \text{(trace is 0)}\\
&\quad +Q^*\phi_2\mathrm{tr}\left(E_+E_-\right)\quad \text{(trace is 1)}
\end{aligned}
$$

We obtain

$$
\begin{aligned}
\mathcal{E}^{a(3)} &= \frac{v^2}{4\kappa^2}\left\{\phi_1^*P + P^*\phi_1 + \phi_2^*Q + Q^*\phi_2\right\} \qquad (202)\\
&= \frac{v^2}{2\kappa^2}\left\{\phi_1^*\left[v^2\phi_1 - 2\left(\rho_1-\rho_2\right)\phi_1\right]+\phi_2^*\left[v^2\phi_2 + 2\left(\rho_1-\rho_2\right)\phi_2\right]\right\}\\
&= \frac{v^2}{2\kappa^2}\left\{\rho_1\left[v^2 - 2\left(\rho_1-\rho_2\right)\right]+\rho_2\left[v^2 + 2\left(\rho_1-\rho_2\right)\right]\right\}\\
&= \frac{v^2}{2\kappa^2}\left[v^2\left(\rho_1+\rho_2\right) - 2\left(\rho_1-\rho_2\right)^2\right]
\end{aligned}
$$

$$
\mathcal{E}^{a(3)} = \frac{v^2}{2\kappa^2}\left[v^2\left(\rho_1+\rho_2\right) - 2\left(\rho_1-\rho_2\right)^2\right] \qquad (203)
$$

Now we collect all results

$$
\begin{aligned}
\mathcal{E}^{a} &= \mathcal{E}^{a(1)} + \mathcal{E}^{a(2)} + \mathcal{E}^{a(3)} \qquad (204)\\
&= -\frac{1}{\kappa^2}\left(\rho_1-\rho_2\right)^2\left[v^2 - 2\left(\rho_1+\rho_2\right)\right]\\
&\quad -\frac{1}{\kappa^2}\left(\rho_1-\rho_2\right)^2\left[v^2 - 2\left(\rho_1+\rho_2\right)\right]\\
&\quad +\frac{v^2}{2\kappa^2}\left[v^2\left(\rho_1+\rho_2\right) - 2\left(\rho_1-\rho_2\right)^2\right]
\end{aligned}
$$

there are three powers of $v$ and we separate the coefficients

$$\begin{aligned} v^0 &: \\ & \frac{1}{\kappa^2}(\rho_1-\rho_2)^2\, 2(\rho_1+\rho_2) + \frac{1}{\kappa^2}(\rho_1-\rho_2)^2\, 2(\rho_1+\rho_2) \\ &= \frac{4}{\kappa^2}(\rho_1-\rho_2)^2(\rho_1+\rho_2) \end{aligned} \tag{205}$$

$$\begin{aligned} v^2 &: \\ & -\frac{1}{\kappa^2}(\rho_1-\rho_2)^2 - \frac{1}{\kappa^2}(\rho_1-\rho_2)^2 - \frac{1}{2\kappa^2}2(\rho_1-\rho_2)^2 \\ &= -\frac{3}{\kappa^2}(\rho_1-\rho_2)^2 \end{aligned} \tag{206}$$

$$\begin{aligned} v^4 &: \\ & \frac{1}{2\kappa^2}(\rho_1+\rho_2) \end{aligned} \tag{207}$$

and the total expression is

$$\mathcal{E}^a = \frac{4}{\kappa^2}(\rho_1-\rho_2)^2(\rho_1+\rho_2) - \frac{3v^2}{\kappa^2}(\rho_1-\rho_2)^2 + \frac{v^4}{2\kappa^2}(\rho_1+\rho_2) \tag{208}$$

or

$$\mathcal{E}^a = \frac{1}{\kappa^2}(\rho_1-\rho_2)^2\left[4(\rho_1+\rho_2)-3v^2\right] + \frac{v^4}{2\kappa^2}(\rho_1+\rho_2) \tag{209}$$

Introducing the normalization

$$\rho \equiv \frac{\rho_1}{v^2/4} = \frac{v^2/4}{\rho_2} \tag{210}$$

$$\begin{aligned} \mathcal{E}^a &= \frac{1}{\kappa^2}\left(\frac{v^2}{4}\right)^2\left(\rho-\frac{1}{\rho}\right)^2\left(\frac{v^2}{4}\right)\left[4\left(\rho+\frac{1}{\rho}\right)-12\right] \\ &+\frac{v^4}{2\kappa^2}\left(\frac{v^2}{4}\right)\left(\rho+\frac{1}{\rho}\right) \end{aligned} \tag{211}$$

$$\begin{aligned} \mathcal{E}^a &= \frac{v^6}{16\kappa^2}\left(\rho-\frac{1}{\rho}\right)^2\left[\left(\rho+\frac{1}{\rho}\right)-3\right] \\ &+\frac{v^6}{8\kappa^2}\left(\rho+\frac{1}{\rho}\right) \end{aligned} \tag{212}$$

$$\begin{aligned}
\mathcal{E}^{a} &= \frac{v^{6}}{4\kappa^{2}}\left(\sinh\psi\right)^{2}\left[2\cosh\psi-3\right]+\frac{v^{6}}{4\kappa^{2}}\cosh\psi \qquad (213)\\
&= \frac{v^{6}}{4\kappa^{2}}\left[2\left(\sinh\psi\right)^{2}\cosh\psi-3\left(\sinh\psi\right)^{2}+\cosh\psi\right]\\
&= \frac{v^{6}}{4\kappa^{2}}\left\{2\cosh\psi\left[\left(\sinh\psi\right)^{2}+1\right]-\cosh\psi-3\left[\left(\sinh\psi\right)^{2}+1\right]+3\right\}\\
&= v^{2}\left(\frac{v^{2}}{\kappa}\right)^{2}\frac{1}{4}\left[2\left(\cosh\psi\right)^{3}-3\left(\cosh\psi\right)^{2}-\cosh\psi+3\right]
\end{aligned}$$

As in the case of the Eq.(167) we will note that the expression must be integrated over the plane, which removes from the coefficient the dimensional factor

$$\left(\frac{v^{2}}{\kappa}\right)^{2}=\frac{1}{\rho_{s}^{2}}$$

and the dimension of the *energy* is given by $v^2 = \Omega_{ci}$.

# 8 Discussion on the versions of the SD equations

The possibility of formulating the expression of the energy as a sum of squared terms plus a (additional) term with topological significance (known as Bogomolnyi formulation) is fundamental for the self-duality. In our case, the CHM fluid/plasma cannot associate a topological significance to the additional energy, a characteristic signalized by **Lee 1991** and by **Dunne**. This induces a certain imprecision in the choice of the way of separating the squared terms, with consequences on the form of the equations, etc. This aspect will only be discussed briefly here, with the only intention to compare few possible choices.

We have shown how to derive two versions of writting the total energy of the system as a sum of squared terms plus an additional (*residual*) term, while this one has no topological significance. After adopting the algebraic ansatz we arrive at two different equations for the scalar function $\psi$ which we associate with the physical streamfunction of the CHM fluid.

The choice of the version that has the correct physical significance should be done on the basis of the supersymmetric invariance of the extended field theoretical model. However, even from this advanced point of view, we can expect at most an indication which will not be applicable directly to our problem. The field theroetical model for the CHM equation shows significant differences compared with topological theories in $2D$. The SD equations, in both versions, lead to time dependent solutions, therefore the stationarity typical for the solutions obtained from the Bogomonlyi form in other theories is here lost. The topological aspect is also lost, the additional energy, in both versions, is not proportional with the total winding number induced by the vortices present in the plane. We must note however that the point of view that results from the SUSY extension of the theory (a natural extension) favorizes the Eq.(148) or possibly, as mentioned below, the Abelian version, Eq.(214). This even if the additional energy does not have a topological meaning.

One possible help comes from looking at the theory of the CHM fluid as being a development of the theory for the Euler fluid. Since the asymptotic states of the latter are goverend by the *sinh*-Poisson equation, we expect that the nonlinear term of the equation for CHM fluid to be of a similar nature. Or, we see that the first set of SD equations leads to a sign which is opposite to the one appearing in the *sinh*-Poisson equation. The physical form and properties of the solutions would be, in that case, completely different. Or, one expects that an ideal fluid without an intrinsic length will transform smootly into a fluid which has an intrinsic length, as the CHM fluid. Although this is not an argument, we take this as a sort of indication that the second choice is more appropriate and we adopt Eq.(182) as the equation governing the asymptotic states of the CHM fluid.

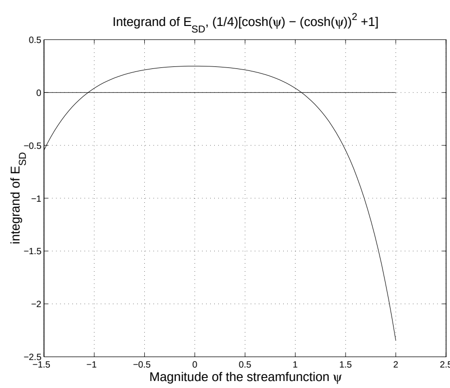


Figure 1: The integrand of the energy $\mathcal{E}_{SD}$ which is the additional term in the *first* Bogomolnyi form.

A different approach can be developed on the basis of the analysis made by **Lee (1991)** of the first set of equations at SD. After adopting the algebraic ansatz, Lee finds that the boundary conditions on the two functions $\phi_1$ and $\phi_2$ lead to restrictive choices : basically it results that the second ladder

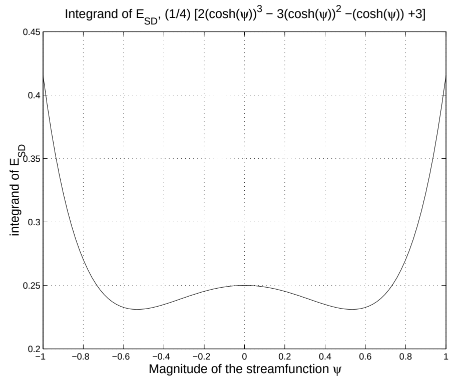


Figure 2: The integrand of the energy $\mathcal{E}_{SD}$ which is the additional term in the *second* Bogomolnyi form.

generator should not be present in the algebraic form of $\phi$. Then the equation resulting from the first set of SD equations is identical with the equation which is derived from the Abelian version of the theory

$$\Delta\psi = \exp(2\psi) - \exp(\psi) \tag{214}$$

Contrary to the Equation (148), Eq.(214) has physically interesting solutions, consisting of rings of vorticity. The way this theory can arise from the *Abelian dominance* in the full field-theoretical (FT) model of the CHM fluid and the physical consequences of this theory will be analysed elsewhere.

Another possible criterion for the choice of one or another Bogomolnyi form is the behavior of the additional energy. In some cases, the solution of the equations at self-duality (*i.e.* by taking to zero the squared terms) show a higher energy due to the residual term, when compared with other choices. In particular we have found that the following form has apparently lower energy for many of the solutions of its associated self-duality equation, in practical applications like the tropical cyclone, etc.:

$$\Delta\psi + \frac{1}{2}\sinh\psi\,(\cosh\psi - 1) = 0 \tag{215}$$

or a version in which the harmonic function is a constant different of 1, *i.e.* taking

$$\rho \equiv \frac{\rho_1}{v^2/(4p)} \tag{216}$$

instead of Eq.(176) we have

$$\Delta\psi + \frac{1}{2p^2}\sinh\psi\,(\cosh\psi - p) = 0 \tag{217}$$

This form arises by taking

$$D_0\phi + \lambda \frac{i}{2\kappa}\left(\left[\left[\phi, \phi^\dagger\right], \phi\right] - v^2\phi\right)$$

in the Eq.(75), with

$$\lambda = \frac{1}{2}$$

and has an associated *residual*, non-topological energy

$$\begin{aligned} \mathcal{E}_3^a &= v^2\left(\frac{v^2}{\kappa}\right)^2 \\ &\quad \times \frac{1}{4}\left[\frac{11}{8}(\sinh\psi)^2(-2+\cosh\psi) + \frac{3}{8}\cosh\psi\right] \end{aligned}$$

which is shown in the Fig.(3).

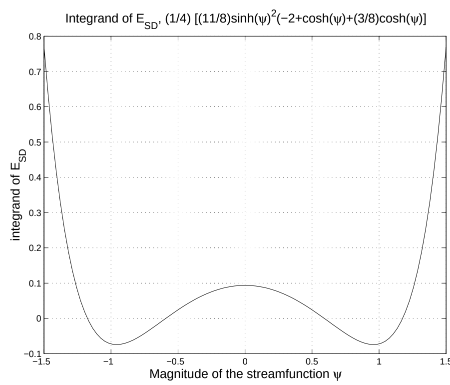


Figure 3: The integrand of the energy $\mathcal{E}_{SD}$ which is the additional term in the *third* (for $\lambda = 1/2$) Bogomolnyi form.

# 9 Various forms of the equation

The crucial step in the derivation of the form Eq.(148) or Eq.(182) is the choice of a solution for the equation (139)

$$\Delta \ln(\rho_1\rho_2) = 0 \tag{218}$$

with the consequence that the product $\rho_1\rho_2$ is the exponential of a harmonic function. This leads to the conclusion that the asymptotic states of the CHM fluid can be described by a *class of differential equations*, parametrized by the harmonic functions. Without developing this aspect here we mention that the fact that instead one equation we find a class of equations has a simple physical significance. The difference resides in the background motion and the equations from this class describes vortices on a background of fluid/plasma motion which has zero *physical* vorticity. Then the "stream-functions" $\psi_1$ and $\psi_2$ that can be introduced separately for $\rho_1$ respectively for $\rho_2$ will differ by a function whose Laplacean is zero, *i.e.* a potential flow. We note that by *physical* vorticity we understand the vorticity perturbation above the condensate background, *i.e.* above $\Omega_{ci}$.

In the following we mention few examples illustrating this freedom of choice of the zero-vorticity background flow.

The separation of variables in polar coordinates

$$\begin{aligned} \Delta\Phi &= 0 \\ \Phi(r,\theta) &\equiv R(r)\Theta(\theta) \end{aligned} \tag{219}$$

with

$$\begin{aligned} \frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \frac{n^2}{r^2}R &= 0 \\ \frac{d^2\Theta}{d\theta^2} + n^2\Theta &= 0 \end{aligned} \tag{220}$$

## 9.1 Solution 1

The first choice was

$$\ln(\rho_1\rho_2) = 0 \tag{221}$$

with the consequence

$$\rho_1\rho_2 = 1 \tag{222}$$

## 9.2 Solution 2

A different (almost arbitrary) choice

$$\Delta h = 0 \tag{223}$$

$$h = \exp(x)\sin(y) \tag{224}$$

Then

$$\ln(\rho_1\rho_2) = \exp(x)\sin(y) \tag{225}$$

$$\rho_1\rho_2 = \exp[\exp(x)\sin(y)] \tag{226}$$

## 9.3 Solution 3 (general solution in cylindrical coordinates)

A different choice for the cylindrical harmonic function $\Phi = \ln(\rho_1 \rho_2)$

$$\begin{aligned}\Phi(r,\theta) &= \left[Ar^n + B\frac{1}{r^n}\right] \\ &\quad \times \left[C\cos(n\theta) + D\sin(n\theta)\right]\end{aligned} \tag{227}$$

and for $n = 0$,

$$\begin{aligned}\Phi(r,\theta) &= (A\theta + B) \\ &\quad \times \left[C\ln r + D\right]\end{aligned} \tag{228}$$

### 9.3.1 Example polar 1

For example,

$$\Phi(r,\theta) = ar\cos\theta + b \tag{229}$$

Consider the choice with a particular value for $b$

$$\ln(\rho_1\rho_2) = ar\cos\theta + b \tag{230}$$

$$\begin{aligned}\rho_1\rho_2 &= \exp(ar\cos\theta + b) \\ &= \frac{v^4}{16p^2}\exp(ar\cos\theta)\end{aligned} \tag{231}$$

for $p$ a positive constant. Define $\rho$ as before

$$\begin{aligned}\rho &\equiv \frac{\rho_1}{\frac{v^2}{4p}\exp\left(\frac{1}{2}ar\cos\theta\right)} \\ &= \frac{1}{\rho_2}\frac{v^2}{4p}\exp\left(\frac{1}{2}ar\cos\theta\right)\end{aligned} \tag{232}$$

$$\begin{aligned}&-\frac{1}{2}\Delta\ln\left[\rho\frac{v^2}{4p}\exp\left(\frac{1}{2}ar\cos\theta\right)\right] \\ &= -\frac{1}{\kappa^2}(\rho_1 - \rho_2)\left[2(\rho_1+\rho_2) - v^2\right]\end{aligned} \tag{233}$$

where

$$\begin{aligned}RHS &\equiv -\frac{1}{\kappa^2}(\rho_1-\rho_2)\left[2(\rho_1+\rho_2)-v^2\right] \\ &= -\frac{1}{\kappa^2}\left[\rho\frac{v^2}{4p}\exp\left(\frac{1}{2}ar\cos\theta\right) - \frac{1}{\rho}\frac{v^2}{4p}\exp\left(\frac{1}{2}ar\cos\theta\right)\right] \\ &\quad \times\left\{2\left[\rho\frac{v^2}{4p}\exp\left(\frac{1}{2}ar\cos\theta\right) - \frac{1}{\rho}\frac{v^2}{4p}\exp\left(\frac{1}{2}ar\cos\theta\right)\right] - v^2\right\}\end{aligned} \tag{234}$$

$$RHS = -\frac{1}{\kappa^2}\left[\frac{v^2}{4p}\exp\left(\frac{1}{2}ar\cos\theta\right)\right]\left(\rho-\frac{1}{\rho}\right) \times\frac{v^2}{4p}\exp\left(\frac{1}{2}ar\cos\theta\right)\left\{2\left(\rho+\frac{1}{\rho}\right)-4p\exp\left(-\frac{1}{2}ar\cos\theta\right)\right\} \tag{235}$$

$$RHS = -\frac{1}{4p^2}\left(\frac{v^2}{\kappa}\right)^2\exp\left(ar\cos\theta\right) \times\left(\rho-\frac{1}{\rho}\right) \times\left[\frac{1}{2}\left(\rho+\frac{1}{\rho}\right)-p\exp\left(-\frac{1}{2}ar\cos\theta\right)\right] \tag{236}$$

and the Right Hand Side

$$\begin{aligned} LHS &= -\frac{1}{2}\Delta\ln\left[\rho\frac{v^2}{4p}\exp\left(\frac{1}{2}ar\cos\theta\right)\right] \\ &= -\frac{1}{2}\Delta\ln\rho+\Delta\ln\left(\frac{v^2}{4p}\right) \\ &\quad -\frac{1}{2}\Delta\left(\frac{1}{2}ar\cos\theta\right) \\ &= -\frac{1}{2}\Delta\rho \end{aligned} \tag{237}$$

The equation is

$$\begin{aligned} &-\frac{1}{2}\Delta\rho \\ = &-\frac{1}{4p^2}\left(\frac{v^2}{\kappa}\right)^2\exp\left(ar\cos\theta\right)\left(\rho-\frac{1}{\rho}\right)\left[\frac{1}{2}\left(\rho+\frac{1}{\rho}\right)-p\exp\left(-\frac{1}{2}ar\cos\theta\right)\right] \end{aligned} \tag{238}$$

However, since we will note

$$\ln\rho\equiv\psi \tag{239}$$

it would have been easier to do that from the beginning.

The equation

$$\begin{aligned} \exp\left(\psi\right) &\equiv \frac{\rho_1}{\frac{v^2}{4p}\exp\left(\frac{1}{2}ar\cos\theta\right)} \\ &= \frac{1}{\rho_2}\frac{v^2}{4p}\exp\left(\frac{1}{2}ar\cos\theta\right) \end{aligned} \tag{240}$$

generates explicit forms for $\rho_{1,2}$ that will be inserted in the equation

$$\begin{aligned} \rho_1 &= \frac{v^2}{4p} \exp\left(\psi + \frac{1}{2} ar \cos\theta\right) \\ \rho_2 &= \frac{v^2}{4p} \exp\left(-\psi + \frac{1}{2} ar \cos\theta\right) \end{aligned} \tag{241}$$

with the same result.

The equation now reads

$$-\Delta\psi + \frac{1}{p^2} \exp\left(ar\cos\theta\right) \sinh\psi \left[\cosh\psi - p \exp\left(-\frac{1}{2} ar\cos\theta\right)\right] = 0 \tag{242}$$

where the space unit is now $\kappa/v^2 \equiv \rho_s$.

### 9.3.2 Example polar 2

The second simple polar choice of $\Delta\Phi = 0$

$$\Phi\left(r,\theta\right) = a\ln r + b \tag{243}$$

$$\begin{aligned} \ln\left(\rho_1\rho_2\right) &= a\ln r + b \\ \rho_1\rho_2 &= \exp\left(a\ln r + b\right) \\ &= \exp\left(b\right) r^a \end{aligned} \tag{244}$$

The normalization suggets

$$\rho_1\rho_2 = \frac{v^4}{16p^2} r^a \tag{245}$$

$$\begin{aligned} \rho &\equiv \frac{\rho_1}{\frac{v^2}{4p} r^{a/2}} \\ &= \frac{1}{\rho_2} \frac{v^2}{4p} r^{a/2} \end{aligned} \tag{246}$$

$$\begin{aligned} \rho_1 &= \rho \frac{v^2}{4p} r^{a/2} \\ \rho_2 &= \frac{1}{\rho} \frac{v^2}{4p} r^{a/2} \end{aligned} \tag{247}$$

The equation is

$$\begin{aligned}
& -\frac{1}{2}\Delta\ln\rho-\frac{1}{2}\Delta\ln\left(r^{a/2}\right) \\
= & -\frac{1}{\kappa^2}\left[\rho\frac{v^2}{4p}r^{a/2}-\frac{1}{\rho}\frac{v^2}{4p}r^{a/2}\right] \\
& \times\left\{2\left[\rho\frac{v^2}{4p}r^{a/2}-\frac{1}{\rho}\frac{v^2}{4p}r^{a/2}\right]-v^2\right\}
\end{aligned} \tag{248}$$

$$\begin{aligned}
& -\frac{1}{2}\Delta\ln\rho \\
= & -\frac{1}{4p^2}\left(\frac{v^2}{\kappa}\right)^2 r^a\left(\rho-\frac{1}{\rho}\right) \\
& \times\left[\frac{1}{2}\left(\rho+\frac{1}{\rho}\right)-pr^{-a/2}\right]
\end{aligned} \tag{249}$$

After introducing the substitution $\ln\rho\equiv\psi$ and measuring the space in units of $\kappa/v^2=\rho_s$ we have

$$-\Delta\psi+\frac{1}{p^2}r^a\sinh\psi\left(\cosh\psi-\frac{p}{r^a}\right)=0 \tag{250}$$

or, better

$$-\Delta\psi+\frac{1}{p^2}\sinh\psi\left(r^a\cosh\psi-p\right)=0 \tag{251}$$

# 10 Discussion on the physical meaning of the model

## 10.1 The short range of the potential

It is considered that the scalar field is very close to the vacuum value

$$\phi\sim v \tag{252}$$

We calculate the current in the region of vanishing space-variation.

$$\begin{aligned}
J^\mu &= -i\left(\left[\phi^\dagger,D^\mu\phi\right]-\left[\left(D^\mu\phi\right)^\dagger,\phi\right]\right) \\
&= -i\left\{\phi^\dagger\left(\partial^\mu\phi+\left[A^\mu,\phi\right]\right)-\left(\partial^\mu\phi+\left[A^\mu,\phi\right]\right)\phi^\dagger\right. \\
&\quad\left.-\left(\partial_\mu\phi^\dagger+\left[\phi^\dagger,A^{\mu\dagger}\right]\right)\phi+\phi\left(\partial_\mu\phi^\dagger+\left[\phi^\dagger,A^{\mu\dagger}\right]\right)\right\} \\
&= -i\left\{\phi^\dagger\left(\partial^\mu\phi\right)-\left(\partial^\mu\phi\right)\phi^\dagger-\left(\partial_\mu\phi^\dagger\right)\phi+\phi\left(\partial_\mu\phi^\dagger\right)\right. \\
&\quad\left.+\phi^\dagger\left[A^\mu,\phi\right]-\left[A^\mu,\phi\right]\phi^\dagger-\left[\phi^\dagger,A^{\mu\dagger}\right]\phi+\phi\left[\phi^\dagger,A^{\mu\dagger}\right]\right\}
\end{aligned} \tag{253}$$

Since we consider that the field $\phi$ is almost constant (and equal to $v$) we can negelct all terms on the first line and obtain

$$J^{\mu} \simeq -i\left(\left[\phi^{\dagger},\left[A^{\mu},\phi\right]\right]+\left[\phi,\left[\phi^{\dagger},A^{\mu\dagger}\right]\right]\right) \tag{254}$$

Let us consider the explicit expressions for the fields

$$\begin{aligned} A_x &= A^x = \frac{1}{2}\left(a-a^*\right)H\ ,\ A^{x\dagger}=\frac{1}{2}\left(a^*-a\right)H \\ A_y &= A^y = \frac{1}{2i}\left(a+a^*\right)H\ ,\ A^{y\dagger}=-\frac{1}{2i}\left(a^*+a\right)H \end{aligned} \tag{255}$$

and

$$\begin{aligned} \phi &= \phi_1 E_+ + \phi_2 E_- \\ \phi^{\dagger} &= \phi_1^* E_- + \phi_2^* E_+ \end{aligned} \tag{256}$$

Then, for $x$, the first part of the formula for the current $j^x$ is

$$\begin{aligned} \left[\phi^{\dagger},\left[A^x,\phi\right]\right] &= \left[\phi^{\dagger},\frac{1}{2}\left(a-a^*\right)\left(\phi_1\left[H,E_+\right]+\phi_2\left[H,E_-\right]\right)\right] \\ &= \frac{1}{2}\left(a-a^*\right)\left[\phi^{\dagger},\left(\phi_1 2E_+ - \phi_2 2E_-\right)\right] \\ &= \left(a-a^*\right)\left[\phi_1^*E_- + \phi_2^*E_+,\phi_1E_+ - \phi_2E_-\right] \\ &= \left(a-a^*\right)\left\{\phi_1^*\phi_1\left[E_-,E_+\right]-\phi_2^*\phi_2\left[E_+,E_-\right]\right\} \\ &= \left(a-a^*\right)\left(-\left|\phi_1\right|^2 H - \left|\phi_2\right|^2 H\right) \\ &= -\left(a-a^*\right)\left(\rho_1+\rho_2\right)H \end{aligned} \tag{257}$$

and the second part

$$\begin{aligned} \left[\phi,\left[\phi^{\dagger},A^{x\dagger}\right]\right] &= \left[\phi,\left[\phi_1^*E_- + \phi_2^*E_+,\frac{1}{2}\left(a^*-a\right)H\right]\right] \\ &= \frac{1}{2}\left(a^*-a\right)\left[\phi,\phi_1^*\left[E_-,H\right]+\phi_2^*\left[E_+,H\right]\right] \\ &= \frac{1}{2}\left(a^*-a\right)\left[\phi_1E_+ + \phi_2E_-,\phi_1^*2E_- - \phi_2^*2E_+\right] \\ &= \left(a^*-a\right)\left\{\phi_1\phi_1^*\left[E_+,E_-\right]-\phi_2\phi_2^*\left[E_-,E_+\right]\right\} \\ &= \left(a^*-a\right)\left(\left|\phi_1\right|^2 H + \left|\phi_2\right|^2 H\right) \\ &= \left(a^*-a\right)\left(\rho_1+\rho_2\right)H \end{aligned} \tag{258}$$

The $x$ component of the current is

$$\begin{aligned} J_x &\simeq -i\left\{-\left(a-a^*\right)\left(\rho_1+\rho_2\right)H + \left(a^*-a\right)\left(\rho_1+\rho_2\right)H\right\} \\ &= 2i(a-a^*)\left(\rho_1+\rho_2\right)H \end{aligned} \tag{259}$$

For the far regions we take the value

$$\rho_1 + \rho_2 \sim v^2/2 \tag{260}$$

We return to the potential notation, $(a - a^*)H = 2A_x$,

$$J_x = 2iv^2 A_x \tag{261}$$

Analogous for the $y$ component of the current,

$$\begin{aligned}
\left[\phi^\dagger, [A^y, \phi]\right] &= \left[\phi^\dagger, \frac{1}{2i}(a+a^*)\left(\phi_1 [H, E_+] + \phi_2 [H, E_-]\right)\right] \\
&= \frac{1}{2i}(a+a^*)\left[\phi^\dagger, (\phi_1 2E_+ - \phi_2 2E_-)\right] \\
&= -i(a+a^*)\left[\phi_1^* E_- + \phi_2^* E_+, \phi_1 E_+ - \phi_2 E_-\right] \\
&= -i(a+a^*)\left\{\phi_1^*\phi_1 [E_-, E_+] - \phi_2^*\phi_2 [E_+, E_-]\right\} \\
&= -i(a+a^*)\left(-|\phi_1|^2 H - |\phi_2|^2 H\right) \\
&= i(a+a^*)(\rho_1+\rho_2) H
\end{aligned} \tag{262}$$

and the second part

$$\begin{aligned}
\left[\phi, \left[\phi^\dagger, A^{y\dagger}\right]\right] &= \left[\phi, \left[\phi_1^* E_- + \phi_2^* E_+, -\frac{1}{2i}(a^*+a) H\right]\right] \\
&= -\frac{1}{2i}(a^*+a)\left[\phi, \phi_1^* [E_-, H] + \phi_2^* [E_+, H]\right] \\
&= -\frac{1}{2i}(a^*+a)\left[\phi_1 E_+ + \phi_2 E_-, \phi_1^* 2E_- - \phi_2^* 2E_+\right] \\
&= i(a^*+a)\left\{\phi_1\phi_1^* [E_+, E_-] - \phi_2\phi_2^* [E_-, E_+]\right\} \\
&= i(a^*+a)\left(|\phi_1|^2 H + |\phi_2|^2 H\right) \\
&= i(a^*+a)(\rho_1+\rho_2) H
\end{aligned} \tag{263}$$

The $x$ component of the current is

$$\begin{aligned}
J_y &\simeq -i\left\{i(a+a^*)(\rho_1+\rho_2) H + i(a^*+a)(\rho_1+\rho_2) H\right\} \\
&= 2(a+a^*)(\rho_1+\rho_2) H
\end{aligned} \tag{264}$$

As before, we replace here $\rho_1 + \rho_2 \sim v^2$ and $(a + a^*)H = 2iA_y$,

$$J_y = 2iv^2 A_y \tag{265}$$

We take the temporal component of the potential in the form

$$\begin{aligned}
A^0 &= bH \\
A^{0\dagger} &= \left(A^0\right)^{*T} = b^* H
\end{aligned} \tag{266}$$

Then the temporal component of the current density is (cf. Eq.(254))

$$J^0 \simeq -i\left\{\left[\phi^\dagger,\left[A^0,\phi\right]\right]+\left[\phi,\left[\phi^\dagger,A^{0\dagger}\right]\right]\right\} \tag{267}$$

and we calculate again the terms, with the particular choice Eq.(266)

$$\begin{aligned}
\left[\phi^\dagger,\left[A^0,\phi\right]\right] &= \left[\phi^\dagger, b\left(\phi_1\left[H,E_+\right]+\phi_2\left[H,E_-\right]\right)\right] \\
&= b\left[\phi^\dagger,\left(\phi_1 2E_+ - \phi_2 2E_-\right)\right] \\
&= 2b\left[\phi_1^* E_- + \phi_2^* E_+, \phi_1 E_+ - \phi_2 E_-\right] \\
&= 2b\left\{\phi_1^*\phi_1\left[E_-,E_+\right] - \phi_2^*\phi_2\left[E_+,E_-\right]\right\} \\
&= 2b\left(-\left|\phi_1\right|^2 H - \left|\phi_2\right|^2 H\right) \\
&= -2b\left(\rho_1+\rho_2\right) H
\end{aligned} \tag{268}$$

and the second part

$$\begin{aligned}
\left[\phi,\left[\phi^\dagger,A^{0\dagger}\right]\right] &= \left[\phi,\left[\phi_1^* E_- + \phi_2^* E_+, b^* H\right]\right] \\
&= b^*\left[\phi, \phi_1^*\left[E_-,H\right]+\phi_2^*\left[E_+,H\right]\right] \\
&= b^*\left[\phi_1 E_+ + \phi_2 E_-, \phi_1^* 2E_- - \phi_2^* 2E_+\right] \\
&= 2b^*\left\{\phi_1\phi_1^*\left[E_+,E_-\right] - \phi_2\phi_2^*\left[E_-,E_+\right]\right\} \\
&= 2b^*\left(\left|\phi_1\right|^2 H + \left|\phi_2\right|^2 H\right) \\
&= 2b^*\left(\rho_1+\rho_2\right) H
\end{aligned} \tag{269}$$

The result for the current in the approximation of the constant background is

$$\begin{aligned}
J^0 &\simeq -i\left\{\left[\phi^\dagger,\left[A^0,\phi\right]\right]+\left[\phi,\left[\phi^\dagger,A^{0\dagger}\right]\right]\right\} \\
&= -i\left\{-2b\left(\rho_1+\rho_2\right)H + 2b^*\left(\rho_1+\rho_2\right)H\right\} \\
&= 2i\left(b-b^*\right)\left(\rho_1+\rho_2\right)H
\end{aligned} \tag{270}$$

We then obtain

$$J^0 = 2iv^2 A^0 \tag{271}$$

Then

$$J^\mu \equiv \left(J^0, J^x, J^y\right) = 2iv^2\left[A^0, A^x, A^y\right] \tag{272}$$

or

$$J^\mu \simeq 2iv^2 A^\mu \tag{273}$$

With this value of the current density we return to equation connecting the gauge field with the matter current. The equation is

$$-\kappa\varepsilon^{\mu\nu\rho}F_{\nu\rho} = iJ^\mu \tag{274}$$

To transfer the antisymmetric tensor $\varepsilon^{\mu\nu\rho}$ in the other side, we multiply by $\varepsilon_{\mu\sigma\tau}$ and sum over repeated indices

$$\begin{aligned}
\varepsilon_{\mu\sigma\tau}\varepsilon^{\mu\nu\rho}F_{\nu\rho} &= -\frac{i}{\kappa}\varepsilon_{\mu\sigma\tau}J^{\mu} \\
\left(\delta_{\sigma\nu}\delta_{\tau\rho}-\delta_{\sigma\rho}\delta_{\tau\nu}\right)F_{\nu\rho} &= -\frac{i}{\kappa}\varepsilon_{\mu\sigma\tau}J^{\mu} \\
F_{\sigma\tau}-F_{\tau\sigma} &= -\frac{i}{\kappa}\varepsilon_{\mu\sigma\tau}J^{\mu} \\
F_{\sigma\tau} &= -\frac{i}{2\kappa}\varepsilon_{\mu\sigma\tau}J^{\mu}
\end{aligned} \tag{275}$$

A direct relation with the previous result is obtained taking the explicit form of $J^{\mu}$ from Eq.(273)

$$\begin{aligned}
F_{\sigma\tau} &= -\frac{i}{2\kappa}\varepsilon_{\mu\sigma\tau}J^{\mu} \\
&= \frac{v^2}{\kappa}\varepsilon_{\mu\sigma\tau}A^{\mu}
\end{aligned} \tag{276}$$

Introducing the expression for the field

$$F_{\sigma\tau}=\partial_{\sigma}A_{\tau}-\partial_{\tau}A_{\sigma}=-\frac{i}{2\kappa}\varepsilon_{\mu\sigma\tau}J^{\mu} \tag{277}$$

we apply the derivative operator $\partial_{\tau}$ and sum over the index $\tau$

$$\begin{aligned}
& \partial_{\tau}\partial_{\sigma}A_{\tau}-\partial_{\tau}\partial_{\tau}A_{\sigma} \\
= & -\frac{i}{2\kappa}\varepsilon_{\mu\sigma\tau}\partial_{\tau}J^{\mu} \\
= & -\frac{i}{2\kappa}\varepsilon_{\mu\sigma\tau}\partial_{\tau}\left(2iv^2A^{\mu}\right) \;\; \text{(from Eq.(273))} \\
= & \frac{v^2}{\kappa}\varepsilon_{\mu\sigma\tau}\partial_{\tau}A^{\mu}
\end{aligned} \tag{278}$$

The term on the right hand side is

$$\varepsilon_{\sigma\tau\mu}\partial_{\tau}A^{\mu}=\frac{1}{2}\varepsilon_{\sigma\tau\mu}F_{\tau}^{\mu} \tag{279}$$

and here we replace, from Eq.(276)

$$F_{\tau}^{\mu}=g^{\mu\alpha}F_{\tau\alpha}=g^{\mu\alpha}\frac{v^2}{\kappa}\varepsilon_{\tau\alpha\eta}A^{\eta} \tag{280}$$

Further the product of the two antisymmetric tensors $\varepsilon$ is expanded

$$\begin{aligned}
\partial_\tau \partial_\sigma A_\tau - \partial_\tau \partial_\tau A_\sigma &= \frac{v^2}{\kappa} \varepsilon_{\mu\sigma\tau} \partial_\tau A^\mu \text{ (from Eq.(278))} \\
&= \frac{v^2}{\kappa} \frac{1}{2} \varepsilon_{\sigma\tau\mu} F_\tau^\mu \text{ (from Eq.(279))} \\
&= \frac{v^2}{\kappa} \frac{1}{2} \varepsilon_{\sigma\tau\mu} g^{\mu\alpha} \frac{v^2}{\kappa} \varepsilon_{\tau\alpha\eta} A^\eta \text{ (from Eq.(280))} \\
&= \frac{1}{2} \left( \frac{v^2}{\kappa} \right)^2 g^{\mu\alpha} \varepsilon_{\sigma\tau\mu} \varepsilon_{\tau\alpha\eta} A^\eta
\end{aligned} \tag{281}$$

The explicit expression for the sum

$$\begin{aligned}
& g^{\mu\alpha} \varepsilon_{\sigma\tau\mu} \varepsilon_{\tau\alpha\eta} \\
=\; & g^{00} \varepsilon_{\sigma\tau 0} \varepsilon_{\tau 0\eta} + g^{11} \varepsilon_{\sigma\tau 1} \varepsilon_{\tau 1\eta} + g^{22} \varepsilon_{\sigma\tau 2} \varepsilon_{\tau 2\eta} \\
=\; & g^{00} \left( \varepsilon_{\sigma 10} \varepsilon_{10\eta} + \varepsilon_{\sigma 20} \varepsilon_{20\eta} \right) \\
& + g^{11} \left( \varepsilon_{\sigma 01} \varepsilon_{01\eta} + \varepsilon_{\sigma 21} \varepsilon_{21\eta} \right) \\
& + g^{22} \left( \varepsilon_{\sigma 02} \varepsilon_{02\eta} + \varepsilon_{\sigma 12} \varepsilon_{12\eta} \right) \\
=\; & g^{00} \left( \delta_{\sigma 2} \delta_{\eta 2} + \delta_{\sigma 1} \delta_{\eta 1} \right) \\
& + g^{11} \left( \delta_{\sigma 2} \delta_{\eta 2} + \delta_{\sigma 0} \delta_{\eta 0} \right) \\
& + g^{22} \left( \delta_{\sigma 1} \delta_{\eta 1} + \delta_{\sigma 0} \delta_{\eta 0} \right) \\
=\; & -\delta_{\sigma 2} \delta_{\eta 2} - \delta_{\sigma 1} \delta_{\eta 1} \\
& + \delta_{\sigma 2} \delta_{\eta 2} + \delta_{\sigma 0} \delta_{\eta 0} + \delta_{\sigma 1} \delta_{\eta 1} + \delta_{\sigma 0} \delta_{\eta 0} \\
=\; & \delta_{\sigma 0} \delta_{\eta 0} + \delta_{\sigma 0} \delta_{\eta 0} \\
=\; & 2 \delta_{\sigma 0} \delta_{\eta 0}
\end{aligned} \tag{282}$$

This is replaced in the Eq.(281)

$$\begin{aligned}
\partial_\tau \partial_\sigma A_\tau - \partial_\tau \partial_\tau A_\sigma &= \frac{1}{2} \left( \frac{v^2}{\kappa} \right)^2 g^{\mu\alpha} \varepsilon_{\sigma\tau\mu} \varepsilon_{\tau\alpha\eta} A^\eta \\
&= \frac{1}{2} \left( \frac{v^2}{\kappa} \right)^2 2 \delta_{\sigma 0} \delta_{\eta 0} A^\eta \\
&= \left( \frac{v^2}{\kappa} \right)^2 \delta_{\sigma 0} A^0 \\
&= -\left( \frac{v^2}{\kappa} \right)^2 \delta_{\sigma 0} A_0
\end{aligned}$$

Taking the gauge condition

$$\partial_\tau A_\tau = 0 \tag{283}$$

it results the equation

$$\partial_\tau \partial_\tau A_0 - \left(\frac{v^2}{\kappa}\right)^2 A_0 = 0 \tag{284}$$

The solution of this equation is, in cylindrical geometry,

$$A_0(r) = K_0(mr)$$

From here we conclude that the mass of the photon is

$$m = \frac{v^2}{\kappa} \tag{285}$$

and this mass is generated via the Higgs mechanism adapted to the Chern-Simons action. The photon gets a mass because it moves in a background where the scalar field is equal with the vacuum value, nonzero value.

From physical considerations, we know that

$$m = \frac{v^2}{\kappa} = \frac{1}{\rho_s} \tag{286}$$

## 10.2 A bound on the energy

We can express the total energy of the system as the space integral of the *time-time* component of the energy-momentum tensor

$$\mathcal{E}_{tot} = \int d^2 r T^{00} \tag{287}$$

A useful formula (Gradshtein 6.561 formula16, [23]) is

$$\int_0^\infty x^\mu K_\nu(ax)\, dx = 2^{\mu-1} a^{-\mu-1} \Gamma\left(\frac{1+\mu+\nu}{2}\right) \Gamma\left(\frac{1+\mu-\nu}{2}\right) \tag{288}$$

for

$$\begin{aligned} \mathrm{Re}(\mu + 1 \pm \nu) &> 0 \\ \mathrm{Re}\, a &> 0 \end{aligned} \tag{289}$$

Then, for $\mu = 1$ and $a = 1$, $\nu = 0$,

$$\begin{aligned} \int_0^\infty x K_0(x)\, dx &= [\Gamma(1)]^2 \\ &= 1 \end{aligned} \tag{290}$$

(cf. Gradshtein 8.338). This must be used with Eq.(10) to calculate the total energy of a system of vortices in plane.

$$\begin{aligned}
W^{cont} &= 2\pi \int d^2 r \omega^2 K \left(m \left|\mathbf{r}_1 - \mathbf{r}_2\right|\right) \quad (291) \\
&= \omega^2 4\pi^2 \frac{1}{m^2} \int_0^\infty (mr)\, d(mr)\, K_0(mr) \\
&= 4\pi^2 \frac{\omega^2}{m^2} = 4\pi^2 \omega^2 \rho_s^2
\end{aligned}$$

Then we have that the $2D$ integral over the plane of the continuum version of the energy of a system with discrete vortices is constant multiplying the square of the elementary quantity of vorticity, which was before associated to each elementary vortex. This corresponds actually to the value of the energy in the field theoretical model, precisely at the self-dual limit, Eq.(68).

## 10.3 Calculation of the flux of the "magnetic field" through the plane

Start with the second differential equation of self-duality Eq.(70)

$$F_{+-} = \frac{1}{\kappa^2} \left[ v^2 \phi - \left[ \left[ \phi, \phi^\dagger \right], \phi \right], \phi^\dagger \right] \quad (292)$$

This has been calculated previously, with the result

$$\left[ v^2 \phi - \left[ \left[ \phi, \phi^\dagger \right], \phi \right], \phi^\dagger \right] = \left( v^2 - 2\left(\rho_1 + \rho_2\right)\right) \left(\rho_1 - \rho_2\right) H \quad (293)$$

Then

$$F_{+-} = \frac{1}{\kappa^2} \left( v^2 - 2\left(\rho_1 + \rho_2\right)\right) \left(\rho_1 - \rho_2\right) H \quad (294)$$

We return to Eq.(48) (the Gauss law constraint)

$$F_{12} = \frac{1}{2\kappa} \left( \left[ \phi^\dagger, D_0 \phi \right] - \left[ \left(D_0 \phi\right)^\dagger, \phi \right] \right) \quad (295)$$

We can express in detail this constraint, using the Eqs.(**??**) and (**??**)

$$\begin{aligned}
F_{12} &= \frac{1}{2\kappa} \left\{ \left[ \phi_1^* E_- + \phi_2^* E_+, -\frac{i}{2\kappa} \left(P E_+ + Q E_-\right) \right] \right. \quad (296) \\
&\quad \left. - \left[ \left( -\frac{i}{2\kappa} \left(P E_+ + Q E_-\right) \right)^\dagger, \phi_1 E_+ + \phi_2 E_- \right] \right\}
\end{aligned}$$

The first term is

$$\begin{aligned}
& \left[\phi_1^* E_- + \phi_2^* E_+, -\frac{i}{2\kappa}\left(PE_+ + QE_-\right)\right] \\
= & \left(-\frac{i}{2\kappa}\right)\left(\phi_1^* P\left[E_-, E_+\right] + \phi_2^* Q\left[E_+, E_-\right]\right) \\
= & \frac{i}{2\kappa}\left(\phi_1^* P - \phi_2^* Q\right) H \\
= & \frac{i}{2\kappa}\left(v^2 - 2\left(\rho_1 + \rho_2\right)\right)\left(\rho_1 - \rho_2\right) H
\end{aligned} \tag{297}$$

The second term is

$$\begin{aligned}
& \left[\left(-\frac{i}{2\kappa}\left(PE_+ + QE_-\right)\right)^\dagger, \phi_1 E_+ + \phi_2 E_-\right] \\
= & \frac{i}{2\kappa}\left[P^* E_- + Q^* E_+, \phi_1 E_+ + \phi_2 E_-\right] \\
= & \frac{i}{2\kappa}\left(P^* \phi_1\left[E_-, E_+\right] + Q^* \phi_2\left[E_+, E_-\right]\right) \\
= & -\frac{i}{2\kappa}\left(P^* \phi_1 - Q^* \phi_2\right) H \\
= & -\frac{i}{2\kappa}\left(v^2 - 2\left(\rho_1 + \rho_2\right)\right)\left(\rho_1 - \rho_2\right) H
\end{aligned} \tag{298}$$

Then

$$\begin{aligned}
F_{12} = & \frac{1}{2\kappa}\left\{\frac{i}{2\kappa}\left(v^2 - 2\left(\rho_1 + \rho_2\right)\right)\left(\rho_1 - \rho_2\right) H\right. \\
& \left. + \frac{i}{2\kappa}\left(v^2 - 2\left(\rho_1 + \rho_2\right)\right)\left(\rho_1 - \rho_2\right) H\right\}
\end{aligned} \tag{299}$$

The result gives us the magnetic field

$$\begin{aligned}
F_{12} = & -B \\
= & \frac{i}{2\kappa^2}\left(v^2 - 2\left(\rho_1 + \rho_2\right)\right)\left(\rho_1 - \rho_2\right) H
\end{aligned} \tag{300}$$

Comparing with

$$\begin{aligned}
F_{+-} = & \frac{1}{\kappa^2}\left(v^2 - 2\left(\rho_1 + \rho_2\right)\right)\left(\rho_1 - \rho_2\right) H \\
= & -\frac{1}{4}\frac{v^4}{\kappa^2}\left(\rho - \frac{1}{\rho}\right)\left[\frac{1}{2}\left(\rho + \frac{1}{\rho}\right) - 1\right] H
\end{aligned} \tag{301}$$

we note the relation

$$F_{12} = \frac{i}{2} F_{+-} \tag{302}$$

The flux is

$$\begin{aligned} \Phi &= \int d^2r \frac{1}{2} \mathrm{tr}\left(H F_{+-}\right) \\ &= \frac{1}{\kappa^2} \int d^2r \left(v^2 - 2\left(\rho_1 + \rho_2\right)\right)\left(\rho_1 - \rho_2\right) \end{aligned} \tag{303}$$

The quantities $\rho_1$ and $\rho_2$ are not normalized, therefore it is preferable to change to the variable $\rho$

$$\rho \equiv \frac{\rho_1}{v^2/4} = \frac{v^2/4}{\rho_2} \tag{304}$$

We note that

$$\begin{aligned} \mathrm{tr}\left(\phi\phi^\dagger\right) &= \rho_1 + \rho_2 \\ \left[\phi, \phi^\dagger\right] &= \left(\rho_1 - \rho_2\right) H \end{aligned} \tag{305}$$

$$\begin{aligned} \mathrm{tr}\left(\phi\phi^\dagger\right) &= \rho_1 + \rho_2 = \left(v^2/4\right)\rho + \frac{v^2/4}{\rho} = \frac{v^2}{4}\left(\rho + \frac{1}{\rho}\right) \\ \left[\phi, \phi^\dagger\right] &= \left(\rho_1 - \rho_2\right) H = \frac{v^2}{4}\left(\rho - \frac{1}{\rho}\right) H \end{aligned} \tag{306}$$

The flux is normalised as

$$\begin{aligned} \Phi &= \frac{1}{\kappa^2} \int d^2r \left(v^2 - 2\left(\rho_1 + \rho_2\right)\right)\left(\rho_1 - \rho_2\right) \\ &= \frac{1}{\kappa^2} \int d^2r \left[v^2 - 2\left(\rho\frac{v^2}{4} + \frac{v^2}{4}\frac{1}{\rho}\right)\right]\left(\rho\frac{v^2}{4} - \frac{v^2}{4}\frac{1}{\rho}\right) \\ &= -\frac{1}{4}\frac{1}{\rho_s^2} \int d^2r \left(\rho - \frac{1}{\rho}\right)\left[\frac{1}{2}\left(\rho + \frac{1}{\rho}\right) - 1\right] \end{aligned} \tag{307}$$

**NOTE**. In the *abelian* relativistic model the following relation exists between the flux and the minimum energy [30]

$$\mathcal{E}_{SD}^{Abelian} = \frac{v^2}{2} \mathrm{tr}\left[\frac{1}{2} H F_{+-}\right]$$

which means

$$\mathcal{E}_{SD}^{Abelian} = \frac{v^2}{2} \Phi \tag{308}$$

**END OF THE NOTE**

We calculate the scalar field self-interaction potential

$$\begin{aligned} V\left(\phi,\phi^{\dagger}\right) &= \frac{1}{4\kappa^2}\mathrm{tr}\left[\left(\left[\left[\phi,\phi^{\dagger}\right],\phi\right]-v^2\phi\right)^{\dagger}\left(\left[\left[\phi,\phi^{\dagger}\right],\phi\right]-v^2\phi\right)\right] \quad (309)\\ &= \frac{1}{4\kappa^2}\mathrm{tr}\left[\left(PE_{+}+QE_{-}\right)^{\dagger}\left(PE_{+}+QE_{-}\right)\right]\\ &= \frac{1}{4\kappa^2}\mathrm{tr}\left[\left(P^{*}E_{-}+Q^{*}E_{+}\right)\left(PE_{+}+QE_{-}\right)\right]\\ &= \frac{1}{4\kappa^2}\left[P^{*}P\mathrm{tr}\left(E_{-}E_{+}\right)+Q^{*}Q\mathrm{tr}\left(E_{+}E_{-}\right)\right]\\ &= \frac{1}{4\kappa^2}\left(P^{*}P+Q^{*}Q\right) \end{aligned}$$

since the other traces are zero. Using the notations $\rho_1$ and $\rho_2$ we have

$$\begin{aligned} V\left(\phi,\phi^{\dagger}\right) &= \frac{1}{4\kappa^2}\left(P^{*}P+Q^{*}Q\right) \quad (310)\\ &= \frac{1}{4\kappa^2}\left\{\left[v^2-2\left(\rho_1-\rho_2\right)\right]^2\rho_1+\left[v^2+2\left(\rho_1-\rho_2\right)\right]^2\rho_2\right\}\\ &= \frac{1}{4\kappa^2}\left\{v^4\rho_1-4v^2\left(\rho_1-\rho_2\right)\rho_1+4\left(\rho_1-\rho_2\right)^2\rho_1\right.\\ &\quad \left.+v^4\rho_2+4v^2\left(\rho_1-\rho_2\right)\rho_2+4\left(\rho_1-\rho_2\right)^2\rho_2\right\}\\ &= \frac{1}{4\kappa^2}\left\{v^4\left(\rho_1+\rho_2\right)-4v^2\left(\rho_1-\rho_2\right)^2+4\left(\rho_1-\rho_2\right)^2\left(\rho_1+\rho_2\right)\right\} \end{aligned}$$

We can express the potential in terms of the normalised variable

$$\begin{aligned} & V\left(\phi,\phi^{\dagger}\right) \quad (311)\\ =\ & \frac{1}{4\kappa^2}\left\{v^4\left(\rho_1+\rho_2\right)-4v^2\left(\rho_1-\rho_2\right)^2+4\left(\rho_1-\rho_2\right)^2\left(\rho_1+\rho_2\right)\right\}\\ =\ & \frac{1}{4\kappa^2}\left\{v^4\left(\rho\frac{v^2}{4}+\frac{v^2}{4}\frac{1}{\rho}\right)\right.\\ & -4v^2\left(\rho\frac{v^2}{4}-\frac{v^2}{4}\frac{1}{\rho}\right)^2\\ & \left.+4\left(\rho\frac{v^2}{4}-\frac{v^2}{4}\frac{1}{\rho}\right)^2\left(\rho\frac{v^2}{4}+\frac{v^2}{4}\frac{1}{\rho}\right)\right\} \end{aligned}$$

$$
\begin{aligned}
& V\left(\phi, \phi^{\dagger}\right) \\
= \; & \frac{1}{4\kappa^2}\frac{v^6}{4}\left\{\left(\rho+\frac{1}{\rho}\right)-\left(\rho-\frac{1}{\rho}\right)^2+\frac{1}{4}\left(\rho-\frac{1}{\rho}\right)^2\left(\rho+\frac{1}{\rho}\right)\right\} \\
= \; & \frac{1}{16}v^2\frac{v^4}{\kappa^2}\left\{\left(\rho+\frac{1}{\rho}\right)-\left(\rho-\frac{1}{\rho}\right)^2+\frac{1}{4}\left(\rho-\frac{1}{\rho}\right)^2\left(\rho+\frac{1}{\rho}\right)\right\}
\end{aligned} \tag{312}
$$

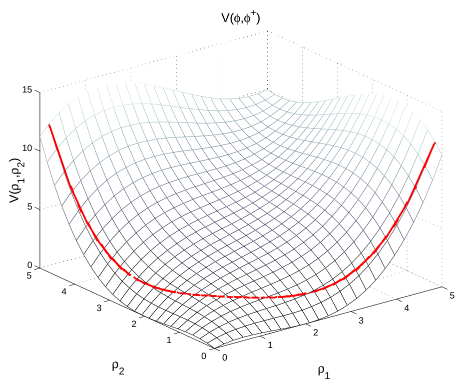


Figure 4: The potential $V(\phi, \phi^{\dagger})$. The points where $\rho_1\rho_2 = 1$ are shown.

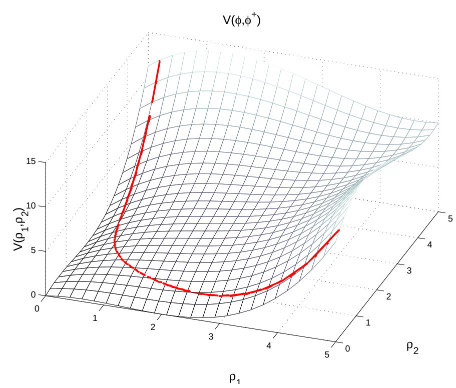


Figure 5: Same as figure 4 with a different view.

## 10.4 Comment on the possible associations between the field-theoretical variables and physical variables

The field theoretical model has been developed as the continuum version of the system of discrete vortices interacting via a short range potential. On the other hand, the original model, represented by the CHM equation, is expressed in terms of three variables

$$\begin{aligned} & \psi \\ \mathbf{v} &= -\nabla \times \widehat{\mathbf{e}}_z \psi \\ \omega &= \nabla \times \mathbf{v} = \Delta \psi \end{aligned}$$

(where $\widehat{\mathbf{e}}_z$ is the versor perpendicular on the plane). These variables have a clear physical meaning. We would like to understand the possible connection between the variables of the field theory $(j^\mu, A_\mu, F_{\mu\nu}, \phi)$ and parameters ($\kappa$ and $v^2$) and the physical variables. We make the observation that the conservation law for the scalar and vorticity field [14] is (including explicitely the normalisation factor $\rho_s$)

$$\iint d^2r \left[\psi^2 + (\rho_s \nabla \psi)^2\right] = \text{const}$$

We make an integration by parts of the second term

$$\begin{aligned} \iint d^2r \, (\nabla\psi)^2 &= \iint d^2r \, (\nabla\psi \cdot \nabla\psi) = \\ &= \iint d^2r \left[\nabla (\psi \cdot \nabla\psi) - \psi \Delta \psi\right] \end{aligned}$$

The integration of the first term is transformed

$$\iint d^2r \nabla (\psi \cdot \nabla\psi) = \oint d\mathbf{l} \cdot (\psi \nabla \psi)$$

Since we have the definition $\mathbf{v} = -\nabla\psi \times \widehat{\mathbf{e}}_z$ we see that the scalar product excludes the component of $\nabla\psi$ that is normal to the contour, which is a circle of very large radius. The diamagnetic effect makes that the velocity of gyration of particles on Larmor orbits generates a macroscopic velocity of the fluid which is tangent to the circle, therefore $\nabla\psi$ has only a nonzero component, normal to the circle. We conclude that there is no contribution from the first term in the integrand. Then we have

$$\iint d^2r \psi \left(\psi - \rho_s^2 \Delta \psi\right) = \text{const}$$

In particular this means that a *vacuum state*, of zero energy, corresponds to

$$\rho_s^2 \Delta\psi - \psi = 0$$

or

$$\rho_s^2\omega - \psi = 0 \tag{313}$$

On the other hand, we have a characterisation of the vacuum state in the field theoretical model, obtained as the asymptotic state of the fields at large distance. There the scalar field $\phi$ is almost constant and the space derivatives are vanishing (this is also known as the large wavelength approximation). It will be shown below that the current $j^\mu$ and the potential $A^\mu$ verify the relation

$$j^\mu - 2iv^2 A^\mu \simeq 0 \tag{314}$$

If these two relations Eqs.(313) and (314) describe the same physics they suggest (ignoring the signs and numerical factor) the following qualitative identifications

$$\begin{aligned} j &\sim \rho_s^2\omega \\ v^2 A &\sim \psi \end{aligned} \tag{315}$$

The second of this equation may seem strange since $A$ (and the covariant derivatives) are vectors. The combination that seems to be plausible is

$$\begin{aligned} D_i\phi_j &= \partial_i\phi_j - \varepsilon_{ik}\psi\phi_k \\ A_i &\sim -\frac{1}{v^2}\varepsilon_{ik}\psi \end{aligned}$$

There is some confirmation from the Abelian version in the Maxwell-Higgs case at self-duality [40]. Eqs.(315) further suggest that the magnetic field of the model can be associated with the physical velocity, $\mathbf{B} \sim \mathbf{v}$ (however in this framework no connection can be made with the Elsasser variables $\mathbf{u} = \mathbf{v} + \mathbf{B}$, $\mathbf{w} = \mathbf{v} - \mathbf{B}$). It is interesting to remark that the magnetic field $B$ can as well be associated with the physical vorticity, since the Chern-Simons Lagrangean has the unique property that connects directly the field tensor $F_{\mu\nu}$ with the current $J^\mu$, as is shown by the second equation of motion, $-\kappa\varepsilon^{\mu\nu\rho}F_{\nu\rho} = iJ^\mu$. Then the relationship which is fundamental for the connection between the field-theoretical framework and the physical model, $\ln\rho = \psi$ (assumed previously as a simple change of variables) appears now consistent with the physical meaning of $B$, since

$$B \sim \Delta \ln\rho = \Delta\psi = \omega \tag{316}$$

This also suggests

$$\kappa B \sim \rho_s^2\omega \tag{317}$$

The detailed form of these identification cannot be made more precise and we limit ourselves to a dimensional analysis. In these relationships there is no factor of dimensionality to intermediate between the two sides. The dimensional factor that multiply the first relation in Eq.(315) must also multiply the second relation, due to Eq.(314). As will be verified below, the factor (we note it $\chi$) must have the dimension

$$[\chi] = L^3 \tag{318}$$

and this implies for the dimansions of the variables ($L$ is length, $T$ is time)

$$\begin{aligned} [\chi]\,[j] &= \left[\rho_s^2\omega\right] = \frac{L^2}{T} \\ L^3\,[j] &= \frac{L^2}{T} \end{aligned} \tag{319}$$

or

$$[j] = \frac{1}{LT} \tag{320}$$

In the second relation of Eq.(315) we have

$$\begin{aligned} [\chi]\left[v^2 A\right] &= [\psi] = \frac{L^2}{T} \\ L^3\left[v^2 A\right] &= \frac{L^2}{T} \end{aligned} \tag{321}$$

As we have mentioned before, the quantity $v^2$ is related with the physical background of vorticity generated by the gyration of the particles. Then its dimension is

$$\left[v^2\right] = \frac{1}{T} \tag{322}$$

from which we derive

$$\left[v^2\right][A] = \frac{1}{TL} \tag{323}$$

or

$$[A] = \frac{1}{L} \tag{324}$$

This further gives

$$[B] = \frac{1}{L^2} \tag{325}$$

All dimensions become coherent if we identify

$$\begin{aligned} \kappa &\equiv c_s \\ v^2 &\equiv \Omega_{ci} \end{aligned} \tag{326}$$

For example, using again the unique dimensional coefficient $\chi$, Eq.(317) is dimensionally correct.

$$\begin{aligned} [\chi]\,[\kappa]\,[B] &= \left[\rho_s^2 \omega\right] \\ L^3 \frac{L}{T}\frac{1}{L^2} &= L^2 \frac{1}{T} \end{aligned} \tag{327}$$

One can now verify that all equations in the field model have coherent dimensions.

We have now a qualitative association between the physical variables and the field model variables and we also have the physical dimensions of the latter. We note that the covariant derivatives (having dimension $L^{-1}$)

$$D_\mu = \partial_\mu + [A_\mu, ] \tag{328}$$

cannot have a clear identification in terms of physical variables. One can only say that the zero component is

$$D_0 = \frac{\partial}{c_s \partial t} + \frac{1}{L^3}\frac{\psi}{\Omega_{ci}} \tag{329}$$

where we have taken into account the second relation from Eq.(315) and included the unknown dimensional factor $L^3$. From the Eqs.(112) and (131), (132) we note that it is not possible to express in terms of the classical ($\psi$, $\mathbf{v}$, $\omega$) variables the potentials $A_x$ and $A_y$.

## 10.5 Comment on the physical constants and normalisations

One of the characteristics of the physical model is the presence of a uniform background of vorticity. In the absence of any excitation we have on any contour in plane a tangential projection of the velocity of the particles performing the Larmor gyration.

An arbitrary contour (say, a large circle of radius $R$) will intersect the circle of the Larmor gyration (of radius $\rho_s$) and one can calculate an average of projection of the velocity onto the tangent at the contour line. Supposing that $R \gg \rho_s$, the contour intercepted by the Larmor circle can be approximated with a stright line that intersects the circle between the angles $\theta_0$ and $\pi - \theta_0$. The contour is a chord and the average $\overline{v}_{\theta_0}$ of the velocity's projection on it,

$v_c(\theta)$, is

$$\begin{aligned}
\overline{v}_{\theta_0} &= \int_{\theta_0}^{\pi-\theta_0} \frac{d\theta}{[(\pi-\theta_0)-\theta_0]} v_c(\theta) \\
&= \frac{1}{\pi - 2\theta_0} \int_{\theta_0}^{\pi-\theta_0} d\theta v_L \sin\theta \\
&= \frac{2v_L}{\pi - 2\theta_0} \cos\theta_0
\end{aligned} \tag{330}$$

where $v_L$ is the velocity on the Larmor circle. Now we can average over the various lengths of the chord inside the Larmor circle,

$$\begin{aligned}
\overline{v} &= \int_0^{\pi} \frac{d\theta_0}{\pi} \frac{2v_L}{\pi - 2\theta_0} \cos\theta_0 \\
&= \frac{2v_L}{\pi} \int_0^{\pi/2} \frac{\sin\tau}{\tau} d\tau \\
&= \frac{2(1.37)}{\pi} v_L
\end{aligned} \tag{331}$$

The symmetric situation will bring a similar factor and finally the average projected velocity is within a factor not far from unity equal to $v_L$. Now consider the definition of the rotational

$$\omega \equiv |\nabla \times \mathbf{v}| = \lim_{A\to 0} \frac{\oint_\Gamma \mathbf{v}\cdot \mathbf{dl}}{A} \tag{332}$$

where $A$ is the area inside the closed contour $\Gamma$. We have, within a unity-size factor

$$\oint_\Gamma \mathbf{v}\cdot\mathbf{dl} \simeq 2\pi R v_L \tag{333}$$

$$A = \pi R^2 \tag{334}$$

Then

$$\omega \sim \lim_{R\to\rho_s} \frac{2v_L}{R} \tag{335}$$

Since

$$v_L = \rho_s \Omega \tag{336}$$

we obtain

$$\begin{aligned}
\omega &\sim \Omega \lim_{R\to\rho_s} \frac{\rho_s}{R} \\
&= \Omega
\end{aligned} \tag{337}$$

*i.e.* we obtain that the value of the vorticity in a region with uniform density of Larmor gyrating particles is $\Omega$, the cyclotronic velocity.

We have in this moment three parallel models, representing the same reality, which we call the Charney-Hasegawa-Mima vortical flow. The connection between these three models implies a comparison of the physical quantities present in each of them. For this reason we have to consider the physical content of the field-theoretical model and in particular we will introduce nondimensional variables. The two physical quantities appearing explicitly in the field-theoretical model are $\kappa$ and $v^2$.

## 10.6 Comparison with numerical simulation and with experiment

The second factor of the nonlinearity *i.e.* $(\cosh \psi - p)$ in all versions of the equation derived above, in particular in Eq.(217) can also be negative, under a certain choice of normalizations. Then, a certain aspect of the graphs resulting from numerical simulations (see Seyler [10]), *i.e.* the presence of two visible symmetric extrema on the graph (*vorticity, streamfunction*) is in agreement with our equation.

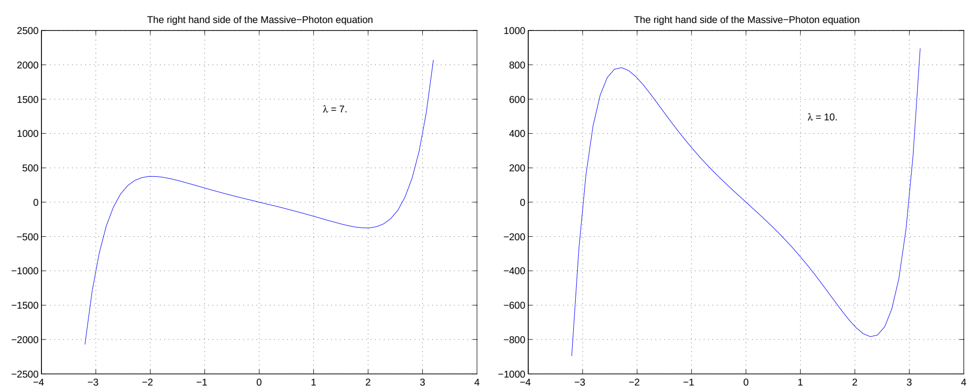


Figure 6: The nonlinear term in the equation, for $p = 7$ and $p = 10$

It is interesting to remark that similar pictures to our result have been found in the study of geostrophic turbulence [41]. The scatter plots for the pair $(\omega, \psi)$ obtained from experimental study of decaying vorticity field and represented in the figure 21 of this reference are very similar to our result (with the choice of inverse sign for the streamfunction).

## 10.7 The vacuum fields

We comment on the meaning of the vacuum value of the matter field. Obviously, it must be related with the presence, in the physical model, of a background of vorticity simply given by the gyration of the particles. This is the sense of the fact that, even at infinity, we have a constant density of matter, $|\phi|^2 = v^2$. The excitations in the form of large vortices take place on this background, whose value of vorticity is very high, the ion cyclotronic frequency, $\Omega_{ci}$. As we said before we can possibly identify

$$\begin{aligned} v^2 &\equiv \Omega_{ci} \\ \kappa &= c_s \end{aligned} \tag{338}$$

The physical vorticity is derived from $F_{+-}$

$$F_{+-} = \frac{1}{\kappa^2} \left( v^2 - 2 \left( \rho_1 + \rho_2 \right) \right) \left( \rho_1 - \rho_2 \right) H \tag{339}$$

## 10.8 The subset of self-dual states of the physical system

We should remind that the identification

$$\psi \equiv \ln \rho \tag{340}$$

was done after the equations of motion in the field theoretical framework have been reduced to the equations of *self-duality* and stationarity. Therefore it is not surprising that, returning with Eq.(340) to the original, Charney-Hasegawa-Mima equation, we find that this one is verified by the functions $\rho$ obeying our equation (144) or Eq.(148) for $\psi$

$$\left[ \left( -\nabla \psi \times \widehat{\mathbf{n}} \right) \cdot \nabla \right] \nabla^2 \psi = 0 \tag{341}$$

because

$$\left[ \left( -\nabla \psi \times \widehat{\mathbf{n}} \right) \cdot \nabla \right] \left[ -\sinh \psi \left( \cosh \psi - 1 \right) \right] = 0$$

The fact that, for any solution $\psi$ of the Eq.(148) the equation of Charney-Hasegawa-Mima is verified at stationarity is useful as a confirmation but is of moderate significance, due to the large space of functions that can verify Eq.(341). The subset of self-dual states is much smaller and precisely defined by Eq.(148).

## 10.9 Comment on the self-duality

In general the self-duality should be seen as a property of a particular geometrico-algebraic object : a fiber bundle. This consists of a basis manifold (on which one has to define an atlas of compatible charts), a fiber attached to every point of the basis manifold (in physics the fiber is seldom said space of internal symmetry) and a group of automorphism of the typical fiber. The local structure on the basis and in the fibre space is Euclidean since they both are manifolds. One can construct the total space of the fiber bundle, which is locally a Cartesian product of an open set of the basis with the space of the fiber, and a projection operator acting in this total space and projecting the points of the total space onto the basis. The transition functions between neighboring charts consist of elements of the group. A connection is a differential one-form defined in every point of the total space and taking values in the algebra of the group. The curvature is the differential two-form obtained by an exterior differentiation of the connection one-form. For a concrete example, the connection is the *potential* $A_\mu$ and the curvature is the *field strength* $F_{\mu\nu}$ like in the electromagnetism, or in any other theory expressed in similar terms. There is a Hodge duality operator, denoted $*$ : applied on a differential $p$-form in a space with $n$ dimensions, it generates a differential $n-p$ form, such as the exterior product of these two forms produces a scalar multiplying the unique $n$ form that can de defined on the $n$-dimensional space, *i.e.* a multiple of the volume form.

The self-duality is the property that consists of the equality between the a differential form and its Hodge dual; this naturally requires that the space be of even dimension. Only the differential forms of the order representing half of the (even) dimension of the space can be self-dual since only in this case their duals will be of the same order. For example in a space with dimension four, differential two-forms can be self-dual.

In particular physical models the differential two-form representing the field strength and its dual are equal at self-duality. But this two-form represents the curvature of the fibre bundle, therefore at self-duality the curvature is equal to its dual. In many cases, this equality is realised by the fact that the curvature is zero and one says that the space is *flat*.

When the self-duality is realised as a condition of *flatness* it is possible to express this equality as the compatibility condition of a system of linear differential equations. This makes possible to introduce a Lax operator and the self-duality equation is exactly integrable by Inverse Scattering Transform. A set of infinite invariants can be found. One example of this type is the classical *sigma* model.

In particular cases, (like ours) the geometrical structure is less clear. The self-duality is expressed by the fact that the *action* functional is minimized *i.e.* the Bogomolnyi limit is saturated.

## 10.10 Comment on the $6^{th}$ order potential

The Abelian version of this theory but with the Maxwell term instead of Chern-Simons is well known from superconductivity theory. It implies a potential of only fourth order which provides the symmetrical vacua of the theory and allows mass generation for the Maxwell photon via the Higgs mechanism.

However in the present theory a sixth order potential is necessary. It has been demonstrated that with only a sixth order potential one can have self-dual states. This has been shown by a simple verification which we have reproduced in the Section about the energy functional related to the Lagrangean density.

However the necessity to include a sixth order potential in the Lagrangean density has a profond origin. This has been shown in series of papers [50], [51], [52].

It has been shown that the Bogomolnyi lower bound for the energy and the first-order-in-time differential equations obtained at self-duality are a property of a classical field theory which possesses a topological charge. The theory is a reduction of a supersymmetric (susy) theory in which the topological charge appears as the central charge of the susy algebraic structure. ( A susy theory is a classical field theory in which besides the usual fields there are other field-variables with the property that they anti-commute, *i.e.* they are classical spinors). It is interesting the way in which this has been shown [51],[52]. First it is shown that any supersymmetric theory which possesses a topological charge necessarily possesses a Bogomolnyi bound and SD equations of motion. Then for a given field theory where a topological conservation charge exists, it is first constructed a supersymmetric extension, adding the anti-commuting variables and other variables that are necessary to close the new algebraic structure. In this extended theory the *central charge* of the susy algebra is the topological charge of the initial theory. The Bogomolnyi bound is identified. Finally it is shown that returning back from the susy extension to the original theory, one still preserves the Bogomolnyi bound. The relation between the potential $W$ in the extended theory and the potential $U$ in the classical non-susy theory is

$$U(\phi) = \sum_a \left( \frac{\partial W}{\partial \phi_a} \right)^2 \tag{342}$$

and for a symmetrical two-vacua potential $W$ we have a potential $U$ of sixth degree in $\phi_a$.

Lee, Lee and Weinberg [53] show explicitly in an Abelian case how this form of the potential is obtained from the requirement that the model can be extended to a $N=2$ supersymmetric model. They begin by constructing an $N=1$ supersymmetric generalization of this Chern-Simons Higgs theory, in which the form of the potential $f\left(|\phi|^2\right)$ is not yet specified. Adding a single pair of Grassmannian varibales $\left(\theta,\overline{\theta}\right)$ to the set of variables of the original model requires to extend the model by introducing additional fields. This is necessary since the supersymmetry transformation must be closed and the original fields are not sufficient. The model will include a matter super-field $\Phi$ which consists of: a complex scalar field $\phi$, a complex spinor field $\psi$ and an auxiliary scalar field $F$; a real spinor field $\Gamma^\alpha$ which contains a real *photon* field $A_\mu$ and a Majorana spinor *photino* field $\lambda$. The action is

$$\begin{aligned}\mathcal{S} &= \int d^2xdt \int d\theta d\overline{\theta} \left\{ -\frac{1}{4}\kappa D^\alpha \Gamma^\beta D_\beta \Gamma_\alpha \right. \\ &\quad -\frac{1}{2}\left(D^\alpha + i\Gamma^a\right)\Phi^*\left(D_\alpha - i\Gamma_\alpha\right)\Phi \\ &\quad \left. +f\left(\Phi^*\Phi\right)\right\}\end{aligned} \tag{343}$$

where the first term is the generalization of the Chern-Simons term. The integration over the Grassmann variables $\theta$ and $\overline{\theta}$ can be done explicitely and in this process it is required to make them visible in the expression of the superfield $\Phi$. In this way there appears in the action density first and second order derivatives of the potential function $f$ since only in this way (by this Taylor expansion) the Grassmann variables will appear explicitely and can be integrated. The action becomes

$$\begin{aligned}\mathcal{S} &= \int d^2xdt \left\{ \frac{1}{4}\kappa\varepsilon^{\mu\nu\lambda}A_\mu F_{\nu\lambda} + \left(D^\mu\phi\right)\left(D_\mu\phi\right) \right. \\ &\quad -\frac{1}{2}\kappa\overline{\lambda}\lambda + i\overline{\psi}\gamma^\mu D_\mu\psi + i\left(\overline{\psi}\lambda\phi - \overline{\lambda}\psi\phi^*\right) + F^*F \\ &\quad +f'\left(F^*\phi + F\phi^*\right) - \frac{1}{2}f''\left(\phi^2\overline{\psi}\psi^c + \phi^{*2}\overline{\psi}^c\psi\right) \\ &\quad \left. -\left(f' + |\phi|^2 f''\right)\overline{\psi}\psi\right\}\end{aligned} \tag{344}$$

where the superscript $c$ means charge conjugate and the *prime* means derivative of the function $f$ to its argument, $|\phi|^2$. The equations of motion for the fields $\lambda$, $F$ and $F^*$ are

$$\begin{aligned}\lambda &= \frac{i}{\kappa}\left(\psi^c\phi - \psi\phi^*\right) \\ F &= -\phi f' \\ F^* &= -\phi^* f'\end{aligned} \tag{345}$$

They permit the replacement of the corresponding functions in the action.

$$\begin{aligned}\mathcal{S} = & \int d^2x dt \left\{ \frac{1}{4}\kappa\varepsilon^{\mu\nu\lambda}A_\mu F_{\nu\lambda} + \left(D^\mu\phi\right)\left(D_\mu\phi\right) \right. \\ & - |\phi|^2 f'^2 + i\overline{\psi}\gamma^\mu D_\mu\psi \\ & -\frac{1}{2}\left(f'' + \frac{1}{\kappa}\right)\left(\phi^2\overline{\psi}\psi^c + \phi^{*2}\overline{\psi}^c\psi\right) \\ & \left. + \left[|\phi|^2\left(\frac{1}{\kappa} - f''\right) - f'\right]\overline{\psi}\psi \right\}\end{aligned} \tag{346}$$

In order this action to be invariant under an $N = 2$ extended supersymmetry it is required that the term on the third line in the above formula vanishes

$$f'' = -\frac{1}{\kappa} \tag{347}$$

or, integrating two times on the variable $\xi \equiv |\Phi|^2$

$$\begin{aligned}f\left(|\Phi|^2\right) &= -\frac{1}{2\kappa}\left(\xi - v^2\right)^2 \\ &= -\frac{1}{2\kappa}\left(|\Phi|^2 - v^2\right)^2\end{aligned} \tag{348}$$

where $v^2$ is a constant. The form of the *potential* term in the action is then obtained from the first term in the second line of Eq.(346)

$$|\phi|^2 f'^2 \rightarrow \frac{1}{2\kappa}|\phi|^2\left(|\phi|^2 - v^2\right)^2 \tag{349}$$

The action contains a bosonic part that has this potential and this is actually the Abelian version of the model discussed in this work.

The fact that at self-duality the theory is a part of a larger supersymmetric theory and that this explains the form of the scalar self-interaction may help us to trace the meaning of the changes we find between the *sinh*-Poisson equation (for the ideal fluid) and the *double-sinh*-Poisson equation, for the fluid of ions with Larmor gyration.

The special Higgs potential that appears in the Lagrangean density and leads to self-dual states has a symmetric minimum which is degenerate with the symmetry-breaking one. This means that the system can have non-topological solitons which verify the same self-dual equations.

# 11 Appendix A : Derivation of the equation

Consider the equations for the ITG model in two-dimensions with adiabatic electrons:

$$\begin{aligned} \frac{\partial n_i}{\partial t} + \nabla \cdot (\mathbf{v}_i n_i) &= 0 \\ \frac{\partial \mathbf{v}_i}{\partial t} + (\mathbf{v}_i \cdot \nabla) \mathbf{v}_i &= \frac{e}{m_i} (-\nabla \phi) + \frac{e}{m_i} \mathbf{v}_i \times \mathbf{B} \end{aligned} \tag{A.1}$$

We assume the quasineutrality

$$n_i \approx n_e \tag{A.2}$$

and the Boltzmann distribution of the electrons along the magnetic field line

$$n_e = n_0 \exp\left(-\frac{|e| \phi}{T_e}\right) \tag{A.3}$$

The velocity of the ion fluid is perpendicular on the magnetic field and is composed of the diamagnetic, electric and polarization drift terms

$$\begin{aligned} \mathbf{v}_i &= \mathbf{v}_{\perp i} \\ &= \mathbf{v}_{dia,i} + \mathbf{v}_E + \mathbf{v}_{pol,i} \\ &= \frac{T_i}{|e| B} \frac{1}{n_i} \frac{dn_i}{dr} \widehat{\mathbf{e}}_y \\ &\quad + \frac{-\nabla \phi \times \widehat{\mathbf{n}}}{B} \\ &\quad - \frac{1}{B \Omega_i} \left( \frac{\partial}{\partial t} + (\mathbf{v}_E \cdot \nabla_\perp) \right) \nabla_\perp \phi \end{aligned} \tag{A.4}$$

The diamagnetic velocity will be neglected. Introducing this velocity into the continuity equation, one obtains an equation for the electrostatic potential $\phi$.

Before writting this equation we introduce new dimensional units for the variables.

$$\phi^{phys} \rightarrow \phi' = \frac{|e| \phi^{phys}}{T_e} \tag{A.5}$$

$$\left(x^{phys}, y^{phys}\right) \rightarrow (x', y') = \left(\frac{x^{phys}}{\rho_s}, \frac{y^{phys}}{\rho_s}\right) \tag{A.6}$$

$$t^{phys} \rightarrow t' = t^{phys} \Omega_i \tag{A.7}$$

The new variables $(t, x, y)$ and the function $\phi$ are non-dimensional. In the following the *primes* are not written. With these variables the equation obtained is

$$\begin{aligned} & \frac{\partial}{\partial t}\left(1-\nabla_{\perp}^{2}\right)\phi \\ & -\left(-\nabla_{\perp}\phi\times\widehat{\mathbf{n}}\right)\cdot\kappa\widehat{\mathbf{e}}_{r} \\ & -\left[\left(-\nabla_{\perp}\phi\times\widehat{\mathbf{n}}\right)\cdot\nabla_{\perp}\right]\nabla_{\perp}^{2}\phi \\ = & \; 0 \end{aligned} \tag{A.8}$$

where

$$\kappa\widehat{\mathbf{e}}_{r}\equiv-\nabla_{\perp}\ln n_{0} \tag{A.9}$$

([57]). Before continuing we compare this equation with the equation of paper [58], Eq.(16). Here taking still the units to be physical, the form of the latter equation is (Eq.(12) from that paper)

$$\begin{aligned} & \frac{\partial}{\partial t}\frac{|e|\,\phi}{T_{e}}-\frac{\partial}{\partial t}\frac{1}{B\Omega_{i}}\nabla_{\perp}^{2}\phi \\ & +\frac{-\nabla_{\perp}\phi\times\widehat{\mathbf{n}}}{B}\cdot\nabla_{\perp}\ln n_{0} \\ & +\frac{-\nabla_{\perp}\phi\times\widehat{\mathbf{n}}}{B}\cdot\nabla_{\perp}\frac{|e|\,\phi}{T_{e}} \\ & -\frac{1}{B^{2}\Omega_{i}}\left[\left(-\nabla_{\perp}\phi\times\widehat{\mathbf{n}}\right)\cdot\nabla_{\perp}\right]\nabla_{\perp}^{2}\phi \\ = & \; 0 \end{aligned} \tag{A.10}$$

The term containing the gradient of the equilibrium density comes from the continuity equation, as convection of the equilibrium density by the fluctuating $E\times B$ velocity. The adiabaticity has been assumed,

$$\frac{\widetilde{n}}{n_{0}}=\frac{|e|\,\phi}{T_{e}} \tag{A.11}$$

and we consider that the temperature is constant (the calculations can easily include a dependence $T_{e}(x)$).

For the second term we have

$$\begin{aligned} \frac{1}{B\Omega_{i}}\nabla_{\perp}^{2}\phi &= \frac{1}{B\Omega_{i}}\frac{T_{e}}{|e|}\nabla_{\perp}^{2}\frac{|e|\,\phi}{T_{e}}=\frac{1}{\Omega_{i}}\frac{1}{\frac{|e|B}{m_{i}}}\frac{T_{e}}{m_{i}}\nabla_{\perp}^{2}\frac{|e|\,\phi}{T_{e}} \\ &= \frac{1}{\Omega_{i}^{2}}c_{s}^{2}\nabla_{\perp}^{2}\frac{|e|\,\phi}{T_{e}}=\rho_{s}^{2}\nabla_{\perp}^{2}\frac{|e|\,\phi}{T_{e}} \end{aligned} \tag{A.12}$$

This will become (with its sign)

$$- \frac{\partial}{\partial t} \nabla'^2_\perp \phi' \tag{A.13}$$

in the new variables Eqs.(A.5)-(A.7).

The third term is

$$\begin{aligned}
\frac{-\nabla_\perp \phi \times \widehat{\mathbf{n}}}{B} \cdot \nabla_\perp \ln n_0 &= \frac{1}{B} \frac{T_e}{|e|} \left( -\nabla_\perp \frac{|e|\,\phi}{T_e} \times \widehat{\mathbf{n}} \right) \cdot \nabla_\perp \ln n_0 \qquad \text{(A.14)} \\
&= \frac{1}{\frac{|e|B}{m_i}} \frac{T_e}{m_i} \left( -\nabla_\perp \frac{|e|\,\phi}{T_e} \times \widehat{\mathbf{n}} \right) \cdot \nabla_\perp \ln n_0 \\
&= \Omega_i \frac{c_s^2}{\Omega_i^2} \left( -\nabla_\perp \frac{|e|\,\phi}{T_e} \times \widehat{\mathbf{n}} \right) \cdot \nabla_\perp \ln n_0 \\
&= \Omega_i \rho_s^2 \left( \nabla_\perp \frac{|e|\,\phi}{T_e} \times \widehat{\mathbf{n}} \right) \cdot (-\nabla_\perp \ln n_0)
\end{aligned}$$

This will become

$$- \Omega_i \left( -\rho_s \nabla_\perp \phi \times \widehat{\mathbf{n}} \right) \cdot \left( -\rho_s \nabla_\perp \ln n_0 \right) \tag{A.15}$$

and in the normalised space variables

$$- \Omega_i \left( -\nabla'_\perp \phi' \times \widehat{\mathbf{n}} \right) \cdot \left( -\nabla'_\perp \ln n_0 \right) \tag{A.16}$$

The last term (with the polarization nonlinearity) is in physical units

$$- \frac{1}{B^2 \Omega_i} \left[ \left( -\nabla_\perp \phi \times \widehat{\mathbf{n}} \right) \cdot \nabla_\perp \right] \nabla_\perp^2 \phi \tag{A.17}$$

This is converted to non-dimensional variables

$$\frac{1}{B^2 \Omega_i} \left[ \left( -\frac{1}{\rho_s} \frac{T_e}{|e|} \rho_s \nabla_\perp \frac{|e|\,\phi}{T_e} \times \widehat{\mathbf{n}} \right) \cdot \frac{1}{\rho_s} \rho_s \nabla_\perp \right] \frac{1}{\rho_s^2} \frac{T_e}{|e|} \left( -\rho_s^2 \nabla_\perp^2 \frac{|e|\,\phi}{T_e} \right) \tag{A.18}$$

Collecting the physical coefficient we have

$$\begin{aligned}
\frac{1}{B^2 \Omega_i} \left( \frac{T_e}{|e|} \right)^2 \frac{1}{\rho_s^4} &= \frac{1}{\left( \frac{|e|B}{m_i} \right)^2} \frac{1}{\Omega_i} \left( \frac{T_e}{m_i} \right)^2 \frac{1}{\rho_s^4} \qquad \text{(A.19)} \\
&= \Omega_i \frac{c_s^4}{\Omega_i^4} \frac{1}{\rho_s^4} \\
&= \Omega_i
\end{aligned}$$

Then, in the normalised variables, this term becomes

$$\Omega_i \left[ \left( -\nabla'_\perp \phi' \times \widehat{\mathbf{n}} \right) \cdot \nabla'_\perp \right] \left( -\nabla'^2_\perp \phi' \right) \tag{A.20}$$

Then the Eqs.(A.10) with the new form of its terms (A.13), (A.16) and (A.20) becomes

$$\begin{aligned} & \frac{\partial}{\partial t}\phi' - \frac{\partial}{\partial t}\nabla'^2_\perp \phi' \\ & -\Omega_i \left( -\nabla'_\perp \phi' \times \widehat{\mathbf{n}} \right) \cdot \left( -\nabla'_\perp \ln n_0 \right) \\ & +\Omega_i \left[ \left( -\nabla'_\perp \phi' \times \widehat{\mathbf{n}} \right) \cdot \nabla'_\perp \right] \left( -\nabla'^2_\perp \phi' \right) \\ = \ & 0 \end{aligned} \tag{A.21}$$

Introducing the time unit $\Omega_i^{-1}$, and eliminating the *primes*

$$\begin{aligned} & \frac{\partial}{\partial t} \left( 1 - \nabla^2_\perp \right) \phi \\ & - \left( -\nabla_\perp \phi \times \widehat{\mathbf{n}} \right) \cdot \left( -\nabla_\perp \ln n_0 \right) \\ & - \left[ \left( -\nabla_\perp \phi \times \widehat{\mathbf{n}} \right) \cdot \nabla_\perp \right] \nabla^2_\perp \phi \\ = \ & 0 \end{aligned} \tag{A.22}$$

The last term is the convection of the vorticity

$$\omega = \nabla^2_\perp \phi \widehat{\mathbf{n}} \tag{A.23}$$

by the velocity field

$$\mathbf{v}_E = -\nabla_\perp \phi \times \widehat{\mathbf{n}} \tag{A.24}$$

We use the definition $\kappa \widehat{\mathbf{e}}_r = \nabla_\perp \ln n_0$ or

$$\kappa \widehat{\mathbf{e}}_y = -\widehat{\mathbf{n}} \times \nabla_\perp \ln n_0 \tag{A.25}$$

Then the resulting equation is

$$\left( 1 - \nabla^2_\perp \right) \frac{\partial \phi}{\partial t} - \kappa \frac{\partial \phi}{\partial y} - \left[ \left( -\nabla_\perp \phi \times \widehat{\mathbf{n}} \right) \cdot \nabla_\perp \right] \nabla^2_\perp \phi = 0 \tag{A.26}$$

The same equation but without the linear (density gradient) term has been derived as the "shielded convective ion cells" [11]

$$\left( 1 - \rho_s^2 \nabla^2_\perp \right) \frac{\partial \phi}{\partial t} - \left[ \phi, \rho_s^2 \nabla^2_\perp \phi \right] = 0$$

and as a possibility to describe the Kelvin-Helmholtz instability modified by the finite parallel electric field $E_\parallel$ and its associated current density $j_\parallel$, with

$$\nabla_\parallel j_\parallel = e \frac{\partial n_e}{\partial t} \simeq \frac{e^2 n_0}{T_e} \frac{\partial \phi^{phys}}{\partial t}$$

where the potential is not normalised yet. The enstropy is conserved

$$U = \int d^2 r \left[ (\rho_s \nabla_\perp \phi)^2 + \left(\rho_s^2 \Delta_\perp \phi\right)^2 \right] = \text{const}$$

# 12 Appendix B : The Euler-Lagrange equations

The calculations from this Appendix should be considered as a guide for a first contact with the methods of the theory of non-Abelian gauge field interacting with nonlinear (= self-interacting) scalar matter field. Here the calculations are not pedagogical (in particular we treat asymmetrically the fields $A_\mu$ and $A_\mu^\dagger$) and we suggest that after these first steps other lectures are necessary, from field-theory genuine sources.

Consider again the Lagrangean density

$$\begin{aligned} \mathcal{L} &= -\kappa \varepsilon^{\mu\nu\rho} \text{tr} \left( \partial_\mu A_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho \right) \\ &\quad - \text{tr} \left[ (D^\mu \phi)^\dagger (D_\mu \phi) \right] \\ &\quad - V\left(\phi, \phi^\dagger\right) \end{aligned} \tag{B.1}$$

The functional variables are

$$\begin{array}{l} A_0, A_0^\dagger, A_1, A_1^\dagger, A_2, A_2^\dagger \\ \phi, \phi^\dagger \end{array} \tag{B.2}$$

and they are all $SU(2)$ matrices with complex entries.

## 12.1 The contributions to the Lagrangean

### 12.1.1 The Chern-Simons term as a differential three-form and the presence of a metric

Apart from a factor, the gauge Lagrangean is the trace of the Chern-Simons differential three-form on a principal bundle with group $SU(2)$.

$$\begin{aligned} \Omega &= \frac{1}{8\pi^2} \text{tr} \left( \mathbf{A} \wedge \mathbf{dA} - \frac{2}{3} \mathbf{A} \wedge \mathbf{A} \wedge \mathbf{A} \right) \\ &= -\frac{1}{16\pi^2} \varepsilon^{\mu\nu\rho} \left( A_\mu^a \partial_\nu A_\rho^a - \frac{1}{3} \varepsilon^{abc} A_\mu^a A_\nu^b A_\rho^c \right) d^3 x \end{aligned} \tag{B.3}$$

The trace of the Chern-Simons form can also be expressed using the exterior differentation and exterior product of forms [48], [46]

$$cs\left(\mathbf{A}\right)=\frac{1}{4\pi}\int_{M^{3}}\mathrm{tr}\left(\mathbf{A}\wedge\mathbf{dA}+\frac{2}{3}\mathbf{A}\wedge\mathbf{A}\wedge\mathbf{A}\right) \tag{B.4}$$

where, for three algebra-valued differential one-forms $A_k$, $k=1,...,3$, one has

$$\mathrm{tr}\left(A_{1}\wedge A_{2}\wedge A_{3}\right)\overset{def}{=}\frac{1}{2}\mathrm{tr}\left(A_{1}\wedge\left[A_{2},A_{3}\right]\right)=\frac{1}{2}\mathrm{tr}\left(\left[A_{1},A_{2}\right]\wedge A_{3}\right) \tag{B.5}$$

A factor of $1/2$ can also be extracted from the first term in (B.4) if we add *minus* its expression but with two of the three indices exchanged. Then

$$cs(\mathbf{A})=\frac{1}{8\pi}\int\varepsilon^{\mu\nu\rho}\mathrm{tr}\left(A_{\mu}\left(\partial_{\nu}A_{\rho}-\partial_{\rho}A_{\nu}\right)+\frac{2}{3}A_{\mu}\left[A_{\nu},A_{\rho}\right]\right) \tag{B.6}$$

The normalizing constant in Eq.(B.6) is related with the fact that the integral of the Chern-Simons form is a topological invariant for adequate boundary conditions and has integer values. For what we need, the gauge field Lagrangean can be taken such as to lead to the gauge-part in the action [49]

$$\mathcal{S}_{1}=\int dt\int d^{2}x\left\{-\frac{1}{2}\kappa\varepsilon^{\mu\nu\rho}\mathrm{tr}\left(A_{\mu}\left(\partial_{\nu}A_{\rho}-\partial_{\rho}A_{\nu}\right)+\frac{2}{3}A_{\mu}\left[A_{\nu},A_{\rho}\right]\right)\right\} \tag{B.7}$$

Therefore we will use the following expression

$$\mathcal{L}_{1}=-\frac{1}{2}\kappa\varepsilon^{\mu\nu\rho}\mathrm{tr}\left(A_{\mu}\left(\partial_{\nu}A_{\rho}-\partial_{\rho}A_{\nu}\right)+\frac{2}{3}A_{\mu}\left[A_{\nu},A_{\rho}\right]\right) \tag{B.8}$$

We write in detail Eq.(B.8). The first term is

$$\begin{aligned}
& \varepsilon^{\mu\nu\rho}\left[A_{\mu}\left(\partial_{\nu}A_{\rho}-\partial_{\rho}A_{\nu}\right)\right] \\
= \; & \varepsilon^{012}A_{0}\left(\partial_{1}A_{2}-\partial_{2}A_{1}\right) \\
& +\varepsilon^{021}A_{0}\left(\partial_{2}A_{1}-\partial_{1}A_{2}\right) \\
& +\varepsilon^{102}A_{1}\left(\partial_{0}A_{2}-\partial_{2}A_{0}\right) \\
& +\varepsilon^{120}A_{1}\left(\partial_{2}A_{0}-\partial_{0}A_{2}\right) \\
& +\varepsilon^{210}A_{2}\left(\partial_{1}A_{0}-\partial_{0}A_{1}\right) \\
& +\varepsilon^{201}A_{2}\left(\partial_{0}A_{1}-\partial_{1}A_{0}\right)
\end{aligned} \tag{B.9}$$

or

$$
\begin{aligned}
& \varepsilon^{\mu\nu\rho}\left[A_\mu\left(\partial_\nu A_\rho-\partial_\rho A_\nu\right)\right] \\
= \; & A_0\left(\partial_1 A_2-\partial_2 A_1\right) \\
& -A_0\left(\partial_2 A_1-\partial_1 A_2\right) \\
& -A_1\left(\partial_0 A_2-\partial_2 A_0\right) \\
& +A_1\left(\partial_2 A_0-\partial_0 A_2\right) \\
& -A_2\left(\partial_1 A_0-\partial_0 A_1\right) \\
& +A_2\left(\partial_0 A_1-\partial_1 A_0\right)
\end{aligned} \tag{B.10}
$$

This is simply two times every distinct term in the sum

$$
\begin{aligned}
& \varepsilon^{\mu\nu\rho}\left[A_\mu\left(\partial_\nu A_\rho-\partial_\rho A_\nu\right)\right] \\
= \; & 2A_0\left(\partial_1 A_2\right)-2A_0\left(\partial_2 A_1\right)-2A_1\left(\partial_0 A_2\right) \\
& +2A_1\left(\partial_2 A_0\right)-2A_2\left(\partial_1 A_0\right)+2A_2\left(\partial_0 A_1\right)
\end{aligned} \tag{B.11}
$$

We continue by calculating the second term in the CS action

$$
\begin{aligned}
\varepsilon^{\mu\nu\rho}A_\mu\left[A_\nu,A_\rho\right] &= \varepsilon^{\mu\nu\rho}A_\mu\left(A_\nu A_\rho-A_\rho A_\nu\right) \\
&= \varepsilon^{012}A_0\left(A_1A_2-A_2A_1\right) \\
&\quad +\varepsilon^{021}A_0\left(A_2A_1-A_1A_2\right) \\
&\quad +\varepsilon^{102}A_1\left(A_0A_2-A_2A_0\right) \\
&\quad +\varepsilon^{120}A_1\left(A_2A_0-A_0A_2\right) \\
&\quad +\varepsilon^{210}A_2\left(A_1A_0-A_0A_1\right) \\
&\quad +\varepsilon^{201}A_2\left(A_0A_1-A_1A_0\right)
\end{aligned} \tag{B.12}
$$

or

$$
\begin{aligned}
\varepsilon^{\mu\nu\rho}A_\mu\left[A_\nu,A_\rho\right] &= \\
&= A_0\left(A_1A_2-A_2A_1\right) \\
&\quad -A_0\left(A_2A_1-A_1A_2\right) \\
&\quad -A_1\left(A_0A_2-A_2A_0\right) \\
&\quad +A_1\left(A_2A_0-A_0A_2\right) \\
&\quad -A_2\left(A_1A_0-A_0A_1\right) \\
&\quad +A_2\left(A_0A_1-A_1A_0\right)
\end{aligned} \tag{B.13}
$$

This is actually two times every distinct term in the sum

$$
\begin{aligned}
\frac{1}{2}\varepsilon^{\mu\nu\rho}A_\mu\left[A_\nu,A_\rho\right] &= A_0A_1A_2-A_0A_2A_1 \\
&\quad -A_1A_0A_2+A_1A_2A_0 \\
&\quad -A_2A_1A_0+A_2A_0A_1
\end{aligned} \tag{B.14}
$$

It results that the form of definition

$$\mathcal{L}_1 = -\frac{1}{2}\kappa\varepsilon^{\mu\nu\rho}\mathrm{tr}\left(A_\mu\left(\partial_\nu A_\rho - \partial_\rho A_\nu\right) + \frac{2}{3}A_\mu\left[A_\nu, A_\rho\right]\right) \tag{B.15}$$

can be written

$$\begin{aligned}\mathcal{L}_1 \;\; = \;\; & -\kappa\mathrm{tr}\left\{A_0\left(\partial_1 A_2\right) - A_0\left(\partial_2 A_1\right) - A_1\left(\partial_0 A_2\right)\right. \\ & +A_1\left(\partial_2 A_0\right) - A_2\left(\partial_1 A_0\right) + A_2\left(\partial_0 A_1\right) \\ & +\frac{2}{3}A_0A_1A_2 - \frac{2}{3}A_0A_2A_1 - \frac{2}{3}A_1A_0A_2 \\ & \left.+\frac{2}{3}A_1A_2A_0 - \frac{2}{3}A_2A_1A_0 + \frac{2}{3}A_2A_0A_1\right\}\end{aligned} \tag{B.16}$$

We recall that every function $A_\mu$ is actually a matrix, in the adjoint representation of the $SU(2)$ algebra. We can use the property of invariance of the operator Trace of a product of matrices to a cyclic permutation of the factors of this product. The third term in the first line without derivatives

$$A_1A_0A_2 \rightarrow A_2A_1A_0 \rightarrow A_0A_2A_1 \tag{B.17}$$

The second term in the second line without derivative

$$A_2A_1A_0 \rightarrow A_0A_2A_1 \tag{B.18}$$

These two terms will add to the second term of the first line, giving

$$-\frac{2}{3}A_0A_2A_1 - \frac{2}{3}A_1A_0A_2 - \frac{2}{3}A_2A_1A_0 \rightarrow -2A_0A_2A_1 \tag{B.19}$$

The first term on the second line without derivatives

$$A_1A_2A_0 \rightarrow A_0A_1A_2 \tag{B.20}$$

The last term

$$A_2A_0A_1 \rightarrow A_1A_2A_0 \rightarrow A_0A_1A_2 \tag{B.21}$$

These two terms are added to the first term of the first line without derivatives

$$\frac{2}{3}A_0A_1A_2 + \frac{2}{3}A_1A_2A_0 + \frac{2}{3}A_2A_0A_1 \rightarrow 2A_0A_1A_2 \tag{B.22}$$

Finnaly, we collect all terms that do not contain derivatives

$$\begin{aligned}& \frac{2}{3}A_0A_1A_2 - \frac{2}{3}A_0A_2A_1 - \frac{2}{3}A_1A_0A_2 \\ & +\frac{2}{3}A_1A_2A_0 - \frac{2}{3}A_2A_1A_0 + \frac{2}{3}A_2A_0A_1 \\ = \;\; & 2A_0A_1A_2 - 2A_0A_2A_1\end{aligned} \tag{B.23}$$

At this point the gauge-field Lagrangean is

$$\begin{aligned}\mathcal{L}_1 &= -\kappa \mathrm{tr}\left\{A_0\left(\partial_1 A_2\right) - A_0\left(\partial_2 A_1\right) - A_1\left(\partial_0 A_2\right)\right. \\ &\quad + A_1\left(\partial_2 A_0\right) - A_2\left(\partial_1 A_0\right) + A_2\left(\partial_0 A_1\right) \\ &\quad \left. + 2A_0 A_1 A_2 - 2A_0 A_2 A_1\right\}\end{aligned} \tag{B.24}$$

**Change of the form of the gauge-field part of the Lagrangean, from integration by parts** Since the Lagrangian density is integrated in order to obtain the action functional, we can consider the effect of integrations by parts. These operations will move the differential operators between the factors of the monomials appearing in the expression of the Lagrangian density and will also generate boundary terms. In general the boundary terms are zero for well-behaved functions, but in our case the presence of a finite condensate of vorticity at infinity can produce finite terms. We will develop below a calculation based on integration by parts, removing the spatial derivatives from acting upon $A_0$ but we will ignore the boundary finite terms. Then the calculation is simply useful for the comparison with other, well known, forms of the CS Lagrangian. We will *not* use the form of the Lagrangian derived from these operations Eq.(B.28), for obtaining the Euler-Lagrange equations, and we will rely on Eq.(B.24).

We turn to the terms containing derivatives. It is possible to make a integrations by parts using the formula

$$\frac{d}{dx}\left(\mathbf{YZ}\right) = \mathbf{Y}^* \frac{d\mathbf{Z}}{dx} + \frac{d\mathbf{Y}^*}{dx}\mathbf{Z} \tag{B.25}$$

We apply this for the following two terms and furthermore we use the cyclic symmetry inside the Trace operator

$$\begin{aligned} A_1\left(\partial_2 A_0\right) &\rightarrow \partial_2\left(A_1^* A_0\right) - \left(\partial_2 A_1\right) A_0 \\ &\rightarrow -A_0\left(\partial_2 A_1\right)\end{aligned} \tag{B.26}$$

$$\begin{aligned} -A_2\left(\partial_1 A_0\right) &\rightarrow -\partial_1\left(A_2^* A_0\right) + \left(\partial_1 A_2\right) A_0 \\ &\rightarrow A_0\left(\partial_1 A_2\right)\end{aligned} \tag{B.27}$$

We collect all terms containing derivatives

$$\begin{aligned} & A_0\left(\partial_1 A_2\right) - A_0\left(\partial_2 A_1\right) - A_1\left(\partial_0 A_2\right) \\ & + A_1\left(\partial_2 A_0\right) - A_2\left(\partial_1 A_0\right) + A_2\left(\partial_0 A_1\right) \\ = \; & 2A_0\left(\partial_1 A_2\right) - 2A_0\left(\partial_2 A_1\right) - A_1\left(\partial_0 A_2\right) + A_2\left(\partial_0 A_1\right)\end{aligned}$$

Finally, the gauge-field Lagrangean density results

$$\begin{aligned} \mathcal{L}_1 &= -\kappa \text{tr} \{2A_0 (\partial_1 A_2) - 2A_0 (\partial_2 A_1) \\ &\quad -A_1 (\partial_0 A_2) + A_2 (\partial_0 A_1) \\ &\quad +2A_0 A_1 A_2 - 2A_0 A_2 A_1\} \end{aligned} \tag{B.28}$$

**Comparison with known forms of the gauge Lagrangean** Let us check this formula by comparing with the situations where the space part is separated [49].

The metric is defined from the general expression of the differential length in the $2+1$ dimensional space, as

$$ds^2 = -dt^2 + h_{ij} dx^i dx^j \tag{B.29}$$

($h^{ij}$ is the space-part of $g^{\mu\nu}$) and one can separate the spatial and temporal parts of the action. In our case the metric is $g = diag\{-1, 1, 1\}$. The action is

$$\mathcal{S}_1 = \int dt \int d^2 x \text{tr} \left\{ -\kappa \varepsilon^{ij} A_i \partial_0 A_j + \kappa \varepsilon^{ij} A_0 F_{ij} \right\} \tag{B.30}$$

where

$$F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j] \tag{B.31}$$

$$F_{i0} = D_i A_0 - \partial_0 A_i \tag{B.32}$$

and

$$\varepsilon^{ij} = \frac{\varepsilon^{0ij}}{\sqrt{h}} \tag{B.33}$$

Then, in the integrand

$$\begin{aligned} & -\varepsilon^{ij} A_i \partial_0 A_j + \varepsilon^{ij} A_0 F_{ij} \\ = & -\varepsilon^{12} A_1 \partial_0 A_2 - \varepsilon^{21} A_2 \partial_0 A_1 \\ & +\varepsilon^{12} A_0 F_{12} + \varepsilon^{21} A_0 F_{21} \end{aligned} \tag{B.34}$$

From Eq.(B.33) where $h$ is the metric, we have

$$\begin{aligned} & -\varepsilon^{ij} A_i \partial_0 A_j + \varepsilon^{ij} A_0 F_{ij} \\ = & -A_1 \partial_0 A_2 + A_2 \partial_0 A_1 \\ & +A_0 F_{12} - A_0 F_{21} \end{aligned} \tag{B.35}$$

Now we replace

$$\begin{aligned} F_{ij} &= \partial_i A_j - \partial_j A_i + [A_i, A_j] \\ &= \partial_i A_j - \partial_j A + A_i A_j - A_j A_i \end{aligned} \tag{B.36}$$

and obtain

$$\begin{aligned}
& -\varepsilon^{ij} A_i \partial_0 A_j + \varepsilon^{ij} A_0 F_{ij} \\
= & -A_1 \partial_0 A_2 + A_2 \partial_0 A_1 \\
& + A_0 \left( \partial_1 A_2 - \partial_2 A_1 + A_1 A_2 - A_2 A_1 \right) \\
& - A_0 \left( \partial_2 A_1 - \partial_1 A_2 + A_2 A_1 - A_1 A_2 \right)
\end{aligned} \tag{B.37}$$

We note that some terms are repeted

$$\begin{aligned}
& -\varepsilon^{ij} A_i \partial_0 A_j + \varepsilon^{ij} A_0 F_{ij} \\
= & -A_1 \partial_0 A_2 + A_2 \partial_0 A_1 + 2A_0 \partial_1 A_2 - 2A_0 \partial_2 A_1 \\
& + 2A_0 A_1 A_2 - 2A_0 A_2 A_1
\end{aligned} \tag{B.38}$$

The result is identical to Eq.(B.28)

$$\begin{aligned}
\mathcal{L}_1 = & -\kappa \mathrm{tr} \left\{ -A_1 \partial_0 A_2 + A_2 \partial_0 A_1 + 2A_0 \partial_1 A_2 - 2A_0 \partial_2 A_1 \right. \\
& \left. + 2A_0 A_1 A_2 - 2A_0 A_2 A_1 \right\}
\end{aligned} \tag{B.39}$$

We also note that until now there was no need to consider summation over components of vectors using the metric coefficients. In case of a product of the form $x_\mu x^\mu$ it will have to consider the metric.

## 12.2 The matter Lagrangean

The form is

$$\mathcal{L}_2 = -\mathrm{tr} \left[ \left( D^\mu \phi \right)^\dagger \left( D_\mu \phi \right) \right] \tag{B.40}$$

Using Eqs.(35), (37) and (39) we can calculate in detail

$$\begin{aligned}
& \mathcal{L}_2 = -\mathrm{tr} \left[ \left( D^\mu \phi \right)^\dagger \left( D_\mu \phi \right) \right] \\
= & -\mathrm{tr} \left[ \left( -\frac{\partial \phi^\dagger}{\partial t} + \phi^\dagger A^{0\dagger} - A^{0\dagger} \phi^\dagger \right) \left( \frac{\partial \phi}{\partial t} + A_0 \phi - \phi A_0 \right) \right. \\
& + \left( \frac{\partial \phi^\dagger}{\partial x} + \phi^\dagger A^{1\dagger} - A^{1\dagger} \phi^\dagger \right) \left( \frac{\partial \phi}{\partial x} + A_1 \phi - \phi A_1 \right) \\
& \left. + \left( \frac{\partial \phi^\dagger}{\partial y} + \phi^\dagger A^{2\dagger} - A^{2\dagger} \phi^\dagger \right) \left( \frac{\partial \phi}{\partial y} + A_2 \phi - \phi A_2 \right) \right]
\end{aligned} \tag{B.41}$$

We have to expand the products

$$
\begin{aligned}
\mathcal{L}_2 \;=\; & -\mathrm{tr}\left\{ -\frac{\partial\phi^\dagger}{\partial t}\frac{\partial\phi}{\partial t} - \frac{\partial\phi^\dagger}{\partial t}A_0\phi + \frac{\partial\phi^\dagger}{\partial t}\phi A_0 \right. \\
& +\phi^\dagger A^{0\dagger}\frac{\partial\phi}{\partial t} + \phi^\dagger A^{0\dagger}A_0\phi - \phi^\dagger A^{0\dagger}\phi A_0 \\
& -A^{0\dagger}\phi^\dagger\frac{\partial\phi}{\partial t} - A^{0\dagger}\phi^\dagger A_0\phi + A^{0\dagger}\phi^\dagger\phi A_0 \\
& +\frac{\partial\phi^\dagger}{\partial x}\frac{\partial\phi}{\partial x} + \frac{\partial\phi^\dagger}{\partial x}A_1\phi - \frac{\partial\phi^\dagger}{\partial x}\phi A_1 \\
& +\phi^\dagger A^{1\dagger}\frac{\partial\phi}{\partial x} + \phi^\dagger A^{1\dagger}A_1\phi - \phi^\dagger A^{1\dagger}\phi A_1 \\
& -A^{1\dagger}\phi^\dagger\frac{\partial\phi}{\partial x} - A^{1\dagger}\phi^\dagger A_1\phi + A^{1\dagger}\phi^\dagger\phi A_1 \\
& +\frac{\partial\phi^\dagger}{\partial y}\frac{\partial\phi}{\partial y} + \frac{\partial\phi^\dagger}{\partial y}A_2\phi - \frac{\partial\phi^\dagger}{\partial y}\phi A_2 \\
& +\phi^\dagger A^{2\dagger}\frac{\partial\phi}{\partial y} + \phi^\dagger A^{2\dagger}A_2\phi - \phi^\dagger A^{2\dagger}\phi A_2 \\
& \left. -A^{2\dagger}\phi^\dagger\frac{\partial\phi}{\partial y} - A^{2\dagger}\phi^\dagger A_2\phi + A^{2\dagger}\phi^\dagger\phi A_2 \right\}
\end{aligned} \tag{B.42}
$$

## 12.3 The Euler-Lagrange equations

The Euler-Lagrange equations

$$
\frac{\partial}{\partial x^\mu}\frac{\delta\mathcal{L}}{\delta\left(\frac{\partial A_\alpha}{\partial x^\mu}\right)} - \frac{\delta\mathcal{L}}{\delta A_\alpha} = 0 \tag{B.43}
$$

We use distinct notations for the three components of the Lagrangean density, $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 - V$ where $\mathcal{L}_1$ is the gauge field part, $\mathcal{L}_2$ is the "matter" part and $V$ is the nonlinear self-interaction potential for the "matter" field. We use the detailed expressions for $\mathcal{L}_1$ from Eq.(B.24) and $\mathcal{L}_2$ is given by the Eq.(B.42). The functional derivations are done separately on these two parts.

### 12.3.1 The formulas for derivation of the Trace of a product of matrices

Use the formulas (see Ref. [47])

$$
\frac{d}{d\mathbf{X}}\mathrm{tr}\left(\mathbf{A}\mathbf{X}\right) = \mathbf{A}^T \tag{B.44}
$$

$$
\frac{d}{d\mathbf{X}}\mathrm{tr}\left(\mathbf{X}\mathbf{A}\right) = \mathbf{A}^T \tag{B.45}
$$

$$\frac{d}{d\mathbf{X}}\mathrm{tr}\left(\mathbf{X}^T\mathbf{A}\right) = \mathbf{A} \tag{B.46}$$

$$\frac{d}{d\mathbf{X}}\mathrm{tr}\left(\mathbf{A}\mathbf{X}^T\right) = \mathbf{A} \tag{B.47}$$

$$\frac{d}{d\mathbf{X}}\mathrm{tr}\left(\mathbf{A}\mathbf{X}\mathbf{B}\right) = \mathbf{A}^T\mathbf{B}^T \tag{B.48}$$

$$\frac{d}{d\mathbf{X}}\mathrm{tr}\left(\mathbf{B}\mathbf{X}^T\mathbf{A}\right) = \mathbf{A}\mathbf{B}$$

$$\frac{d}{d\mathbf{X}}\mathrm{tr}\left(\mathbf{X}\mathbf{A}\mathbf{X}^T\right) = \mathbf{X}\left(\mathbf{A}+\mathbf{A}^T\right) \tag{B.49}$$

$$\frac{d}{d\mathbf{X}}\mathrm{tr}\left(\mathbf{X}^T\mathbf{A}\mathbf{X}\right) = \left(\mathbf{A}+\mathbf{A}^T\right)\mathbf{X} \tag{B.50}$$

$$\frac{d}{d\mathbf{X}}\mathrm{tr}\left(\mathbf{A}\mathbf{X}\mathbf{B}\mathbf{X}\right) = \mathbf{A}^T\mathbf{X}^T\mathbf{B}^T + \mathbf{B}^T\mathbf{X}^T\mathbf{A}^T \tag{B.51}$$

$$\frac{d}{d\mathbf{X}}\mathrm{tr}\left(\mathbf{A}\mathbf{X}\mathbf{B}\mathbf{X}^T\mathbf{C}\right) = \mathbf{A}^T\mathbf{C}^T\mathbf{X}\mathbf{B}^T + \mathbf{C}\mathbf{A}\mathbf{X}\mathbf{B} \tag{B.52}$$

where **A**, **B**, **C**, **X** are arbitrary complex matrices.

## 12.4 The Euler-Lagrange equations for the gauge field

### 12.4.1 The variation to $A_0$

The equation of motion resulting from the variation to $A_0$ is

$$\frac{\partial}{\partial x^\mu}\frac{\delta\mathcal{L}}{\delta\left(\frac{\partial A_0}{\partial x^\mu}\right)} - \frac{\delta\mathcal{L}}{\delta A_0} = 0 \tag{B.53}$$

or

$$\frac{\partial}{\partial x^0}\frac{\delta\mathcal{L}}{\delta\left(\partial_0 A_0\right)} + \frac{\partial}{\partial x^1}\frac{\delta\mathcal{L}}{\delta\left(\partial_1 A_0\right)} + \frac{\partial}{\partial x^2}\frac{\delta\mathcal{L}}{\delta\left(\partial_2 A_0\right)} - \frac{\delta\mathcal{L}}{\delta A_0} = 0 \tag{B.54}$$

**Functional derivatives at $A_0$ of the gauge field Lagrangean** The gauge field Lagrangean is Eq.(B.24)

$$\begin{aligned}\mathcal{L}_1 = & -\kappa\mathrm{tr}\left\{A_0\left(\partial_1 A_2\right) - A_0\left(\partial_2 A_1\right) - A_1\left(\partial_0 A_2\right)\right. \\ & +A_1\left(\partial_2 A_0\right) - A_2\left(\partial_1 A_0\right) + A_2\left(\partial_0 A_1\right) \\ & \left.+2A_0A_1A_2 - 2A_0A_2A_1\right\}\end{aligned} \tag{B.55}$$

and we have to calculate

$$\frac{\partial}{\partial x^0}\frac{\delta\mathcal{L}_1}{\delta\left(\partial_0 A_0\right)} + \frac{\partial}{\partial x^1}\frac{\delta\mathcal{L}_1}{\delta\left(\partial_1 A_0\right)} + \frac{\partial}{\partial x^2}\frac{\delta\mathcal{L}_1}{\delta\left(\partial_2 A_0\right)} - \frac{\delta\mathcal{L}_1}{\delta A_0} \tag{B.56}$$

The first term in the Euler-Lagrange equation (B.56) for $A_0$ is zero since

$$\frac{\delta \mathcal{L}_1}{\delta\left(\partial_0 A_0\right)} = 0$$

For the second term there is only one contribution

$$\begin{aligned} \frac{\partial}{\partial x^1}\frac{\delta \mathcal{L}_1}{\delta\left(\frac{\partial A_0}{\partial x^1}\right)} &= \frac{\partial}{\partial x^1}\frac{\delta}{\delta\left(\partial_1 A_0\right)}(-\kappa)\operatorname{tr}\left\{-A_2\left(\partial_1 A_0\right)\right\} \\ &= -\kappa\frac{\partial}{\partial x^1}\left\{-A_2^T\right\} \\ &= (-\kappa)\left(-\partial_1 A_2^T\right) \end{aligned} \tag{B.57}$$

The third term also consists of one contribution

$$\begin{aligned} \frac{\partial}{\partial x^2}\frac{\delta \mathcal{L}_1}{\delta\left(\frac{\partial A_0}{\partial x^2}\right)} &= \frac{\partial}{\partial x^2}\frac{\delta}{\delta\left(\partial_2 A_0\right)}(-\kappa)\operatorname{tr}\left\{A_1\left(\partial_2 A_0\right)\right\} \\ &= (-\kappa)\frac{\partial}{\partial x^2}\left\{A_1^T\right\} \\ &= (-\kappa)\left(\partial_2 A_1^T\right) \end{aligned} \tag{B.58}$$

The last term in Eq.(**??**) is the derivative of $\mathcal{L}_1$ to the functional variable $A_0$,

$$\begin{aligned} \frac{\delta \mathcal{L}_1}{\delta A_0} = & -\kappa\frac{\delta}{\delta A_0}\operatorname{tr}\left\{A_0\left(\partial_1 A_2\right) - A_0\left(\partial_2 A_1\right) - A_1\left(\partial_0 A_2\right)\right. \\ & +A_1\left(\partial_2 A_0\right) - A_2\left(\partial_1 A_0\right) + A_2\left(\partial_0 A_1\right) \\ & \left.+2A_0A_1A_2 - 2A_0A_2A_1\right\} \end{aligned} \tag{B.59}$$

In detail, every term

$$\frac{\delta}{\delta A_0}\operatorname{tr}\left\{A_0\left(\partial_1 A_2\right)\right\} = \left(\partial_1 A_2\right)^T \tag{B.60}$$

$$\frac{\delta}{\delta A_0}\operatorname{tr}\left\{-A_0\left(\partial_2 A_1\right)\right\} = -\left(\partial_2 A_1\right)^T \tag{B.61}$$

$$\frac{\delta}{\delta A_0}\operatorname{tr}\left\{-A_1\left(\partial_0 A_2\right)\right\} = 0 \tag{B.62}$$

$$\frac{\delta}{\delta A_0}\operatorname{tr}\left\{A_1\left(\partial_2 A_0\right)\right\} = 0 \tag{B.63}$$

$$\frac{\delta}{\delta A_0}\operatorname{tr}\left\{-A_2\left(\partial_1 A_0\right)\right\} = 0 \tag{B.64}$$

$$\frac{\delta}{\delta A_0}\text{tr}\left\{A_2\left(\partial_0 A_1\right)\right\} = 0 \tag{B.65}$$

$$\frac{\delta}{\delta A_0}\text{tr}\left\{2A_0 A_1 A_2\right\} = 2\left(A_1 A_2\right)^T \tag{B.66}$$

$$\frac{\delta}{\delta A_0}\text{tr}\left\{-2A_0 A_2 A_1\right\} = -2\left(A_2 A_1\right)^T \tag{B.67}$$

Collecting these formulas we find

$$\begin{aligned}
\frac{\delta \mathcal{L}_1}{\delta A_0} &= (-\kappa)\left\{\left(\partial_1 A_2\right)^T - \left(\partial_2 A_1\right)^T + 2\left(A_1 A_2\right)^T - 2\left(A_2 A_1\right)^T\right\} \\
&= (-\kappa)\{\partial_1 A_2 - \partial_2 A_1 + A_1 A_2 - A_2 A_1 \\
&\quad A_1 A_2 - A_2 A_1\}^T \\
&= (-\kappa)\left(\partial_1 A_2 - \partial_2 A_1 + [A_1, A_2]\right)^T \\
&\quad + (-\kappa)\left([A_1, A_2]\right)^T
\end{aligned} \tag{B.68}$$

The result of the variation of the gauge-part of the Lagrangian $\mathcal{L}_1$, to the functional variable $A_0$ is obtained from the results Eqs.(B.57), (B.58) and (B.68)

$$\begin{aligned}
& \frac{\partial}{\partial x^1}\frac{\delta \mathcal{L}_1}{\delta\left(\partial_1 A_0\right)} + \frac{\partial}{\partial x^2}\frac{\delta \mathcal{L}_1}{\delta\left(\partial_2 A_0\right)} - \frac{\delta \mathcal{L}_1}{\delta A_0} \\
&= (-\kappa)\left(-\partial_1 A_2^T\right) + (-\kappa)\left(\partial_2 A_1^T\right) \\
&\quad - (-\kappa)\left(\partial_1 A_2 - \partial_2 A_1 + [A_1, A_2]\right)^T \\
&\quad - (-\kappa)\left([A_1, A_2]\right)^T \\
&= \kappa\left(\partial_1 A_2 - \partial_2 A_1 + [A_1, A_2]\right)^T \\
&\quad + (-\kappa)\left\{-\partial_1 A_2^T + \partial_2 A_1^T - [A_1, A_2]^T\right\} \\
&= 2\kappa\left(\partial_1 A_2 - \partial_2 A_1 + [A_1, A_2]\right)^T \\
&= 2\kappa\left(F_{12}\right)^T
\end{aligned} \tag{B.69}$$

**Functional derivative with respect to $A_0$ of the "matter" Lagrangean**
We continue with the variation to the functional variable $A_0$ of the of the "matter" part of the Lagrangean $\mathcal{L}_2$ is

$$\frac{\partial}{\partial x^0}\frac{\delta \mathcal{L}_2}{\delta\left(\partial_0 A_0\right)} + \frac{\partial}{\partial x^1}\frac{\delta \mathcal{L}_2}{\delta\left(\partial_1 A_0\right)} + \frac{\partial}{\partial x^2}\frac{\delta \mathcal{L}_2}{\delta\left(\partial_2 A_0\right)} - \frac{\delta \mathcal{L}_2}{\delta A_0} \tag{B.70}$$

where

$$\mathcal{L}_2 = -\text{tr}\left[\left(D^\mu \phi\right)^\dagger \left(D_\mu \phi\right)\right]$$

has the detailed expression given in Eq.(B.42). The Lagrangean is

$$
\begin{aligned}
\mathcal{L}_2 \;=\; & -\mathrm{tr}\left\{ -\frac{\partial \phi^\dagger}{\partial t}\frac{\partial \phi}{\partial t} - \frac{\partial \phi^\dagger}{\partial t} A_0 \phi + \frac{\partial \phi^\dagger}{\partial t} \phi A_0 \right. \\
& + \phi^\dagger A^{0\dagger} \frac{\partial \phi}{\partial t} + \phi^\dagger A^{0\dagger} A_0 \phi - \phi^\dagger A^{0\dagger} \phi A_0 \\
& - A^{0\dagger} \phi^\dagger \frac{\partial \phi}{\partial t} - A^{0\dagger} \phi^\dagger A_0 \phi + A^{0\dagger} \phi^\dagger \phi A_0 \\
& + \frac{\partial \phi^\dagger}{\partial x}\frac{\partial \phi}{\partial x} + \frac{\partial \phi^\dagger}{\partial x} A_1 \phi - \frac{\partial \phi^\dagger}{\partial x} \phi A_1 \\
& + \phi^\dagger A^{1\dagger} \frac{\partial \phi}{\partial x} + \phi^\dagger A^{1\dagger} A_1 \phi - \phi^\dagger A^{1\dagger} \phi A_1 \\
& - A^{1\dagger} \phi^\dagger \frac{\partial \phi}{\partial x} - A^{1\dagger} \phi^\dagger A_1 \phi + A^{1\dagger} \phi^\dagger \phi A_1 \\
& + \frac{\partial \phi^\dagger}{\partial y}\frac{\partial \phi}{\partial y} + \frac{\partial \phi^\dagger}{\partial y} A_2 \phi - \frac{\partial \phi^\dagger}{\partial y} \phi A_2 \\
& + \phi^\dagger A^{2\dagger} \frac{\partial \phi}{\partial y} + \phi^\dagger A^{2\dagger} A_2 \phi - \phi^\dagger A^{2\dagger} \phi A_2 \\
& \left. - A^{2\dagger} \phi^\dagger \frac{\partial \phi}{\partial y} - A^{2\dagger} \phi^\dagger A_2 \phi + A^{2\dagger} \phi^\dagger \phi A_2 \right\}
\end{aligned}
$$

**Calculation of the variation of the *matter* Lagrangian to the field** $A_0$ The first term is

$$\frac{\partial}{\partial x^0} \frac{\delta \mathcal{L}_2}{\delta (\partial_0 A_0)} = 0$$

since there is no explicit dependence of $\mathcal{L}_2$ on $(\partial_0 A_0)$.

The next two terms in the variation of $\mathcal{L}_2$ are

$$
\begin{aligned}
\frac{\partial}{\partial x^1} \frac{\delta \mathcal{L}_2}{\delta (\partial_1 A_0)} &= 0 \\
\frac{\partial}{\partial x^2} \frac{\delta \mathcal{L}_2}{\delta (\partial_2 A_0)} &= 0
\end{aligned}
$$

Again, there is no dependence of $\mathcal{L}_2$ with respect to $\partial_1 A_0$ and $\partial_2 A_0$ and these contributions are zero.

The last term is

$$\frac{\delta \mathcal{L}_2}{\delta A_0} = -\frac{\delta}{\delta A_0} \mathrm{tr}\left[ (D^\mu \phi)^\dagger (D_\mu \phi) \right] \tag{B.71}$$

Only few terms from $\mathcal{L}_2$ have non-zero contributions

$$\begin{aligned}\frac{\delta \mathcal{L}_2}{\delta A_0} &= -\frac{\delta}{\delta A_0}\mathrm{tr}\left\{-\frac{\partial \phi^\dagger}{\partial t}A_0\phi + \frac{\partial \phi^\dagger}{\partial t}\phi A_0\right. \\ &\quad +\phi^\dagger A^{0\dagger}A_0\phi - \phi^\dagger A^{0\dagger}\phi A_0 \\ &\quad \left. -A^{0\dagger}\phi^\dagger A_0\phi + A^{0\dagger}\phi^\dagger\phi A_0\right\}\end{aligned} \tag{B.72}$$

We calculate in detail every term

$$-\frac{\delta}{\delta A_0}\mathrm{tr}\left\{-\frac{\partial \phi^\dagger}{\partial t}A_0\phi\right\} = \left(\frac{\partial \phi^\dagger}{\partial t}\right)^T (\phi)^T \tag{B.73}$$

$$-\frac{\delta}{\delta A_0}\mathrm{tr}\left\{\frac{\partial \phi^\dagger}{\partial t}\phi A_0\right\} = -\left(\frac{\partial \phi^\dagger}{\partial t}\phi\right)^T \tag{B.74}$$

$$-\frac{\delta}{\delta A_0}\mathrm{tr}\left\{\phi^\dagger A^{0\dagger}A_0\phi\right\} = -\left(\phi^\dagger A^{0\dagger}\right)^T (\phi)^T \tag{B.75}$$

In this is formula we have applied Eq.(B.48) with $\mathbf{A} \equiv \phi^\dagger A^{0\dagger}$, $\mathbf{B} \equiv \phi$ since the functional variables $A^{0\dagger}$ and $A_0$ are independent.

$$-\frac{\delta}{\delta A_0}\mathrm{tr}\left\{-\phi^\dagger A^{0\dagger}\phi A_0\right\} = \left(\phi^\dagger A^{0\dagger}\phi\right)^T \tag{B.76}$$

In this is formula we have applied Eq.(B.45) with $\mathbf{A} \equiv \phi^\dagger A^{0\dagger}\phi$, as explained above.

$$-\frac{\delta}{\delta A_0}\mathrm{tr}\left\{-A^{0\dagger}\phi^\dagger A_0\phi\right\} = \left(A^{0\dagger}\phi^\dagger\right)^T (\phi)^T \tag{B.77}$$

This is formula (B.48) with $\mathbf{A} \equiv A^{0\dagger}\phi^\dagger$, $\mathbf{B} \equiv \phi$.

$$-\frac{\delta}{\delta A_0}\mathrm{tr}\left\{A^{0\dagger}\phi^\dagger\phi A_0\right\} = -\left(A^{0\dagger}\phi^\dagger\phi\right)^T \tag{B.78}$$

Here the Eq.(B.45) has been used with $\mathbf{A} \equiv A_0^\dagger\phi^\dagger\phi$.

Now we sum the results from Eqs.(B.73) to (B.78)

$$\begin{aligned}\frac{\delta \mathcal{L}_2}{\delta A_0} &= \left(\frac{\partial \phi^\dagger}{\partial t}\right)^T (\phi)^T - \left(\frac{\partial \phi^\dagger}{\partial t}\phi\right)^T \\ &\quad -\left(\phi^\dagger A^{0\dagger}\right)^T (\phi)^T \\ &\quad +\left(\phi^\dagger A^{0\dagger}\phi\right)^T \\ &\quad +\left(A^{0\dagger}\phi^\dagger\right)^T (\phi)^T \\ &\quad -\left(A^{0\dagger}\phi^\dagger\phi\right)^T\end{aligned} \tag{B.79}$$

From the first, third and fifth terms we separate to the right the factor $(\phi)^T$, and similarly in the other terms.

$$\frac{\delta\mathcal{L}_2}{\delta A_0} = \left\{\left(\frac{\partial\phi^\dagger}{\partial t}\right)^T - \left(\phi^\dagger A^{0\dagger}\right)^T + \left(A^{0\dagger}\phi^\dagger\right)^T\right\}(\phi)^T + (\phi)^T\left\{-\left(\frac{\partial\phi^\dagger}{\partial t}\right)^T + \left(\phi^\dagger A^{0\dagger}\right)^T - \left(A^{0\dagger}\phi^\dagger\right)^T\right\} \tag{B.80}$$

Now, taking the transpose operator out of the paranthesis,

$$\frac{\delta\mathcal{L}_2}{\delta A_0} = \left\{\phi\left(\frac{\partial\phi^\dagger}{\partial t}\right) - \phi\left(\phi^\dagger A^{0\dagger}\right) + \phi\left(A^{0\dagger}\phi^\dagger\right) - \left(\frac{\partial\phi^\dagger}{\partial t}\right)\phi + \left(\phi^\dagger A^{0\dagger}\right)\phi - \left(A^{0\dagger}\phi^\dagger\right)\phi\right\}^T \tag{B.81}$$

$$\frac{\delta\mathcal{L}_2}{\delta A_0} = \left\{\phi\left[\frac{\partial\phi^\dagger}{\partial t} - \left(\phi^\dagger A^{0\dagger}\right) + \left(A^{0\dagger}\phi^\dagger\right)\right]\right\}^T + \left\{\left[-\frac{\partial\phi^\dagger}{\partial t} + \left(\phi^\dagger A^{0\dagger}\right) - \left(A^{0\dagger}\phi^\dagger\right)\right]\phi\right\}^T \tag{B.82}$$

We now change the upper index 0 into the low index 0 for the potential $A^\dagger$,

$$A^{0\dagger} = -A_0^\dagger$$

and obtain

$$\begin{aligned}\frac{\delta\mathcal{L}_2}{\delta A_0} &= \left\{\phi\left[\frac{\partial\phi^\dagger}{\partial t} + \phi^\dagger A_0^\dagger - A_0^\dagger\phi^\dagger\right]\right\}^T + \left\{\left[-\frac{\partial\phi^\dagger}{\partial t} - \phi^\dagger A_0^\dagger + A_0^\dagger\phi^\dagger\right]\phi\right\}^T \\ &= \left\{\phi\left[\frac{\partial\phi^\dagger}{\partial t} + (A_0\phi)^\dagger - (\phi A_0)^\dagger\right]\right\}^T + \left\{\left[-\frac{\partial\phi^\dagger}{\partial t} - (A_0\phi)^\dagger + (\phi A_0)^\dagger\right]\phi\right\}^T \\ &= \left\{\phi\left[\frac{\partial\phi}{\partial t} + A_0\phi - \phi A_0\right]^\dagger\right\}^T + \left\{-\left[\frac{\partial\phi}{\partial t} + A_0\phi - \phi A_0\right]^\dagger\phi\right\}^T \\ &= \left\{\phi\left(D_0\phi\right)^\dagger\right\}^T - \left\{\left(D_0\phi\right)^\dagger\phi\right\}^T \\ &= \left\{\left[\phi, \left(D_0\phi\right)^\dagger\right]\right\}^T\end{aligned} \tag{B.83}$$

Collecting these results

$$\frac{\partial}{\partial x^0}\frac{\delta\mathcal{L}_2}{\delta(\partial_0 A_0)}+\frac{\partial}{\partial x^1}\frac{\delta\mathcal{L}_2}{\delta(\partial_1 A_0)}+\frac{\partial}{\partial x^2}\frac{\delta\mathcal{L}_2}{\delta(\partial_2 A_0)}-\frac{\delta\mathcal{L}_2}{\delta A_0} \tag{B.84}$$
$$= -\left\{\left[\phi,(D_0\phi)^\dagger\right]\right\}^T$$

**Calculation of the variation with respect to $A_0$ of the scalar potential** This is extremely simple since the scalar potential does not depend on $A_0$ nor of its derivatives to $x^\mu$.

$$\frac{\delta V}{\delta A_0}\equiv 0 \tag{B.85}$$

**Final result for the variation of the Lagrangian with respect to $A_0$** We assemble the partial results: Eq.(B.69) and Eq.(B.84)

$$\left(\frac{\partial}{\partial x^0}\frac{\delta}{\delta(\partial_0 A_0)}+\frac{\partial}{\partial x^1}\frac{\delta}{\delta(\partial_1 A_0)}+\frac{\partial}{\partial x^2}\frac{\delta}{\delta(\partial_2 A_0)}-\frac{\delta}{\delta A_0}\right)(\mathcal{L}_1+\mathcal{L}_2+V) \tag{B.86}$$
$$= 2\kappa(F_{12})^T-\left\{\left[\phi,(D_0\phi)^\dagger\right]\right\}^T$$
$$= 0$$

The equation is

$$2\kappa(F_{12})^T=\left\{\left[\phi,(D_0\phi)^\dagger\right]\right\}^T \tag{B.87}$$

or

$$2\kappa F_{12}=\left[\phi,(D_0\phi)^\dagger\right] \tag{B.88}$$

The left hand side can be written

$$2\kappa F_{12}=\kappa\varepsilon^{0\nu\rho}F_{\nu\rho} \tag{B.89}$$

and we change the order of the terms in the commutator

$$\kappa\varepsilon^{0\nu\rho}F_{\nu\rho} = -\left[(D_0\phi)^\dagger,\phi\right] \tag{B.90}$$
$$= i\times i\left[(D_0\phi)^\dagger,\phi\right]$$

and multiplying with $-1$,

$$-\kappa\varepsilon^{0\nu\rho}F_{\nu\rho}=-i\times\left\{i\left[(D_0\phi)^\dagger,\phi\right]\right\} \tag{B.91}$$

We note that in the right hand side we have a part of the expression of the current

$$J_0 \sim -i\left\{-\left[(D_0\phi)^\dagger, \phi\right]\right\} \tag{B.92}$$

according to the definition of the current

$$\begin{aligned} -\kappa\varepsilon^{0\nu\rho}F_{\nu\rho} &= -iJ_0 \\ &= iJ^0 \end{aligned} \tag{B.93}$$

Then at this point of the derivation it is suggested the following form of the $\mu = 0$ component of the equation of motion

$$-\kappa\varepsilon^{0\nu\rho}F_{\nu\rho} = iJ^0 \tag{B.94}$$

Below we will join to this part of functional variation another part, resulting from the functional varaition with respect to $A^{0\dagger}$.

### 12.4.2 The functional variation with respect to the variable $A^{0\dagger}$

We have to calculate

$$\left(\frac{\partial}{\partial x^0}\frac{\delta}{\delta\left(\partial_0 A^{0\dagger}\right)} + \frac{\partial}{\partial x^1}\frac{\delta}{\delta\left(\partial_1 A^{0\dagger}\right)} + \frac{\partial}{\partial x^2}\frac{\delta}{\delta\left(\partial_2 A^{0\dagger}\right)} - \frac{\delta}{\delta A^{0\dagger}}\right)\left(\mathcal{L}_1 + \mathcal{L}_2 - V\right) = 0 \tag{B.95}$$

Each term is calculated separately.

**Functional derivatives of the gauge field Lagrangian with respect to $A^{0\dagger}$** The gauge-field part of the Lagrangian is

$$\begin{aligned} \mathcal{L}_1 = & -\kappa\mathrm{tr}\left\{A_0\left(\partial_1 A_2\right) - A_0\left(\partial_2 A_1\right) - A_1\left(\partial_0 A_2\right)\right. \\ & + A_1\left(\partial_2 A_0\right) - A_2\left(\partial_1 A_0\right) + A_2\left(\partial_0 A_1\right) \\ & \left. + 2A_0 A_1 A_2 - 2A_0 A_2 A_1\right\} \end{aligned} \tag{B.96}$$

We note that the gauge field Lagrangian $\mathcal{L}_1$ is not expressed in terms of $A_0^\dagger$

$$\left(\frac{\partial}{\partial x^0}\frac{\delta}{\delta\left(\partial_0 A^{0\dagger}\right)} + \frac{\partial}{\partial x^1}\frac{\delta}{\delta\left(\partial_1 A^{0\dagger}\right)} + \frac{\partial}{\partial x^2}\frac{\delta}{\delta\left(\partial_2 A^{0\dagger}\right)} - \frac{\delta}{\delta A^{0\dagger}}\right)\mathcal{L}_1 = 0 \tag{B.97}$$

**Functional derivatives of the "matter" Lagrangian with respect to $A^{0\dagger}$** For the *matter* part of the Lagrangian we have to calculate

$$\left(\frac{\partial}{\partial x^0}\frac{\delta}{\delta\left(\partial_0 A^{0\dagger}\right)}+\frac{\partial}{\partial x^1}\frac{\delta}{\delta\left(\partial_1 A^{0\dagger}\right)}+\frac{\partial}{\partial x^2}\frac{\delta}{\delta\left(\partial_2 A^{0\dagger}\right)}-\frac{\delta}{\delta A^{0\dagger}}\right)\mathcal{L}_2$$

where $\mathcal{L}_2$ is given in Eq.(B.42).

We have

$$\frac{\partial}{\partial x^0}\frac{\delta}{\delta\left(\partial_0 A^{0\dagger}\right)}\mathcal{L}_2=0 \tag{B.98}$$

$$\frac{\partial}{\partial x^1}\frac{\delta}{\delta\left(\partial_1 A^{0\dagger}\right)}\mathcal{L}_2=0 \tag{B.99}$$

$$\frac{\partial}{\partial x^2}\frac{\delta}{\delta\left(\partial_2 A^{0\dagger}\right)}\mathcal{L}_2=0 \tag{B.100}$$

Few of the terms in Eq.(B.42) can provide a non-zero contribution

$$\begin{aligned}\frac{\delta\mathcal{L}_2}{\delta A^{0\dagger}} \;=\; & -\frac{\delta}{\delta A^{0\dagger}}\mathrm{tr}\left\{\phi^\dagger A^{0\dagger}\frac{\partial\phi}{\partial t}-A^{0\dagger}\phi^\dagger\frac{\partial\phi}{\partial t}\right.\\ & +\phi^\dagger A^{0\dagger}A_0\phi-\phi^\dagger A^{0\dagger}\phi A_0\\ & \left.-A^{0\dagger}\phi^\dagger A_0\phi+A^{0\dagger}\phi^\dagger\phi A_0\right\}\end{aligned} \tag{B.101}$$

In detail, the terms are

$$-\frac{\delta}{\delta A^{0\dagger}}\mathrm{tr}\left\{\phi^\dagger A^{0\dagger}\frac{\partial\phi}{\partial t}\right\}=-\left(\phi^\dagger\right)^T\left(\frac{\partial\phi}{\partial t}\right)^T \tag{B.102}$$

$$-\frac{\delta}{\delta A^{0\dagger}}\mathrm{tr}\left\{-A^{0\dagger}\phi^\dagger\frac{\partial\phi}{\partial t}\right\}=\left(\phi^\dagger\frac{\partial\phi}{\partial t}\right)^T \tag{B.103}$$

$$-\frac{\delta}{\delta A^{0\dagger}}\mathrm{tr}\left\{\phi^\dagger A^{0\dagger}A_0\phi\right\}=-\left(\phi^\dagger\right)^T\left(A_0\phi\right)^T \tag{B.104}$$

$$-\frac{\delta}{\delta A^{0\dagger}}\mathrm{tr}\left\{-\phi^\dagger A^{0\dagger}\phi A_0\right\}=\left(\phi^\dagger\right)^T\left(\phi A_0\right)^T \tag{B.105}$$

$$-\frac{\delta}{\delta A^{0\dagger}}\mathrm{tr}\left\{-A^{0\dagger}\phi^\dagger A_0\phi\right\}=\left(\phi^\dagger A_0\phi\right)^T \tag{B.106}$$

$$-\frac{\delta}{\delta A^{0\dagger}}\mathrm{tr}\left\{A^{0\dagger}\phi^\dagger\phi A_0\right\}=-\left(\phi^\dagger\phi A_0\right)^T \tag{B.107}$$

The results are now added

$$
\begin{aligned}
\frac{\delta \mathcal{L}_2}{\delta A^{0\dagger}} \;=\; & -\left(\phi^\dagger\right)^T \left(\frac{\partial \phi}{\partial t}\right)^T \\
& + \left(\phi^\dagger \frac{\partial \phi}{\partial t}\right)^T \\
& - \left(\phi^\dagger\right)^T \left(A_0 \phi\right)^T \\
& + \left(\phi^\dagger\right)^T \left(\phi A_0\right)^T \\
& + \left(\phi^\dagger A_0 \phi\right)^T \\
& - \left(\phi^\dagger \phi A_0\right)^T
\end{aligned}
\tag{B.108}
$$

In three terms we left-factorize $\left(\phi^\dagger\right)^T$

$$
\begin{aligned}
& \left(\phi^\dagger\right)^T \left\{ -\left(\frac{\partial \phi}{\partial t}\right)^T - \left(A_0 \phi\right)^T + \left(\phi A_0\right)^T \right\} \\
= \; & \left(\phi^\dagger\right)^T \left\{ -\left(\frac{\partial \phi}{\partial t}\right)^T - \left(\phi^T A_0^T - A_0^T \phi^T\right) \right\} \\
= \; & \left(\phi^\dagger\right)^T \left\{ -\left(\frac{\partial \phi}{\partial t}\right)^T - \left[\phi^T, A_0^T\right] \right\} \\
= \; & -\left(\phi^\dagger\right)^T \left\{ \frac{\partial \phi}{\partial t} + \left[A_0, \phi\right] \right\}^T \\
= \; & -\left(\phi^\dagger\right)^T \left(D_0 \phi\right)^T
\end{aligned}
\tag{B.109}
$$

For the other three terms the result is similar

$$
\begin{aligned}
& \left\{ \left(\frac{\partial \phi}{\partial t}\right)^T + \left(A_0 \phi\right)^T - \left(\phi A_0\right)^T \right\} \left(\phi^\dagger\right)^T \\
= \; & \left\{ \left(\frac{\partial \phi}{\partial t}\right)^T + \left(\phi^T A_0^T - A_0^T \phi^T\right) \right\} \left(\phi^\dagger\right)^T \\
= \; & \left\{ \left(\frac{\partial \phi}{\partial t}\right)^T + \left[\phi^T, A_0^T\right] \right\} \left(\phi^\dagger\right)^T \\
= \; & \left\{ \left(\frac{\partial \phi}{\partial t}\right)^T + \left[A_0, \phi\right]^T \right\} \left(\phi^\dagger\right)^T \\
= \; & \left(D_0 \phi\right)^T \left(\phi^\dagger\right)^T
\end{aligned}
\tag{B.110}
$$

Then

$$\begin{aligned}\frac{\delta\mathcal{L}_2}{\delta A^{0\dagger}} &= -\left(\phi^\dagger\right)^T (D_0\phi)^T + (D_0\phi)^T \left(\phi^\dagger\right)^T \\ &= \left[(D_0\phi)^T, \left(\phi^\dagger\right)^T\right] \\ &= \left[\phi^\dagger, D_0\phi\right]^T\end{aligned} \tag{B.111}$$

The Euler Lagrange equation is, for $A^{0\dagger}$

$$\frac{\partial}{\partial x^\mu}\frac{\delta\mathcal{L}}{\delta\left(\frac{\partial A^{0\dagger}}{\partial x^\mu}\right)} - \frac{\delta\mathcal{L}}{\delta A^{0\dagger}} = 0 \tag{B.112}$$

The Euler Lagrange equation in the case of $A^{0\dagger}$ is reduced to only the last term

$$-\frac{\delta\mathcal{L}}{\delta A^{0\dagger}} = -\frac{\delta\mathcal{L}_2}{\delta A^{0\dagger}} \tag{B.113}$$

and is written from Eq.(B.111)

$$-\frac{\delta\mathcal{L}_2}{\delta A^{0\dagger}} = -\left[\phi^\dagger, D_0\phi\right]^T \tag{B.114}$$

### 12.4.3 The Euler-Lagrange equation derived from functional variation to $A_0$

We now collect the results of the functional derivatives from both the gauge and the matter parts of the Lagrangean, in the Euler-Lagrange equation for $A_0$ and *add* the zero-valued term resulted from the functional variation with respect to $A^{0\dagger}$. The formulas to be used are Eq.(B.69), Eq.(B.84) and Eq.(B.114)

$$\begin{aligned}\left\{-\left(\frac{\delta\mathcal{L}_1}{\delta A_0} + \frac{\delta\mathcal{L}_2}{\delta A_0}\right)\right\} + \left\{-\frac{\delta\mathcal{L}_2}{\delta A^{0\dagger}}\right\} &= 0 \\ 2\kappa\left(F_{12}\right)^T - \left[\phi, (D_0\phi)^\dagger\right]^T - \left[\phi^\dagger, D_0\phi\right]^T &= 0\end{aligned} \tag{B.115}$$

or,

$$-2\kappa F_{12} = -\left[\phi^\dagger, D_0\phi\right] - \left[\phi, (D_0\phi)^\dagger\right]$$

and interchanging the factors in the second commutator

$$-2\kappa F_{12} = -\left[\phi^\dagger, D_0\phi\right] + \left[(D_0\phi)^\dagger, \phi\right] \tag{B.116}$$

The left hand side is the zero component of a tensorial contraction

$$
\begin{aligned}
-\kappa\left(\varepsilon^{012}F_{12}+\varepsilon^{021}F_{21}\right) &= -\left\{\left[\phi^{\dagger},D_0\phi\right]-\left[\left(D_0\phi\right)^{\dagger},\phi\right]\right\} \qquad (B.117)\\
&= -i\times\left\{-i\left(\left[\phi^{\dagger},D_0\phi\right]-\left[\left(D_0\phi\right)^{\dagger},\phi\right]\right)\right\}
\end{aligned}
$$

We will identify the right hand side as the covariant 0-component of a current

$$
J_0=-i\left\{\left[\phi^{\dagger},D_0\phi\right]-\left[\left(D_0\phi\right)^{\dagger},\phi\right]\right\} \qquad (B.118)
$$

and this equation takes the form of the Gauss law constraint from the main text

$$
-\kappa\varepsilon^{0\mu\nu}F_{\mu\nu}=-iJ_0=iJ^0 \qquad (B.119)
$$

Therefore we conclude that we have derived the 0 component of the equation

$$
-\kappa\varepsilon^{\mu\nu\rho}F_{\nu\rho}=iJ^{\mu}
$$

### 12.4.4 The Euler-Lagrange equation from the variation to $A_1$

The detailed equation is

$$
\frac{\partial}{\partial x^{\mu}}\frac{\delta\mathcal{L}}{\delta\left(\frac{\partial A_1}{\partial x^{\mu}}\right)}-\frac{\delta\mathcal{L}}{\delta A_1}=0 \qquad (B.120)
$$

and shows a difference compared to the case of $A_0$: now there is a dependence of $\mathcal{L}_1$ on the derivatives of the field $A_1$.

We have to calculate

$$
\left(\frac{\partial}{\partial x^0}\frac{\delta}{\delta\left(\partial_0 A_1\right)}+\frac{\partial}{\partial x^1}\frac{\delta}{\delta\left(\partial_1 A_1\right)}+\frac{\partial}{\partial x^2}\frac{\delta}{\delta\left(\partial_2 A_1\right)}-\frac{\delta}{\delta A_1}\right)\left(\mathcal{L}_1+\mathcal{L}_2-V\right)=0
$$

**The functional variation with respect to $A_1$ of the *gauge*-field part of Lagrangian** This part is

$$
\left(\frac{\partial}{\partial x^0}\frac{\delta}{\delta\left(\partial_0 A_1\right)}+\frac{\partial}{\partial x^1}\frac{\delta}{\delta\left(\partial_1 A_1\right)}+\frac{\partial}{\partial x^2}\frac{\delta}{\delta\left(\partial_2 A_1\right)}-\frac{\delta}{\delta A_1}\right)\mathcal{L}_1 \qquad (B.121)
$$

and $\mathcal{L}_1$ is given by Eq.(B.24)

$$
\begin{aligned}
\mathcal{L}_1 = & -\kappa\mathrm{tr}\left\{A_0\left(\partial_1 A_2\right)-A_0\left(\partial_2 A_1\right)-A_1\left(\partial_0 A_2\right)\right.\\
& +A_1\left(\partial_2 A_0\right)-A_2\left(\partial_1 A_0\right)+A_2\left(\partial_0 A_1\right)\\
& \left.+2A_0A_1A_2-2A_0A_2A_1\right\}
\end{aligned}
$$

Term by term

$$\frac{\partial}{\partial x^0}\frac{\delta \mathcal{L}_1}{\delta\left(\partial_0 A_1\right)} = \frac{\partial}{\partial x^0}\frac{\delta}{\delta\left(\partial_0 A_1\right)}(-\kappa)\,\mathrm{tr}\left\{A_2\left(\partial_0 A_1\right)\right\} = -\kappa\left(\partial_0 A_2^T\right) \tag{B.122}$$

The second term in Eq.(B.121)

$$\frac{\partial}{\partial x^1}\frac{\delta \mathcal{L}_1}{\delta\left(\partial_1 A_1\right)} = 0 \tag{B.123}$$

The third term from the Eq.(B.121)

$$\frac{\partial}{\partial x^2}\frac{\delta \mathcal{L}_1}{\delta\left(\partial_2 A_1\right)} = \frac{\partial}{\partial x^2}\frac{\delta}{\delta\left(\partial_2 A_1\right)}(-\kappa)\,\mathrm{tr}\left\{-A_0\left(\partial_2 A_1\right)\right\} = \kappa\left(\partial_2 A_0^T\right) \tag{B.124}$$

The last term in Eq.(B.121) is

$$\frac{\delta \mathcal{L}_1}{\delta A_1} = \frac{\delta}{\delta A_1}(-\kappa)\left\{-A_1\left(\partial_0 A_2\right)+A_1\left(\partial_2 A_0\right)+2A_0A_1A_2-2A_0A_2A_1\right\} \tag{B.125}$$

In detail

$$\frac{\delta}{\delta A_1}\left\{-A_1\left(\partial_0 A_2\right)\right\} = -\left(\partial_0 A_2\right)^T$$

$$\frac{\delta}{\delta A_1}\left\{A_1\left(\partial_2 A_0\right)\right\} = \left(\partial_2 A_0\right)^T$$

$$\frac{\delta}{\delta A_1}\left\{2A_0A_1A_2\right\} = 2A_0^TA_2^T$$

$$\frac{\delta}{\delta A_1}\left\{-2A_0A_2A_1\right\} = -2\left(A_0A_2\right)^T$$

It results

$$\frac{\delta \mathcal{L}_1}{\delta A_1} = (-\kappa)\left\{-\left(\partial_0 A_2\right)^T+\left(\partial_2 A_0\right)^T+2A_0^TA_2^T-2\left(A_0A_2\right)^T\right\}$$

Collecting the results for all the *gauge* field Lagrangian

$$\left(\frac{\partial}{\partial x^0}\frac{\delta}{\delta\left(\partial_0 A_1\right)}+\frac{\partial}{\partial x^1}\frac{\delta}{\delta\left(\partial_1 A_1\right)}+\frac{\partial}{\partial x^2}\frac{\delta}{\delta\left(\partial_2 A_1\right)}-\frac{\delta}{\delta A_1}\right)\mathcal{L}_1 \tag{B.126}$$

$$= -\kappa\left(\partial_0 A_2^T\right)+\kappa\left(\partial_2 A_0^T\right)-(-\kappa)\left\{-\left(\partial_0 A_2\right)^T+\left(\partial_2 A_0\right)^T+2A_0^TA_2^T-2\left(A_0A_2\right)^T\right\}$$

$$= 2\kappa\left\{-\left(\partial_0 A_2^T\right)+\left(\partial_2 A_0^T\right)+A_0^TA_2^T-A_2^TA_0^T\right\}$$

We can write

$$\begin{aligned}
& 2\kappa\left\{-\left(\partial_0 A_2^T\right)+\left(\partial_2 A_0^T\right)+A_0^T A_2^T-A_2^T A_0^T\right\} \\
= & -2\kappa\left\{\partial_0 A_2-\partial_2 A_0-A_2 A_0+A_0 A_2\right\}^T \\
= & -2\kappa\left\{\partial_0 A_2-\partial_2 A_0+\left[A_0, A_2\right]\right\}^T \\
= & -2\kappa F_{02}^T
\end{aligned}$$

**The functional variation at $A_1$ of the *matter* part of the Lagrangean**
The calculations are similar to the previous case for $A_0$.

We have to calculate

$$\left(\frac{\partial}{\partial x^0}\frac{\delta}{\delta\left(\partial_0 A_1\right)}+\frac{\partial}{\partial x^1}\frac{\delta}{\delta\left(\partial_1 A_1\right)}+\frac{\partial}{\partial x^2}\frac{\delta}{\delta\left(\partial_2 A_1\right)}-\frac{\delta}{\delta A_1}\right)\mathcal{L}_2$$

where $\mathcal{L}_2$ is given in Eq.(B.42).

We have

$$\frac{\partial}{\partial x^0}\frac{\delta}{\delta\left(\partial_0 A_1\right)}\mathcal{L}_2=0$$

$$\frac{\partial}{\partial x^1}\frac{\delta}{\delta\left(\partial_1 A_1\right)}\mathcal{L}_2=0$$

$$\frac{\partial}{\partial x^2}\frac{\delta}{\delta\left(\partial_2 A_1\right)}\mathcal{L}_2=0$$

$$\begin{aligned}
\frac{\delta\mathcal{L}_2}{\delta A_1} &= -\frac{\delta}{\delta A_1}\operatorname{tr}\left[\left(D^\mu\phi\right)^\dagger\left(D_\mu\phi\right)\right] \qquad (B.127)\\
&= -\frac{\delta}{\delta A_1}\operatorname{tr}\left[-\left(D_0\phi\right)^\dagger\left(D_0\phi\right)+\left(D_1\phi\right)^\dagger\left(D_1\phi\right)+\left(D_2\phi\right)^\dagger\left(D_2\phi\right)\right]
\end{aligned}$$

$$\frac{\delta\mathcal{L}_2}{\delta A_1}=-\frac{\delta}{\delta A_1}\operatorname{tr}\left\{\left(D_1\phi\right)^\dagger\left(D_1\phi\right)\right\} \qquad (B.128)$$

$$\begin{aligned}
\frac{\delta\mathcal{L}_2}{\delta A_1} = & -\frac{\delta}{\delta A_1}\operatorname{tr}\left\{\frac{\partial\phi^\dagger}{\partial x}A_1\phi-\frac{\partial\phi^\dagger}{\partial x}\phi A_1\right. \qquad (B.129)\\
& +\phi^\dagger A^{1\dagger}A_1\phi-\phi^\dagger A^{1\dagger}\phi A_1 \\
& \left.-A^{1\dagger}\phi^\dagger A_1\phi+A^{1\dagger}\phi^\dagger\phi A_1\right\}
\end{aligned}$$

Term by term

$$-\frac{\delta}{\delta A_1}\operatorname{tr}\left\{\frac{\partial\phi^\dagger}{\partial x^1}A_1\phi\right\}=-\left(\frac{\partial\phi^\dagger}{\partial x^1}\right)^T(\phi)^T \qquad (B.130)$$

$$- \frac{\delta}{\delta A_1} \mathrm{tr} \left\{ - \frac{\partial \phi^\dagger}{\partial x^1} \phi A_1 \right\} = \left( \frac{\partial \phi^\dagger}{\partial x^1} \phi \right)^T \tag{B.131}$$

$$- \frac{\delta}{\delta A_1} \mathrm{tr} \left\{ \phi^\dagger A^{1\dagger} A_1 \phi \right\} = - \left( \phi^\dagger A^{1\dagger} \right)^T (\phi)^T \tag{B.132}$$

$$- \frac{\delta}{\delta A_1} \mathrm{tr} \left\{ - \phi^\dagger A^{1\dagger} \phi A_1 \right\} = \left( \phi^\dagger A^{1\dagger} \phi \right)^T \tag{B.133}$$

$$- \frac{\delta}{\delta A_1} \mathrm{tr} \left\{ - A^{1\dagger} \phi^\dagger A_1 \phi \right\} = \left( A^{1\dagger} \phi^\dagger \right)^T (\phi)^T \tag{B.134}$$

$$- \frac{\delta}{\delta A_1} \mathrm{tr} \left\{ A^{1\dagger} \phi^\dagger \phi A_1 \right\} = - \left( A^{1\dagger} \phi^\dagger \phi \right)^T \tag{B.135}$$

Now we collect all these contributions

$$\begin{aligned} \frac{\delta \mathcal{L}_2}{\delta A_1} = & - \left( \frac{\partial \phi^\dagger}{\partial x^1} \right)^T (\phi)^T + \left( \frac{\partial \phi^\dagger}{\partial x^1} \phi \right)^T \\ & - \left( \phi^\dagger A^{1\dagger} \right)^T (\phi)^T \\ & + \left( \phi^\dagger A^{1\dagger} \phi \right)^T \\ & + \left( A^{1\dagger} \phi^\dagger \right)^T (\phi)^T \\ & - \left( A^{1\dagger} \phi^\dagger \phi \right)^T \end{aligned} \tag{B.136}$$

As in the case of $A_0$ equation, we right-factorise from the first, third and fifth terms

$$\begin{aligned} & - \left( \frac{\partial \phi^\dagger}{\partial x^1} \right)^T (\phi)^T - \left( \phi^\dagger A^{1\dagger} \right)^T (\phi)^T + \left( A^{1\dagger} \phi^\dagger \right)^T (\phi)^T \\ = & - \left\{ \left( \frac{\partial \phi^\dagger}{\partial x^1} \right)^T + \left( \phi^\dagger A^{1\dagger} \right)^T - \left( A^{1\dagger} \phi^\dagger \right)^T \right\} (\phi)^T \\ = & - \left\{ \frac{\partial \phi^\dagger}{\partial x^1} + \left[ \phi^\dagger, A^{1\dagger} \right] \right\}^T (\phi)^T \\ = & - \left\{ (D_1 \phi)^\dagger \right\}^T (\phi)^T \end{aligned} \tag{B.137}$$

In a similar way we have from the second, fourth and sixth terms

$$\begin{aligned}
& \left(\frac{\partial \phi^\dagger}{\partial x^1}\phi\right)^T + \left(\phi^\dagger A^{1\dagger}\phi\right)^T - \left(A^{1\dagger}\phi^\dagger\phi\right)^T \\
= \;& (\phi)^T \left\{ \left(\frac{\partial \phi^\dagger}{\partial x^1}\right)^T + \left(\phi^\dagger A^{1\dagger}\right)^T - \left(A^{1\dagger}\phi^\dagger\right)^T \right\} \\
= \;& (\phi)^T \left\{ \left(\frac{\partial \phi^\dagger}{\partial x^1}\right)^T + \left(\phi^\dagger A^{1\dagger} - A^{1\dagger}\phi^\dagger\right)^T \right\} \\
= \;& (\phi)^T \left\{ \frac{\partial \phi^\dagger}{\partial x^1} + \left[\phi^\dagger, A^{1\dagger}\right] \right\}^T \\
= \;& (\phi)^T \left\{ (D_1\phi)^\dagger \right\}^T
\end{aligned} \tag{B.138}$$

Then, finally

$$\begin{aligned}
\frac{\delta \mathcal{L}_2}{\delta A_1} &= -\left\{(D_1\phi)^\dagger\right\}^T (\phi)^T + (\phi)^T \left\{(D_1\phi)^\dagger\right\}^T \\
&= \left\{(D_1\phi)^\dagger \phi\right\}^T - \left\{\phi\,(D_1\phi)^\dagger\right\}^T \\
&= \left\{\left[(D_1\phi)^\dagger, \phi\right]\right\}^T
\end{aligned}$$

Adding all contributions

$$\frac{\partial}{\partial x^\mu}\frac{\delta \mathcal{L}_2}{\delta\left(\frac{\partial A_1}{\partial x^\mu}\right)} - \frac{\delta \mathcal{L}_2}{\delta A_1} = -\left\{\left[(D_1\phi)^\dagger, \phi\right]\right\}^T \tag{B.139}$$

The total functional variation (gauge and matter parts)

$$\begin{aligned}
& \left(\frac{\partial}{\partial x^0}\frac{\delta}{\delta(\partial_0 A_1)} + \frac{\partial}{\partial x^1}\frac{\delta}{\delta(\partial_1 A_1)} + \frac{\partial}{\partial x^2}\frac{\delta}{\delta(\partial_2 A_1)} - \frac{\delta}{\delta A_1}\right)\mathcal{L}_1 \\
& + \left(\frac{\partial}{\partial x^0}\frac{\delta}{\delta(\partial_0 A_1)} + \frac{\partial}{\partial x^1}\frac{\delta}{\delta(\partial_1 A_1)} + \frac{\partial}{\partial x^2}\frac{\delta}{\delta(\partial_2 A_1)} - \frac{\delta}{\delta A_1}\right)\mathcal{L}_2 \\
= \;& 0
\end{aligned}$$

$$\begin{aligned}
& -2\kappa F_{02}^T \\
& -\left\{\left[(D_1\phi)^\dagger, \phi\right]\right\}^T \\
= \;& 0
\end{aligned} \tag{B.140}$$

Before concluding the calculation for $A_1$ , and having in mind a possible more symmetrical form, we study the functional variations to the field $A^{1\dagger}$.

### 12.4.5 Functional variations to the field $A^{1\dagger}$

The equation is

$$\frac{\partial}{\partial x^\mu}\frac{\delta\left(\mathcal{L}_1+\mathcal{L}_2\right)}{\delta\left(\frac{\partial A^{1\dagger}}{\partial x^\mu}\right)}-\frac{\delta\left(\mathcal{L}_1+\mathcal{L}_2\right)}{\delta A^{1\dagger}}=0 \tag{B.141}$$

since only from gauge and matter we expect contributions. But we note that the gauge part $\mathcal{L}_1$ has no dependence on the fields $\partial_\mu A^{1\dagger}$ and $A^{1\dagger}$

$$\begin{aligned}\frac{\delta\mathcal{L}_1}{\delta\left(\partial_\mu A^{1\dagger}\right)} &= 0 \\ \frac{\delta\mathcal{L}_1}{\delta A^{1\dagger}} &= 0\end{aligned} \tag{B.142}$$

**The variation at $A_1^\dagger$ of the matter part of the Lagrangean** The matter part does not contain any term with $\partial_\mu A^{1\dagger}$ which means that it also cannot contribute to the equation. The only contribution may arise from the variation of matter part, $\mathcal{L}_2$, to the field $A^{1\dagger}$.The formulas start with

$$\begin{aligned}\frac{\delta\mathcal{L}_2}{\delta A^{1\dagger}} &= -\frac{\delta}{\delta A^{1\dagger}}\mathrm{tr}\left[\left(D^\mu\phi\right)^\dagger\left(D_\mu\phi\right)\right] \\ &= -\frac{\delta}{\delta A^{1\dagger}}\mathrm{tr}\left[-\left(D_0\phi\right)^\dagger\left(D_0\phi\right)+\left(D_1\phi\right)^\dagger\left(D_1\phi\right)+\left(D_2\phi\right)^\dagger\left(D_2\phi\right)\right]\end{aligned} \tag{B.143}$$

Only the second term must be retained

$$\frac{\delta\mathcal{L}_2}{\delta A^{1\dagger}}=-\frac{\delta}{\delta A^{1\dagger}}\mathrm{tr}\left\{\left(D_1\phi\right)^\dagger\left(D_1\phi\right)\right\} \tag{B.144}$$

$$\begin{aligned}\frac{\delta\mathcal{L}_2}{\delta A^{1\dagger}} = & -\frac{\delta}{\delta A^{1\dagger}}\mathrm{tr}\left\{\phi^\dagger A^{1\dagger}\frac{\partial\phi}{\partial x^1}+\phi^\dagger A^{1\dagger}A_1\phi-\phi^\dagger A^{1\dagger}\phi A_1\right. \\ & \left.-A^{1\dagger}\phi^\dagger\frac{\partial\phi}{\partial x^1}-A^{1\dagger}\phi^\dagger A_1\phi+A^{1\dagger}\phi^\dagger\phi A_1\right\}\end{aligned} \tag{B.145}$$

Term by term

$$-\frac{\delta}{\delta A^{1\dagger}}\mathrm{tr}\left\{\phi^\dagger A^{1\dagger}\frac{\partial\phi}{\partial x^1}\right\}=-\left(\phi^\dagger\right)^T\left(\frac{\partial\phi}{\partial x^1}\right)^T \tag{B.146}$$

$$-\frac{\delta}{\delta A^{1\dagger}}\mathrm{tr}\left\{\phi^\dagger A^{1\dagger}A_1\phi\right\}=-\left(\phi^\dagger\right)^T\left(A_1\phi\right)^T \tag{B.147}$$

$$-\frac{\delta}{\delta A^{1\dagger}}\mathrm{tr}\left\{-\phi^\dagger A^{1\dagger}\phi A_1\right\}=\left(\phi^\dagger\right)^T\left(\phi A_1\right)^T \tag{B.148}$$

$$-\frac{\delta}{\delta A^{1\dagger}}\mathrm{tr}\left\{-A^{1\dagger}\phi^\dagger\frac{\partial\phi}{\partial x^1}\right\} = \left(\phi^\dagger\frac{\partial\phi}{\partial x^1}\right)^T \tag{B.149}$$

$$-\frac{\delta}{\delta A^{1\dagger}}\mathrm{tr}\left\{-A^{1\dagger}\phi^\dagger A_1\phi\right\} = \left(\phi^\dagger A_1\phi\right)^T \tag{B.150}$$

$$-\frac{\delta}{\delta A^{1\dagger}}\mathrm{tr}\left\{A^{1\dagger}\phi^\dagger\phi A_1\right\} = -\left(\phi^\dagger\phi A_1\right)^T \tag{B.151}$$

Finally the sum of contributions

$$\begin{aligned}\frac{\delta\mathcal{L}_2}{\delta A_1^\dagger} &= -\left(\phi^\dagger\right)^T\left(\frac{\partial\phi}{\partial x^1}\right)^T \\ &\quad -\left(\phi^\dagger\right)^T(A_1\phi)^T \\ &\quad +\left(\phi^\dagger\right)^T(\phi A_1)^T \\ &\quad +\left(\phi^\dagger\frac{\partial\phi}{\partial x^1}\right)^T \\ &\quad +\left(\phi^\dagger A_1\phi\right)^T \\ &\quad -\left(\phi^\dagger\phi A_1\right)^T\end{aligned} \tag{B.152}$$

The first three terms can be written in the compact form

$$\begin{aligned}&-\left(\phi^\dagger\right)^T\left(\frac{\partial\phi}{\partial x^1}\right)^T-\left(\phi^\dagger\right)^T(A_1\phi)^T+\left(\phi^\dagger\right)^T(\phi A_1)^T \\ =\ &-\left(\phi^\dagger\right)^T\left\{\left(\frac{\partial\phi}{\partial x^1}\right)^T+(A_1\phi)^T-(\phi A_1)^T\right\} \\ =\ &-\left(\phi^\dagger\right)^T\left\{\left(\frac{\partial\phi}{\partial x^1}\right)^T+[A_1,\phi]^T\right\} \\ =\ &-\left(\phi^\dagger\right)^T(D_1\phi)^T\end{aligned} \tag{B.153}$$

and the other three terms

$$\begin{aligned}&\left(\phi^\dagger\frac{\partial\phi}{\partial x^1}\right)^T+\left(\phi^\dagger A_1\phi\right)^T-\left(\phi^\dagger\phi A_1\right)^T \\ =\ &\left\{\left(\frac{\partial\phi}{\partial x^1}\right)^T+(A_1\phi)^T-(\phi A_1)^T\right\}\left(\phi^\dagger\right)^T \\ =\ &\left\{\left(\frac{\partial\phi}{\partial x^1}\right)^T+[A_1,\phi]^T\right\}\left(\phi^\dagger\right)^T \\ =\ &(D_1\phi)^T\left(\phi^\dagger\right)^T\end{aligned} \tag{B.154}$$

The full result

$$\begin{aligned}\frac{\delta\mathcal{L}_2}{\delta A^{1\dagger}} &= -\left(\phi^\dagger\right)^T (D_1\phi)^T + (D_1\phi)^T \left(\phi^\dagger\right)^T \qquad \text{(B.155)}\\ &= \left\{\phi^\dagger (D_1\phi) - (D_1\phi)\,\phi^\dagger\right\}^T \\ &= -\left\{\left[D_1\phi, \phi^\dagger\right]\right\}^T\end{aligned}$$

and the contribution of $A^{1\dagger}$ to the Euler Lagrange equation has the form

$$-\frac{\delta\mathcal{L}_2}{\delta A^{1\dagger}} = -\left\{\left[D_1\phi, \phi^\dagger\right]\right\}^T \qquad \text{(B.156)}$$

### 12.4.6 The final form of the Euler-Lagrange equation derived from functional variation to $A_1$

We now collect the results of the functional derivatives from both the gauge and the matter parts of the Lagrangean, in the Euler-Lagrange equation for $A_1$ and add the zero-valued term from the functional derivation to $A^{1\dagger}$, Eqs. (B.140), (B.156). Therefore we have to combine the following two results

$$-2\kappa F_{02}^T - \left\{\left[(D_1\phi)^\dagger, \phi\right]\right\}^T \qquad \text{(B.157)}$$

and

$$\left\{\left[D_1\phi, \phi^\dagger\right]\right\}^T \qquad \text{(B.158)}$$

We can subtract the two terms which leads to

$$\begin{aligned} & -2\kappa F_{02}^T \qquad \text{(B.159)}\\ & -\left\{\left[(D_1\phi)^\dagger, \phi\right]\right\}^T - \left\{\left[D_1\phi, \phi^\dagger\right]\right\}^T \\ = \; & 0\end{aligned}$$

$$-2\kappa F_{02} = \left[(D_1\phi)^\dagger, \phi\right] + \left[D_1\phi, \phi^\dagger\right]$$

We have

$$\begin{aligned}\varepsilon^{1\mu\nu}F_{\mu\nu} &= \varepsilon^{102}F_{02} + \varepsilon^{120}F_{20}\\ &= -F_{02} + F_{20}\\ &= -2F_{02}\end{aligned}$$

from which

$$-2\kappa F_{02} = \kappa\varepsilon^{1\mu\nu}F_{\mu\nu}$$

We can freely replace the covariant with contravariant indices in the right hand side

$$\begin{aligned}\kappa\varepsilon^{1\mu\nu}F_{\mu\nu} &= \left[\left(D^1\phi\right)^\dagger,\phi\right]+\left[D^1\phi,\phi^\dagger\right]\\ &= i\left\{-i\left(\left[\left(D^1\phi\right)^\dagger,\phi\right]-\left[\phi^\dagger,D^1\phi\right]\right)\right\}\\ &= -i\left\{-i\left(\left[\phi^\dagger,D^1\phi\right]-\left[\left(D^1\phi\right)^\dagger,\phi\right]\right)\right\}\\ &= -iJ^1\end{aligned}$$

and the equation is

$$-\kappa\varepsilon^{1\mu\nu}F_{\mu\nu}=iJ^1 \tag{B.160}$$

where

$$J^1=-i\left(\left[\phi^\dagger,D^1\phi\right]-\left[\left(D^1\phi\right)^\dagger,\phi\right]\right)$$

as in the definition Eq.(44). Then the equation represents the component 1 of the equation of motion, Together with Eq.(B.119) we have

$$-\kappa\varepsilon^{\mu\nu\rho}F_{\nu\rho}=iJ^\mu$$

where

$$J^\mu=-i\left\{\left[\phi^\dagger,D^\mu\phi\right]-\left[\left(D^\mu\phi\right)^\dagger,\phi\right]\right\}$$

*i.e.* Eq.(42).

## 12.5 The Euler-Lagrange equation for the matter fields

This equation is obtained by functional variation of the action at the matter fields, $\phi$ and respectively $\phi^\dagger$.

$$\frac{\partial}{\partial x^\mu}\frac{\delta\mathcal{L}}{\delta\left(\frac{\partial\phi}{\partial x^\mu}\right)}-\frac{\delta\mathcal{L}}{\delta\phi}=0 \tag{B.161}$$

and

$$\frac{\partial}{\partial x^\mu}\frac{\delta\mathcal{L}}{\delta\left(\frac{\partial\phi^\dagger}{\partial x^\mu}\right)}-\frac{\delta\mathcal{L}}{\delta\phi^\dagger}=0 \tag{B.162}$$

The matter Lagrangean consists of the kinematical part

$$\mathcal{L}_2=-\mathrm{tr}\left[\left(D^\mu\phi\right)^\dagger\left(D_\mu\phi\right)\right]$$

and the potential of self-interaction for the matter field

$$V\left(\phi,\phi^\dagger\right)=\frac{1}{4\kappa^2}\mathrm{tr}\left[\left(\left[\left[\phi,\phi^\dagger\right],\phi\right]-v^2\phi\right)^\dagger\left(\left[\left[\phi,\phi^\dagger\right],\phi\right]-v^2\phi\right)\right] \tag{B.163}$$

$$\mathcal{L}_{matter} = \mathcal{L}_2 - V\left(\phi, \phi^\dagger\right) \tag{B.164}$$

Let us consider the second Euler-Lagrange equation and calculate the functional derivatives at $\partial_\mu \phi^\dagger$. We have

$$\frac{\delta \mathcal{L}}{\delta\left(\frac{\partial \phi^\dagger}{\partial x^\mu}\right)} = \frac{\delta \mathcal{L}_2}{\delta\left(\frac{\partial \phi^\dagger}{\partial x^\mu}\right)} \tag{B.165}$$

$$\begin{aligned}
\frac{\delta \mathcal{L}_2}{\delta\left(\partial_0 \phi^\dagger\right)} &= -\frac{\delta}{\delta\left(\partial_0 \phi^\dagger\right)} \mathrm{tr}\left[\frac{\partial \phi^\dagger}{\partial t}\frac{\partial \phi}{\partial t} + \frac{\partial \phi^\dagger}{\partial t} A_0 \phi - \frac{\partial \phi^\dagger}{\partial t} \phi A_0\right] \\
&= -\left[\frac{\partial \phi}{\partial t} + A_0 \phi - \phi A_0\right]^T \\
&= -\left(D_0 \phi\right)^T
\end{aligned} \tag{B.166}$$

Analog calculations give

$$\begin{aligned}
\frac{\delta \mathcal{L}_2}{\delta\left(\partial_1 \phi^\dagger\right)} &= -\frac{\delta}{\delta\left(\partial_1 \phi^\dagger\right)} \mathrm{tr}\left[\left(D_1 \phi\right)^\dagger \left(D_1 \phi\right)\right] \\
&= -\left(D_1 \phi\right)^T
\end{aligned} \tag{B.167}$$

and

$$\begin{aligned}
\frac{\delta \mathcal{L}_2}{\delta\left(\partial_2 \phi^\dagger\right)} &= -\frac{\delta}{\delta\left(\partial_2 \phi^\dagger\right)} \mathrm{tr}\left[\left(D_2 \phi\right)^\dagger \left(D_2 \phi\right)\right] \\
&= -\left(D_2 \phi\right)^T
\end{aligned} \tag{B.168}$$

The other term in the Euler-Lagrange calculation implies the derivatives

$$\frac{\delta \mathcal{L}}{\delta \phi^\dagger} = \frac{\delta \mathcal{L}_2}{\delta \phi^\dagger} - \frac{\delta V}{\delta \phi^\dagger} \tag{B.169}$$

The first term is

$$\frac{\delta \mathcal{L}_2}{\delta \phi^\dagger} = -\frac{\delta}{\delta \phi^\dagger} \mathrm{tr}\left[-\left(D_0 \phi\right)^\dagger \left(D_0 \phi\right) + \left(D_i \phi\right)^\dagger \left(D_i \phi\right)\right] \tag{B.170}$$

$$\begin{aligned}
\mathcal{L}_2 = -\mathrm{tr}&\left[\left(D^\mu \phi\right)^\dagger \left(D_\mu \phi\right)\right] \\
= \quad -\mathrm{tr}&\left[\left(\frac{\partial \phi^\dagger}{\partial t} + \phi^\dagger A^{0\dagger} - A^{0\dagger}\phi^\dagger\right)\left(\frac{\partial \phi}{\partial t} + A_0 \phi - \phi A_0\right)\right. \\
&+ \left(\frac{\partial \phi^\dagger}{\partial x} + \phi^\dagger A^{1\dagger} - A^{1\dagger}\phi^\dagger\right)\left(\frac{\partial \phi}{\partial x} + A_1 \phi - \phi A_1\right) \\
&\left. + \left(\frac{\partial \phi^\dagger}{\partial y} + \phi^\dagger A^{2\dagger} - A^{2\dagger}\phi^\dagger\right)\left(\frac{\partial \phi}{\partial y} + A_2 \phi - \phi A_2\right)\right]
\end{aligned} \tag{B.171}$$

and we will calculate it in detail.

$$
\begin{aligned}
& -\frac{\delta}{\delta\phi^\dagger}\mathrm{tr}\left[-\left(D_0\phi\right)^\dagger\left(D_0\phi\right)\right] \qquad \text{(B.172)}\\
= & -\frac{\delta}{\delta\phi^\dagger}\mathrm{tr}\left[\left(\frac{\partial\phi^\dagger}{\partial t}+\phi^\dagger A^{0\dagger}-A^{0\dagger}\phi^\dagger\right)\left(\frac{\partial\phi}{\partial t}+A_0\phi-\phi A_0\right)\right]\\
= & -\frac{\delta}{\delta\phi^\dagger}\mathrm{tr}\left[\phi^\dagger A^{0\dagger}\frac{\partial\phi}{\partial t}+\phi^\dagger A^{0\dagger}A_0\phi-\phi^\dagger A^{0\dagger}\phi A_0\right.\\
& \left.-A^{0\dagger}\phi^\dagger\frac{\partial\phi}{\partial t}-A^{0\dagger}\phi^\dagger A_0\phi+A^{0\dagger}\phi^\dagger\phi A_0\right]
\end{aligned}
$$

Term by term

$$
\begin{aligned}
-\frac{\delta}{\delta\phi^\dagger}\mathrm{tr}\left[\phi^\dagger A^{0\dagger}\frac{\partial\phi}{\partial t}\right] &= -\left(A^{0\dagger}\frac{\partial\phi}{\partial t}\right)^T\\
-\frac{\delta}{\delta\phi^\dagger}\mathrm{tr}\left[\phi^\dagger A^{0\dagger}A_0\phi\right] &= -\left(A^{0\dagger}A_0\phi\right)^T\\
-\frac{\delta}{\delta\phi^\dagger}\mathrm{tr}\left[-\phi^\dagger A^{0\dagger}\phi A_0\right] &= \left(A^{0\dagger}\phi A_0\right)^T\\
-\frac{\delta}{\delta\phi^\dagger}\mathrm{tr}\left[-A^{0\dagger}\phi^\dagger\frac{\partial\phi}{\partial t}\right] &= \left(A^{0\dagger}\right)^T\left(\frac{\partial\phi}{\partial t}\right)^T\\
-\frac{\delta}{\delta\phi^\dagger}\mathrm{tr}\left[-A^{0\dagger}\phi^\dagger A_0\phi\right] &= \left(A^{0\dagger}\right)^T\left(A_0\phi\right)^T\\
-\frac{\delta}{\delta\phi^\dagger}\mathrm{tr}\left[A^{0\dagger}\phi^\dagger\phi A_0\right] &= -\left(A^{0\dagger}\right)^T\left(\phi A_0\right)^T
\end{aligned}
$$

Summing up the terms

$$
\begin{aligned}
& -\frac{\delta}{\delta\phi^\dagger}\mathrm{tr}\left[-\left(D_0\phi\right)^\dagger\left(D_0\phi\right)\right]\\
= & -\left(A^{0\dagger}\frac{\partial\phi}{\partial t}\right)^T-\left(A^{0\dagger}A_0\phi\right)^T+\left(A^{0\dagger}\phi A_0\right)^T\\
& +\left(A^{0\dagger}\right)^T\left(\frac{\partial\phi}{\partial t}\right)^T+\left(A^{0\dagger}\right)^T\left(A_0\phi\right)^T-\left(A^{0\dagger}\right)^T\left(\phi A_0\right)^T
\end{aligned}
$$

We apply the transpose and factorize

$$
\begin{aligned}
& -\frac{\delta}{\delta\phi^\dagger}\operatorname{tr}\left[-\left(D_0\phi\right)^\dagger\left(D_0\phi\right)\right] \\
= & -\left[\left(\frac{\partial\phi}{\partial t}\right)^T+\left(A_0\phi\right)^T-\left(\phi A_0\right)^T\right]\left(A^{0\dagger}\right)^T \\
& +\left(A^{0\dagger}\right)^T\left[\left(\frac{\partial\phi}{\partial t}\right)^T+\left(A_0\phi\right)^T-\left(\phi A_0\right)^T\right] \\
= & -\left(D_0\phi\right)^T\left(A^{0\dagger}\right)^T+\left(A^{0\dagger}\right)^T\left(D_0\phi\right)^T \\
= & \left[\left(A^{0\dagger}\right)^T,\left(D_0\phi\right)^T\right]
\end{aligned}
$$

or

$$
-\frac{\delta}{\delta\phi^\dagger}\operatorname{tr}\left[-\left(D_0\phi\right)^\dagger\left(D_0\phi\right)\right]=\left[D_0\phi,A^{0\dagger}\right]^T
$$

By analogue calculations we obtain

$$
-\frac{\delta}{\delta\phi^\dagger}\operatorname{tr}\left[\left(D_1\phi\right)^\dagger\left(D_1\phi\right)\right]=\left[D_1\phi,A_1^\dagger\right]^T
$$

and

$$
-\frac{\delta}{\delta\phi^\dagger}\operatorname{tr}\left[\left(D_2\phi\right)^\dagger\left(D_2\phi\right)\right]=\left[D_2\phi,A_2^\dagger\right]^T
$$

Then

$$
\begin{aligned}
\frac{\delta\mathcal{L}_2}{\delta\phi^\dagger} &= -\frac{\delta}{\delta\phi^\dagger}\operatorname{tr}\left[-\left(D_0\phi\right)^\dagger\left(D_0\phi\right)+\left(D_i\phi\right)^\dagger\left(D_i\phi\right)\right] \qquad \text{(B.173)} \\
&= \left[D_0\phi,A_0^\dagger\right]^T+\left[D_1\phi,A_1^\dagger\right]^T+\left[D_2\phi,A_2^\dagger\right]^T
\end{aligned}
$$

Further

$$
\frac{\delta\mathcal{L}}{\delta\phi^\dagger}=\frac{\delta\mathcal{L}_2}{\delta\phi^\dagger}-\frac{\delta V}{\delta\phi^\dagger}
$$

and the Euler-Lagrange equation results by combining Eqs.(B.166), (B.167), (B.168) and (B.173)

$$\begin{aligned}
& \frac{\partial}{\partial x^{\mu}} \frac{\delta \mathcal{L}}{\delta \left( \frac{\partial \phi^{\dagger}}{\partial x^{\mu}} \right)} - \frac{\delta \mathcal{L}}{\delta \phi^{\dagger}} \\
= \; & \frac{\partial}{\partial x^{0}} \left( D_0 \phi \right)^T + \frac{\partial}{\partial x^{1}} \left( D_1 \phi \right)^T + \frac{\partial}{\partial x^{2}} \left( D_2 \phi \right)^T \\
& + \left[ D_0 \phi, A_0^{\dagger} \right]^T + \left[ D_1 \phi, A_1^{\dagger} \right]^T + \left[ D_2 \phi, A_2^{\dagger} \right]^T \\
& + \frac{\delta V}{\delta \phi^{\dagger}} \\
= \; & 0
\end{aligned}$$

We note that

$$\begin{aligned}
& \frac{\partial}{\partial x^{0}} \left( D_0 \phi \right)^T + \left[ D_0 \phi, A_0^{\dagger} \right]^T \\
= \; & \left\{ \left( \frac{\partial}{\partial x^{0}} + \left[ , A_0^{\dagger} \right] \right) \left( D_0 \phi \right) \right\}^T \\
= \; & \left( D_0^{\dagger} D_0 \phi \right)^T \\
= \; & - \left( D^{0\dagger} D_0 \phi \right)^T
\end{aligned}$$

The other terms have similar form and we obtain

$$- \left( D^{\mu\dagger} D_{\mu} \phi \right)^T + \frac{\delta V}{\delta \phi^{\dagger}} = 0$$

# 13 Appendix C : Derivation of the second self-duality equation

The gauge field equation in terms of $\pm$ variables (Dunne [30])

Let us calculate

$$F_{+-} = \partial_+ A_- - \partial_- A_+ + \left[ A_+, A_- \right] \tag{C.1}$$

using the space variables $(1,2) \equiv (x,y)$.

$$\begin{aligned}
F_{+-} \; = \; & \left( \frac{\partial}{\partial x^{1}} + i \frac{\partial}{\partial x^{2}} \right) \left( A_1 - i A_2 \right) - \\
& - \left( \frac{\partial}{\partial x^{1}} - i \frac{\partial}{\partial x^{2}} \right) \left( A_1 + i A_2 \right) + \\
& + \left[ A_1 + i A_2, A_1 - i A_2 \right]
\end{aligned} \tag{C.2}$$

$$F_{+-} = \frac{\partial A_1}{\partial x^1} + i\frac{\partial A_1}{\partial x^2} - i\frac{\partial A_2}{\partial x^1} + \frac{\partial A_2}{\partial x^2}$$
$$-\frac{\partial A_1}{\partial x^1} + i\frac{\partial A_1}{\partial x^2} - i\frac{\partial A_2}{\partial x^1} - \frac{\partial A_2}{\partial x^2}$$
$$-i\left[A_1, A_2\right] + i\left[A_2, A_1\right] \tag{C.3}$$

$$\begin{aligned} F_{+-} &= -2i\left\{\frac{\partial A_2}{\partial x^1} - \frac{\partial A_1}{\partial x^2} + \left[A_1, A_2\right]\right\} \\ &= -2iF_{12} \\ &= -2i\varepsilon^{012}F_{12} \end{aligned} \tag{C.4}$$

On the other hand, we have the equation of motion Eq.(46)

$$-2\kappa\varepsilon^{012}F_{12} = iJ^0 \tag{C.5}$$

from which we derive

$$\begin{aligned} -2\varepsilon^{012}F_{12} &= \frac{1}{\kappa}iJ^0 \\ &= \frac{1}{i}F_{+-} \end{aligned}$$

$$\begin{aligned} F_{+-} &= -\frac{J^0}{\kappa} = \frac{J_0}{\kappa} \\ &= \frac{1}{\kappa}\left\{-i\left(\left[\phi^\dagger, D_0\phi\right] - \left[(D_0\phi)^\dagger, \phi\right]\right)\right\} \end{aligned} \tag{C.6}$$

where we can use the second of the Eqs.(69), valid at self-duality

$$\begin{aligned} D_0\phi &= \frac{i}{2\kappa}\left(\left[\left[\phi, \phi^\dagger\right], \phi\right] - v^2\phi\right) \\ (D_0\phi)^\dagger &= -\frac{i}{2\kappa}\left(\left(\left[\left[\phi, \phi^\dagger\right], \phi\right]\right)^\dagger - v^2\phi^\dagger\right) \end{aligned} \tag{C.7}$$

We calculate separately the terms

$$\left[\phi^\dagger, D_0\phi\right] = \frac{i}{2\kappa}\left\{\left[\phi^\dagger, \left[\left[\phi, \phi^\dagger\right], \phi\right]\right] - v^2\left[\phi^\dagger, \phi\right]\right\} \tag{C.8}$$

$$\left[(D_0\phi)^\dagger, \phi\right] = -\frac{i}{2\kappa}\left\{\left[\left(\left[\left[\phi, \phi^\dagger\right], \phi\right]\right)^\dagger, \phi\right] - v^2\left[\phi^\dagger, \phi\right]\right\} \tag{C.9}$$

We can prove by expanding the commutators the equality of the first terms from the curly brackets of right hand sides of the two equations. From Eq.(C.8)

$$\begin{aligned}\left[\phi^\dagger,\left[\left[\phi,\phi^\dagger\right],\phi\right]\right] &= \left[\phi^\dagger,\left[\phi,\phi^\dagger\right]\phi-\phi\left[\phi,\phi^\dagger\right]\right] \\ &= \left[\phi^\dagger,\left[\phi,\phi^\dagger\right]\phi\right]-\left[\phi^\dagger,\phi\left[\phi,\phi^\dagger\right]\right] \\ &= \phi^\dagger\left[\phi,\phi^\dagger\right]\phi \\ &\quad -\left[\phi,\phi^\dagger\right]\phi\phi^\dagger \\ &\quad -\phi^\dagger\phi\left[\phi,\phi^\dagger\right] \\ &\quad +\phi\left[\phi,\phi^\dagger\right]\phi^\dagger\end{aligned} \tag{C.10}$$

From Eq.(C.9)

$$\begin{aligned}\left[\left(\left[\left[\phi,\phi^\dagger\right],\phi\right]\right)^\dagger,\phi\right] &= \left[\left[\phi^\dagger,\left[\phi,\phi^\dagger\right]^\dagger\right],\phi\right] \\ &= \left[\left[\phi^\dagger,\left[\phi,\phi^\dagger\right]\right],\phi\right]\end{aligned}$$

since $\left[\phi,\phi^\dagger\right]^\dagger=\left[\phi,\phi^\dagger\right]$ ; then

$$\begin{aligned}\left[\left(\left[\left[\phi,\phi^\dagger\right],\phi\right]\right)^\dagger,\phi\right] &= \left[\phi^\dagger\left[\phi,\phi^\dagger\right]-\left[\phi,\phi^\dagger\right]\phi^\dagger,\phi\right] \\ &= \left[\phi^\dagger\left[\phi,\phi^\dagger\right],\phi\right]-\left[\left[\phi,\phi^\dagger\right]\phi^\dagger,\phi\right] \\ &= \phi^\dagger\left[\phi,\phi^\dagger\right]\phi \\ &\quad -\phi\phi^\dagger\left[\phi,\phi^\dagger\right] \\ &\quad -\left[\phi,\phi^\dagger\right]\phi^\dagger\phi \\ &\quad +\phi\left[\phi,\phi^\dagger\right]\phi^\dagger\end{aligned} \tag{C.11}$$

We note that the first and the last terms in (C.10) and (C.11) are the same. The other terms in (C.10) are

$$\begin{aligned}&-\left[\phi,\phi^\dagger\right]\phi\phi^\dagger-\phi^\dagger\phi\left[\phi,\phi^\dagger\right] \\ =\;& -\phi\phi^\dagger\phi\phi^\dagger+\phi^\dagger\phi\phi\phi^\dagger \\ & -\phi^\dagger\phi\phi\phi^\dagger+\phi^\dagger\phi\phi^\dagger\phi \\ =\;& -\phi\phi^\dagger\phi\phi^\dagger+\phi^\dagger\phi\phi^\dagger\phi\end{aligned} \tag{C.12}$$

and from (C.11)

$$\begin{aligned}&-\phi\phi^\dagger\left[\phi,\phi^\dagger\right]-\left[\phi,\phi^\dagger\right]\phi^\dagger\phi \\ =\;& -\phi\phi^\dagger\phi\phi^\dagger+\phi\phi^\dagger\phi^\dagger\phi \\ & -\phi\phi^\dagger\phi^\dagger\phi+\phi^\dagger\phi\phi^\dagger\phi \\ =\;& -\phi\phi^\dagger\phi\phi^\dagger+\phi^\dagger\phi\phi^\dagger\phi\end{aligned} \tag{C.13}$$

and the two expressions are identical. This means that

$$\left[\phi^\dagger, \left[\left[\phi, \phi^\dagger\right], \phi\right]\right] - v^2 \left[\phi^\dagger, \phi\right] = \left[\left(\left[\left[\phi, \phi^\dagger\right], \phi\right]\right)^\dagger, \phi\right] - v^2 \left[\phi^\dagger, \phi\right] \tag{C.14}$$

and

$$\left[\phi^\dagger, D_0\phi\right] = -\left[\left(D_0\phi\right)^\dagger, \phi\right] \tag{C.15}$$

and

$$\begin{aligned} F_{+-} &= -\frac{i}{\kappa}\left\{\left(\left[\phi^\dagger, D_0\phi\right] - \left[\left(D_0\phi\right)^\dagger, \phi\right]\right)\right\} \\ &= -\frac{2i}{\kappa}\left[\phi^\dagger, D_0\phi\right] \end{aligned} \tag{C.16}$$

where we replace the expression of $D_0\phi$

$$\begin{aligned} F_{+-} &= -\frac{2i}{\kappa}\left[\phi^\dagger, \frac{i}{2\kappa}\left(\left[\left[\phi, \phi^\dagger\right], \phi\right] - v^2\phi\right)\right] \\ &= \frac{1}{\kappa^2}\left[\phi^\dagger, \left[\left[\phi, \phi^\dagger\right], \phi\right] - v^2\phi\right] \\ &= \frac{1}{\kappa^2}\left[v^2\phi - \left[\left[\phi, \phi^\dagger\right], \phi\right], \phi^\dagger\right] \end{aligned} \tag{C.17}$$

# 14 Appendix D : Notes on definitions

The information about the algebraic structure invoked in the present model can be found in [32]. A **simple group** is a group that does not have invariant subgroups, except of the identity and the whole group.

A **simple algebra** is an algebra that does not have *proper ideals*.

A **semi-simple algebra** is an algebra that can be written as a direct sum of simple algebras.

$U(1)$ is *not* simple.

The **dimension** of a simple Lie algebra is the total *number of linearly independent generators*.

The **rank of the algebra**, $r$, is the maximum *number of simultaneously diagonalisable generators* of a simple Lie algebra.

In the Cartan-Weil analysis the generators are written in a basis where they can be devided into two sets:

- the **Cartan subalgebra**, which is the *maximal Abelian subalgebra* of $G$. It contains $r$ diagonalisable generators $H_i$, $i = 1, ..., r$

$$\left[H_i, H_j\right] = 0 \;,\; i, j = 1, ..., r \tag{D.1}$$

- the remaining generators of the algebra $G$ are defined such as they satisfy the eigenvalue problems

$$[H_i, E_\mu] = \alpha_i E_\mu \ , \ i = 1, ..., r \tag{D.2}$$

It results that the constants $\alpha_i$ can be considered *structure constants* of the algebra in the Cartan-Weil basis.

For each generator $E_\mu$ there are $r$ constant numbers, $\alpha_i$, $i = 1, ..., r$ ; if we consider a space with dimension $r$, then the set of points $(\alpha_1, \alpha_2, ..., \alpha_r)$ corresponding to one generator $E_\mu$ is a point in this space. This space is called **root space** and the name **root** comes from the fact the the vector $(\alpha_1, \alpha_2, ..., \alpha_r)$ is obtained by solving the equation (D.2), an eigenvalue problem.

Two problems are connected and are treated together using the **Dynkin diagrams** of the simple algebras:

1. **to classify all possible systems of roots** for the algebras of a given rank $r$;

2. **to find all possible irreducible representations** of a simple group $G$. This means to identify a system of physical states on which the generators $E_\mu$ are acting (the states belong to a Hilbert space) with the property that these states are transformed between them (or, the system of states is closed under the action of the generators $E_\mu$). These states are taken as the basis for an irreducible representation.

Considering the physical states which are the basis of the irreducible representation, $|\lambda\rangle$, they can be labelled by the $r$ eigenvalues of the *diagonalisable generators* $H_i$

$$H_i \left|\lambda\right\rangle = \lambda_i \left|\lambda\right\rangle \ , \ i = 1, ..., r$$

The set $\lambda$ is called the **weight** of the representation vector.

# 15 Appendix E : Expanded form of the first equation of motion

The first equation of motion is

$$D_\mu D^\mu \phi = \frac{\partial V}{\partial \phi^\dagger} \tag{E.1}$$

As explained by Dunne [33] the derivative of the potential $V$ is obtained from the functional variation to $\phi^\dagger$ and for this we need the expanded form of $V$. The potential is given initially in terms of the trace of the operators

$$V\left(\phi,\phi^\dagger\right)=\frac{1}{4\kappa^2}\mathrm{tr}\left[\left(\left[\left[\phi,\phi^\dagger\right],\phi\right]-v^2\phi\right)^\dagger\left(\left[\left[\phi,\phi^\dagger\right],\phi\right]-v^2\phi\right)\right] \tag{E.2}$$

In the equations of motion we will treat separately each term:

$$\begin{aligned}
& \left(\left[\left[\phi,\phi^\dagger\right],\phi\right]-v^2\phi\right)^\dagger\left(\left[\left[\phi,\phi^\dagger\right],\phi\right]-v^2\phi\right) \\
= & \left(\left[\left[\phi,\phi^\dagger\right],\phi\right]^\dagger-v^2\phi^\dagger\right)\left(\left[\left[\phi,\phi^\dagger\right],\phi\right]-v^2\phi\right) \\
= & \left[\left[\phi,\phi^\dagger\right],\phi\right]^\dagger\left[\left[\phi,\phi^\dagger\right],\phi\right] \\
& -v^2\phi^\dagger\left[\left[\phi,\phi^\dagger\right],\phi\right] \\
& -v^2\left[\left[\phi,\phi^\dagger\right],\phi\right]^\dagger\phi \\
& +v^4\phi^\dagger\phi
\end{aligned} \tag{E.3}$$

The first term, of sixth degree

$$\begin{aligned}
& \left[\left[\phi,\phi^\dagger\right],\phi\right]^\dagger\left[\left[\phi,\phi^\dagger\right],\phi\right] \\
= & \left[\phi\phi^\dagger-\phi^\dagger\phi,\phi\right]^\dagger\left[\phi\phi^\dagger-\phi^\dagger\phi,\phi\right] \\
= & \left(\phi\phi^\dagger\phi-\phi^\dagger\phi\phi-\phi\phi\phi^\dagger+\phi\phi^\dagger\phi\right)^\dagger \\
& \times\left(\phi\phi^\dagger\phi-\phi^\dagger\phi\phi-\phi\phi\phi^\dagger+\phi\phi^\dagger\phi\right) \\
= & \left(\phi^\dagger\phi\phi^\dagger-\phi^\dagger\phi^\dagger\phi-\phi\phi^\dagger\phi^\dagger+\phi^\dagger\phi\phi^\dagger\right) \\
& \times\left(\phi\phi^\dagger\phi-\phi^\dagger\phi\phi-\phi\phi\phi^\dagger+\phi\phi^\dagger\phi\right) \\
= & \left(2\phi^\dagger\phi\phi^\dagger-\phi^\dagger\phi^\dagger\phi-\phi\phi^\dagger\phi^\dagger\right) \\
& \times\left(2\phi\phi^\dagger\phi-\phi^\dagger\phi\phi-\phi\phi\phi^\dagger\right)
\end{aligned} \tag{E.4}$$

$$\begin{aligned}
& \left[\left[\phi,\phi^\dagger\right],\phi\right]^\dagger\left[\left[\phi,\phi^\dagger\right],\phi\right] \\
= & 4\phi^\dagger\phi\phi^\dagger\phi\phi^\dagger\phi \\
& -2\phi^\dagger\phi\phi^\dagger\phi^\dagger\phi\phi \\
& -2\phi^\dagger\phi\phi^\dagger\phi\phi\phi^\dagger \\
& -2\phi^\dagger\phi^\dagger\phi\phi\phi^\dagger\phi \\
& +\phi^\dagger\phi^\dagger\phi\phi^\dagger\phi\phi \\
& +\phi^\dagger\phi^\dagger\phi\phi\phi\phi^\dagger \\
& -2\phi\phi^\dagger\phi^\dagger\phi\phi^\dagger\phi \\
& +\phi\phi^\dagger\phi^\dagger\phi^\dagger\phi\phi \\
& +\phi\phi^\dagger\phi^\dagger\phi\phi\phi^\dagger
\end{aligned} \tag{E.5}$$

We remark that in Eq.(E.5) the terms five and seven

$$\begin{aligned} & \mathrm{tr}\left(-2\phi^\dagger\phi\phi^\dagger\phi\phi\phi^\dagger + \phi^\dagger\phi^\dagger\phi\phi^\dagger\phi\phi\right) \\ = & -\mathrm{tr}\left(\phi^\dagger\phi\phi^\dagger\phi\phi\phi^\dagger\right) \end{aligned}$$

terms six and eight

$$\begin{aligned} & \mathrm{tr}\left(\phi^\dagger\phi^\dagger\phi\phi\phi\phi^\dagger + \phi\phi^\dagger\phi^\dagger\phi^\dagger\phi\phi\right) \\ = & 2\mathrm{tr}\left(\phi^\dagger\phi^\dagger\phi\phi\phi\phi^\dagger\right) \end{aligned}$$

terms second and nine

$$\begin{aligned} & \mathrm{tr}\left(-2\phi^\dagger\phi\phi^\dagger\phi^\dagger\phi\phi + \phi\phi^\dagger\phi^\dagger\phi\phi\phi^\dagger\right) \\ = & -\mathrm{tr}\left(\phi^\dagger\phi\phi^\dagger\phi^\dagger\phi\phi\right) \end{aligned} \tag{E.6}$$

third

$$-2\phi^\dagger\phi\phi^\dagger\phi\phi\phi^\dagger \tag{E.7}$$

four

$$-2\phi^\dagger\phi^\dagger\phi\phi\phi^\dagger\phi \tag{E.8}$$

can be grouped. Collecting the terms we have

$$\begin{aligned} & \left[\left[\phi, \phi^\dagger\right], \phi\right]^\dagger \left[\left[\phi, \phi^\dagger\right], \phi\right] \\ = & -\mathrm{tr}\left(\phi^\dagger\phi\phi^\dagger\phi\phi\phi^\dagger\right) \\ & +2\mathrm{tr}\left(\phi^\dagger\phi^\dagger\phi\phi\phi\phi^\dagger\right) \\ & -\mathrm{tr}\left(\phi^\dagger\phi\phi^\dagger\phi^\dagger\phi\phi\right) \\ & -2\mathrm{tr}\left(\phi^\dagger\phi\phi^\dagger\phi\phi\phi^\dagger\right) \\ & -2\mathrm{tr}\left(\phi^\dagger\phi^\dagger\phi\phi\phi^\dagger\phi\right) \end{aligned} \tag{E.9}$$

The third term, after two permutations of the first two factors, is identical to the fifth term

$$\begin{aligned} & -\mathrm{tr}\left(\phi^\dagger\phi\phi^\dagger\phi^\dagger\phi\phi\right) - 2\mathrm{tr}\left(\phi^\dagger\phi^\dagger\phi\phi\phi^\dagger\phi\right) \\ = & -\mathrm{tr}\left(\phi^\dagger\phi^\dagger\phi\phi\phi^\dagger\phi\right) - 2\mathrm{tr}\left(\phi^\dagger\phi^\dagger\phi\phi\phi^\dagger\phi\right) \\ = & -3\mathrm{tr}\left(\phi^\dagger\phi^\dagger\phi\phi\phi^\dagger\phi\right) \end{aligned} \tag{E.10}$$

The first and the fourth factors are equal

$$\begin{aligned} & -\mathrm{tr}\left(\phi^\dagger\phi\phi^\dagger\phi\phi\phi^\dagger\right) - 2\mathrm{tr}\left(\phi^\dagger\phi\phi^\dagger\phi\phi\phi^\dagger\right) \\ = & -3\mathrm{tr}\left(\phi^\dagger\phi\phi^\dagger\phi\phi\phi^\dagger\right) \end{aligned} \tag{E.11}$$

Then from the first, sixth degree product, we obtain

$$\begin{aligned} & \left[\left[\phi, \phi^\dagger\right], \phi\right]^\dagger \left[\left[\phi, \phi^\dagger\right], \phi\right] \\ = \; & -3\mathrm{tr}\left(\phi^\dagger\phi^\dagger\phi\phi\phi^\dagger\phi\right) - 3\mathrm{tr}\left(\phi^\dagger\phi\phi^\dagger\phi\phi\phi^\dagger\right) + 2\mathrm{tr}\left(\phi^\dagger\phi^\dagger\phi\phi\phi\phi^\dagger\right) \end{aligned} \tag{E.12}$$

The next two terms in the potential (proportional with $(-v^2)$) are expanded

$$\begin{aligned} & -v^2\phi^\dagger \left[\left[\phi, \phi^\dagger\right], \phi\right] - v^2 \left[\left[\phi, \phi^\dagger\right], \phi\right]^\dagger \phi \\ = \; & \left(-v^2\right) \left\{\phi^\dagger \left(\phi\phi^\dagger\phi - \phi^\dagger\phi\phi - \phi\phi\phi^\dagger + \phi\phi^\dagger\phi\right) \right. \\ & \left. \left(\phi\phi^\dagger\phi - \phi^\dagger\phi\phi - \phi\phi\phi^\dagger + \phi\phi^\dagger\phi\right)^\dagger \phi\right\} \\ = \; & \left(-v^2\right) \left\{2\phi^\dagger\phi\phi^\dagger\phi - \phi^\dagger\phi^\dagger\phi\phi - \phi^\dagger\phi\phi\phi^\dagger \right. \\ & \left. \left(2\phi\phi^\dagger\phi - \phi^\dagger\phi\phi - \phi\phi\phi^\dagger\right)^\dagger \phi\right\} \\ = \; & \left(-v^2\right) \left\{2\phi^\dagger\phi\phi^\dagger\phi - \phi^\dagger\phi^\dagger\phi\phi - \phi^\dagger\phi\phi\phi^\dagger \right. \\ & \left. 2\phi^\dagger\phi\phi^\dagger\phi - \phi^\dagger\phi^\dagger\phi\phi - \phi\phi^\dagger\phi^\dagger\phi\right\} \\ = \; & \left(-v^2\right) \left\{4\phi^\dagger\phi\phi^\dagger\phi - 2\phi^\dagger\phi^\dagger\phi\phi - \phi^\dagger\phi\phi\phi^\dagger - \phi\phi^\dagger\phi^\dagger\phi\right\} \end{aligned} \tag{E.13}$$

The last term is unchanged

$$v^4\phi^\dagger\phi \tag{E.14}$$

Now we invoke two properties of the **Trace** operator:

1. the symmetry to cyclic permutation

$$\mathrm{tr}\left(ABCD\right) = \mathrm{tr}\left(DABC\right) = \mathrm{tr}\left(CDAB\right) = \mathrm{tr}\left(BCDA\right) \tag{E.15}$$

2. the linearity for sum of arguments

$$\mathrm{tr}\left(A + B\right) = \mathrm{tr}\left(A\right) + \mathrm{tr}\left(B\right) \tag{E.16}$$

Then we remark in Eq.(E.13) that the last three terms can be grouped, so that the final form for it is

$$\begin{aligned} & -v^2\phi^\dagger \left[\left[\phi, \phi^\dagger\right], \phi\right] - v^2 \left[\left[\phi, \phi^\dagger\right], \phi\right]^\dagger \phi \\ = \; & \left(-v^2\right) \mathrm{tr} \left\{4\phi^\dagger\phi\phi^\dagger\phi - 2\phi^\dagger\phi^\dagger\phi\phi - \phi^\dagger\phi\phi\phi^\dagger - \phi\phi^\dagger\phi^\dagger\phi\right\} \\ = \; & \left(-v^2\right) 4\mathrm{tr} \left\{\phi^\dagger\phi\phi^\dagger\phi - \phi^\dagger\phi^\dagger\phi\phi\right\} \end{aligned} \tag{E.17}$$

Adding the contributions to the potential

$$\begin{aligned}
& 4\kappa^2 V\left(\phi,\phi^\dagger\right) && \text{(E.18)}\\
= \; & -3\mathrm{tr}\left(\phi^\dagger\phi^\dagger\phi\phi\phi^\dagger\phi\right) - 3\mathrm{tr}\left(\phi^\dagger\phi\phi^\dagger\phi\phi\phi^\dagger\right) + 2\mathrm{tr}\left(\phi^\dagger\phi^\dagger\phi\phi\phi\phi^\dagger\right)\\
& + \left(-v^2\right) 4\mathrm{tr}\left\{\phi^\dagger\phi\phi^\dagger\phi - \phi^\dagger\phi^\dagger\phi\phi\right\}\\
& + v^4\mathrm{tr}\left(\phi^\dagger\phi\right)
\end{aligned}$$

Consider now the variation of $V\left(\phi,\phi^\dagger\right)$ to the function $\phi^\dagger$

$$\frac{\delta}{\delta\phi^\dagger}V\left(\phi,\phi^\dagger\right) \tag{E.19}$$

This will be calculated by adding a small functional variation to $\phi^\dagger$ and retaining the first oder:

$$\begin{aligned}
& \text{perturbed sixth order part} && \text{(E.20)}\\
= \; & \mathrm{tr}\left[-3\left(\phi^\dagger+\delta\phi^\dagger\right)\phi^\dagger\phi\phi\phi^\dagger\phi\right.\\
& -3\phi^\dagger\left(\phi^\dagger+\delta\phi^\dagger\right)\phi\phi\phi^\dagger\phi\\
& -3\phi^\dagger\phi^\dagger\phi\phi\left(\phi^\dagger+\delta\phi^\dagger\right)\phi\\
& -3\left(\phi^\dagger+\delta\phi^\dagger\right)\phi\phi^\dagger\phi\phi\phi^\dagger\\
& -3\phi^\dagger\phi\left(\phi^\dagger+\delta\phi^\dagger\right)\phi\phi\phi^\dagger\\
& -3\phi^\dagger\phi\phi^\dagger\phi\phi\left(\phi^\dagger+\delta\phi^\dagger\right)+\\
& +2\left(\phi^\dagger+\delta\phi^\dagger\right)\phi^\dagger\phi\phi\phi\phi^\dagger\\
& +2\phi^\dagger\left(\phi^\dagger+\delta\phi^\dagger\right)\phi\phi\phi\phi^\dagger\\
& \left.+2\phi^\dagger\phi^\dagger\phi\phi\phi\left(\phi^\dagger+\delta\phi^\dagger\right)\right]
\end{aligned}$$

This gives, after permuting the small $\delta\phi^\dagger$ to the left

$$\begin{aligned}
& \text{perturbed sixth degree part} && \text{(E.21)}\\
= \; & \text{sixth degree part} +\\
& +\mathrm{tr}\left[-3\phi^\dagger\phi\phi\phi^\dagger\phi\left(\delta\phi^\dagger\right)\right.\\
& -3\phi\phi\phi^\dagger\phi\phi^\dagger\left(\delta\phi^\dagger\right)\\
& -3\phi\phi^\dagger\phi^\dagger\phi\phi\left(\delta\phi^\dagger\right)\\
& -3\phi\phi^\dagger\phi\phi\phi^\dagger\left(\delta\phi^\dagger\right)\\
& -3\phi\phi\phi^\dagger\phi^\dagger\phi\left(\delta\phi^\dagger\right)\\
& -3\phi^\dagger\phi\phi^\dagger\phi\phi\left(\delta\phi^\dagger\right)\\
& +2\phi^\dagger\phi\phi\phi\phi^\dagger\left(\delta\phi^\dagger\right)\\
& +2\phi\phi\phi\phi^\dagger\phi^\dagger\left(\delta\phi^\dagger\right)\\
& \left.+2\phi^\dagger\phi^\dagger\phi\phi\phi\left(\delta\phi^\dagger\right)\right]
\end{aligned}$$

This is symbolically written

$$\begin{aligned} & \text{perturbed sixth degree part} \\ = & \text{ sixth degree part} \\ & +\mathrm{tr}\left[A\left(\delta\phi^{\dagger}\right)\right] \end{aligned} \tag{E.22}$$

where $A$ is given in (E.34).

The fourth order part is

$$\begin{aligned} & \text{perturbed fourth degree part} \\ = & \left(-v^{2}\right)4\mathrm{tr}\left[\left(\phi^{\dagger}+\delta\phi^{\dagger}\right)\phi\phi^{\dagger}\phi\right. \\ & +\phi^{\dagger}\phi\left(\phi^{\dagger}+\delta\phi^{\dagger}\right)\phi \\ & -\left(\phi^{\dagger}+\delta\phi^{\dagger}\right)\phi^{\dagger}\phi\phi \\ & \left.-\phi^{\dagger}\left(\phi^{\dagger}+\delta\phi^{\dagger}\right)\phi\phi\right] \end{aligned} \tag{E.23}$$

or

$$\begin{aligned} & \text{perturbed fourth degree part} \\ = & \text{ fourth degree part} + \\ & +\left(-v^{2}\right)4\mathrm{tr}\left[\phi\phi^{\dagger}\phi\left(\delta\phi^{\dagger}\right)\right. \\ & +\phi\phi^{\dagger}\phi\left(\delta\phi^{\dagger}\right) \\ & -\phi^{\dagger}\phi\phi\left(\delta\phi^{\dagger}\right) \\ & \left.-\phi\phi\phi^{\dagger}\left(\delta\phi^{\dagger}\right)\right] \end{aligned} \tag{E.24}$$

This can be written

$$\begin{aligned} & \text{perturbed fourth degree part} \\ = & \text{ fourth degree part} + \\ & \left(-v^{2}\right)4\mathrm{tr}\left[B\left(\delta\phi^{\dagger}\right)\right] \end{aligned} \tag{E.25}$$

where

$$B \equiv \phi\phi^{\dagger}\phi + \phi\phi^{\dagger}\phi - \phi^{\dagger}\phi\phi - \phi\phi\phi^{\dagger} \tag{E.26}$$

The last part

$$\begin{aligned} & \text{perturbed second degree part} \\ = & \, v^{4}\mathrm{tr}\left[\left(\phi^{\dagger}+\delta\phi^{\dagger}\right)\phi\right] \\ = & \text{ second degree part} \\ & +v^{4}\mathrm{tr}\left[\phi\left(\delta\phi^{\dagger}\right)\right] \end{aligned} \tag{E.27}$$

The three terms have the sum

$$
\begin{aligned}
V\left(\phi, \phi^{\dagger}+\delta\phi^{\dagger}\right) &= V\left(\phi, \phi^{\dagger}\right) \\
&+\operatorname{tr}\left[A\left(\delta\phi^{\dagger}\right)\right] \\
&+\left(-v^{2}\right) 4\operatorname{tr}\left[B\left(\delta\phi^{\dagger}\right)\right] \\
&+v^{4}\operatorname{tr}\left[\phi\left(\delta\phi^{\dagger}\right)\right]
\end{aligned}
\tag{E.28}
$$

We introduce a short notation

$$
C \equiv A+\left(-4v^{2}\right) B+v^{4}\phi \tag{E.29}
$$

and we have

$$
V\left(\phi, \phi^{\dagger}+\delta\phi^{\dagger}\right)=V\left(\phi, \phi^{\dagger}\right)+\operatorname{tr}\left[C\left(\delta\phi^{\dagger}\right)\right] \tag{E.30}
$$

The last term is

$$
\begin{aligned}
& \operatorname{tr}\left(\begin{array}{cc} C_{11} & C_{12} \\ C_{21} & C_{22} \end{array}\right)\left(\begin{array}{cc} \delta\phi_{11}^{\dagger} & \delta\phi_{12}^{\dagger} \\ \delta\phi_{21}^{\dagger} & \delta\phi_{22}^{\dagger} \end{array}\right) \\
= & \operatorname{tr}\left(\begin{array}{cc} C_{11}\delta\phi_{11}^{\dagger}+C_{12}\delta\phi_{21}^{\dagger} & C_{11}\delta\phi_{12}^{\dagger}+C_{12}\delta\phi_{22}^{\dagger} \\ C_{21}\delta\phi_{11}^{\dagger}+C_{22}\delta\phi_{21}^{\dagger} & C_{21}\delta\phi_{12}^{\dagger}+C_{22}\delta\phi_{22}^{\dagger} \end{array}\right) \\
= & C_{11}\delta\phi_{11}^{\dagger}+C_{12}\delta\phi_{21}^{\dagger}+C_{21}\delta\phi_{12}^{\dagger}+C_{22}\delta\phi_{22}^{\dagger}
\end{aligned}
\tag{E.31}
$$

From here we can derive

$$
\begin{aligned}
\frac{\delta V}{\delta\phi^{\dagger}} &= \left(\begin{array}{cc} \frac{\delta V}{\left(\delta\phi^{\dagger}\right)_{11}} & \frac{\delta V}{\left(\delta\phi^{\dagger}\right)_{12}} \\ \frac{\delta V}{\left(\delta\phi^{\dagger}\right)_{21}} & \frac{\delta V}{\left(\delta\phi^{\dagger}\right)_{22}} \end{array}\right) \\
&= \left(\begin{array}{cc} C_{11} & C_{21} \\ C_{12} & C_{22} \end{array}\right) \\
&= C^{T}
\end{aligned}
\tag{E.32}
$$

In detailed form

$$
\begin{aligned}
C^{T} &= \left[A+\left(-4v^{2}\right) B+v^{4}\phi\right]^{T} \\
&= A^{T}+\left(-4v^{2}\right) B^{T}+v^{4}\phi^{T}
\end{aligned}
\tag{E.33}
$$

where

$$
\begin{aligned}
A = & -3\phi^{\dagger}\phi\phi\phi^{\dagger}\phi-3\phi\phi\phi^{\dagger}\phi\phi^{\dagger}-3\phi\phi^{\dagger}\phi^{\dagger}\phi\phi \\
& -3\phi\phi^{\dagger}\phi\phi\phi^{\dagger}-3\phi\phi\phi^{\dagger}\phi^{\dagger}\phi-3\phi^{\dagger}\phi\phi^{\dagger}\phi\phi \\
& +2\phi^{\dagger}\phi\phi\phi\phi^{\dagger}+2\phi\phi\phi\phi^{\dagger}\phi^{\dagger}+2\phi^{\dagger}\phi^{\dagger}\phi\phi\phi
\end{aligned}
\tag{E.34}
$$

and

$$B = 2\phi\phi^{\dagger}\phi - \phi^{\dagger}\phi\phi - \phi\phi\phi^{\dagger} \tag{E.35}$$

and the last term is

$$\phi \tag{E.36}$$

We must make a rearrangement of terms in

$$C = A + \left(-4v^2\right) B + v^4\phi$$

in order to obtain the last form of the equation of motion.

**NOTE**

For comparison and for an easier analysis we can use the Abelian version as a suggestion. The *Abelian* version of this arrangement is

$$\left(|\phi|^2 - v^2\right)\left(3|\phi|^2 - v^2\right)\phi$$

This may work for example for the second part (proportional with $(-v^2)$)

$$\begin{aligned} & \left(-4v^2\right) B \\ = & \left(-4v^2\right)\left(2\phi\phi^{\dagger}\phi - \phi^{\dagger}\phi\phi - \phi\phi\phi^{\dagger}\right) \\ = & \left(-4v^2\right)\left[\left[\phi, \phi^{\dagger}\right], \phi\right] \end{aligned}$$

and this is similar with the Abelian version

$$\left(-v^2\right) 4|\phi|^2 \phi$$

Obviously the last term is the same

$$v^4\phi \leftrightarrow \left(-v^2\right)^2 \phi$$

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