On the late phase of relaxation of two-dimensional fluids: turbulence of unitons

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Abstract. The two-dimensional ideal fluid and the plasma confined by a strong magnetic field exhibit an intrinsic tendency to organization due to the inverse spectral cascade. In the asymptotic states reached at relaxation the turbulence has vanished and there are only coherent vortical structures. We are interested in the regime that precedes these ordered flow patterns, in which there still is turbulence and imperfect but robust structures have emerged. To develop an analytical description we propose to start from the stationary coherent states and (in the direction opposite to relaxation) explore the space of configurations before the extremum of the functional that defines the structures has been reached. We find necessary to assemble different but related models: point-like vortices, its field theoretical formulation as interacting matter and gauge fields, chiral model and surfaces with constant mean curvature. These models are connected by the similar ability to describe randomly interacting coherent structures. They derive exactly the same equation for the asymptotic state (sinh-Poisson equation, confirmed by numerical calculation of fluid flows). The chiral model, to which one can arrive from self-duality equation of the field theoretical model for fluid and from constant mean curvature surface equations, appears to be the suitable analytical framework. Its solutions, the unitons, aquire dynamics when the system is not at the extremum of the action. In the present work we provide arguments that the underlying common nature of these models can be used to develop an approach to fluid and plasma states of turbulence interacting with structures.

Keywords: turbulence, coherent structures, self-duality, constant mean curvature surfaces, chiral model, unitons

1. Introduction

A characteristic of the dynamics of the plasma immersed in magnetic field is the dominant role of two space lengths: of the gradient of parameters (density, temperature), L, and of the Larmor gyration ρ_L. The first governs the reservoir of free energy and the second the rate of extraction of this energy through instabilities. A strong magnetic field determines quasi-two-dimensionality which favors the inverse spectral transfer, *i.e.* of energy from small to large spatial scales. This is manifested by a tendency to generate coherent pattern of flow, at the scale L (convective cells in confined plasma). Then, although the first manifestation of the nonlinearities is the turbulence, there are also robust, quasi-coherent structures of flow. The mixture of turbulence and structures depends on the magnitude of the drive and of the region of spectrum where energy is injected. It is only at relaxation (absence of sources) from an initial turbulent state that the ordered pattern of flow produced by the inverse spectral cascade is clearly visible. This evolution is very complex and it may be useful to benefit from the study of the similar problem for the ideal $2D$ neutral (Euler) fluids. The absence of ρ_L rises doubts on the possibility to draw such a parallel, as the complex set of equations (fluid and kinetic) of the plasma are mapped to the non-dissipative Navier-Stokes (*i.e.* Euler) equation. There are however many cases where the Euler equation appears to be an adequate description of the convective processes in plasma [1], [2].

quasi-coherent structures are a component of the fluctuating field even deep in the turbulent regime. In general they are randomly and transiently generated, being a signature of a tendency of organization that competes with the turbulence. Experiments and numerical simulations show that for ideal $2D$ Euler fluid the relaxation from an initial turbulent state leads to a highly coherent structure of flow. The process consists of separation and clusterization of the positive respectively negative vorticity via like-sign vortex mergings. The asymptotic state consists of a dipole and it is known that the streamfunction verifies the sinh-Poisson equation.

The late phase of the relaxation, *i.e.* the regime that precedes the final ordered state, is a mixture of coherent structures and turbulence. This state cannot be studied with methods of statistical physics since the number of active degrees of freedom (as shown for example by a Karhunen-Loeve analysis) decreases while the number of cumulants (irreducible correlations) increases, due to progressive formation of quasi-coherent structures. Then any perturbative expansion intended to remain close to the Gaussian statistics cannot converge [3]. The study of regimes where turbulence and robust coherent structures are simultaneously present requires a framework in which they both can be represented analytically. Since they are as different as random fluctuations versus solitons/vortices, the specific analytical methods (statistical physics versus geometric-algebraic methods of exact integrability) should find a common platform. This is possible since, individually, the turbulent field and the coherent structures are extrema of an unique action functional. This can be written starting from the basic equations and applying procedures like Martin-Siggia-Rose method [4]. It is more transparent to use path integral formulation [5] and to define a generating functional from which the correlations (turbulent field in the presence of structures) can be obtained by functional derivation. This has been applied for a drift wave vortex perturbed by turbulence [6] and for a collection of quasi-coherent vortices embedded in a weak drift wave turbulent field [7] in magnetized plasma. Technically, this approach benefits of the powerful Feynman diagrammatic methods but the analytical developments can become cumbersome and the applications are limited.

In this work we examine the advanced phases of relaxation of the two-dimensional ideal (Euler) fluid. This state consists of few robust vortices interacting with the turbulent field [8], [9]. In the asymptotic state the turbulent field has vanishing amplitude and only the static large scale vortical dipolar structure exist. The streamfunction verifies the sinh-Poisson equation. In the following we adopt this as the reference state and try to explore the regime that precedes it: there is still turbulence, quasi-coherent structures have emerged and the streamfunction does not yet verify the sinh-Poisson equation.

The reference state suggests for the $2D$ Euler fluid connections that may prove to be useful: (1) statistical physics of point-like vortices (SPV), (2) non-Abelian field theory (FT) of matter interacting with a Chern-Simons gauge potential, and (3) surfaces with constant mean curvature (CMC) in the $3D$ Euclidean space. For each system the function that describes the state verifies the sinh-Poisson equation and the corresponding (asymptotic) states are as exceptional as our reference state for fluid. The systems evolve to the highly organized asymptotic states driven by different mechanisms: inverse cascade for the fluid; extremum of entropy at negative temperature for the statistical ensemble of point-like vortices; extremum of the action functional for the field theory model; curvature flow with area dissipation at fixed volume, *i.e.* minimization of the capillarity energy for surfaces. Regarding the asymptotic states, their special nature has individual characterization for these systems: static dipolar coherent structure of flow, for the $2D$ fluid (the reference state); maximum combinatorial entropy at fixed energy, for point-like vortices; self-duality state, for the FT model; Constant Mean Curvature, for the surface. As different as they may seem, the dynamical evolution toward the final states and respectively the characteristics of the final states are however strongly related. The mappings that can be established between them give us the hope that asking a question in one framework we can get a hint from another one, if it happens that the formulation is more favorable there.

To examine in parallel the asymptotic states (dipolar vortex, Self-Duality, Constant Mean Curvature) based on their similar nature can be a useful instrument of investigation and an interesting problem in itself. However our objective is to explore the regime that precedes the limiting ("final", *i.e.* asymptotic) states, when turbulence is still active. Or, the final states are all determined from variational calculus which identifies the state but is unable to describe the regime preceding it. Here however we are helped by the special nature of the final states: like solitons or topological solutions, they are robust and can be found in the preceding evolution regime even if they are not exact solutions. There are interaction between them and with the turbulent field. Models based on such inferred representation of special solutions in regimes where they are just emerging are rather usual: turbulence of Langmuir solitons in plasma [10], turbulence of topological defects in superconductivity [11], filaments of nonlinear Schrodinger equation in optical turbulence [12], instanton plasma [13] or even random topological changes in models of baryogenesis [14], *i.e.* turbulence of sphalerons. In our case the adequate framework would be the field theoretical model with solutions of the self-dual state. Or, here there is a problem. The self-dual states represent the absolute extremum of the action functional of the FT model. Analytically they are described as solutions of the sinh-Poisson equation, the same equation for the streamfunction of the physical $2D$ Euler fluid. The equation is exactly integrable and solutions are known. We can imagine using these solutions in the preceding regime as coherent vortices interacting with the turbulent vorticity field. However the exact state identified by the extremum of the action is self-duality and the sinh-Poisson equation is derived under a particular hypothesis ("ansatz") about the algebraic content of the matter and gauge fields. Then one can ask if, leaving the final (reference) state and pulling-back to explore the regime which precedes it, we are - or we are not - allowed to maintain this particular algebraic structure. The functions that are not yet at exact self-duality may have an algebraic content which is different of that simplified "ansatz" which has led us to the sinh-Poisson equation. On the other hand we know that the asymptotic state is correctly identified since it is confirmed that the physical fluid reaches the static structure that is solution of this equation. The FT system, evolving toward SD will have two tasks: to progressively alter the larger algebraic content by reducing it to the simplified "ansatz" and in the same time to decrease the magnitude of the action such that the asymptotic state is static self-duality. If this is the case then we have to consider not the exact solutions of the sinh-Poisson equation (since it has been derived under a simplified "ansatz") but the solutions of the self-duality equations that are less restrictive with the algebraic content. These solutions exist and are called *unitons*. Therefore we are led to consider the turbulence of the structures manifested at self-duality, without adopting the simplified algebraic "ansatz", and this means turbulence of unitons.

We briefly review the models and show how it is derived the equation *sinh*-Poisson in each of them. This will make plausible the mappings between models. The connections between integrable equations, geometry of surfaces and field theory have been much investigated in the mathematics and physics literature. We limit ourselves to reveal some physical interpretations of these connections which are potentially useful to the subject of turbulence and structures in fluid flow. The purpose is to evaluate the possibility to construct an effective analytical apparatus for the study of structures embedded in turbulence.

2. Euler fluid and system of point-like vortices

The ideal (non-dissipative) incompressible fluid in two - dimensions, which we will shortly call $2D$ Euler fluid, can be described by three functions $(\psi, \mathbf{v}, \omega)$. From the scalar field streamfunction $\psi(x, y, t)$ one derives the velocity vector field $\mathbf{v}(x, y, t) = -\nabla\psi \times \widehat{\mathbf{e}}_z$ (here $\widehat{\mathbf{e}}_z$ is the versor perpendicular on the plane (x, y)) and the vorticity $\omega\widehat{\mathbf{e}}_z = \nabla \times \mathbf{v} = \Delta\psi\widehat{\mathbf{e}}_z$. The Euler equation is the advection of the vorticity by its own velocity field

$$\frac{d\omega}{dt} = \frac{\partial}{\partial t}\Delta\psi + \left[\left(-\nabla\psi \times \widehat{\mathbf{e}}_z\right)\cdot\nabla\right]\Delta\psi = 0. \tag{1}$$

In $2D$ there is flow of energy in the spectrum from small spatial scales toward the large spatial scales, *i.e.* inverse cascade. The numerical simulations confirm this behavior [8], [9]. Adding just a small viscosity and starting from a state of turbulence, the fluid evolves to a state of highly ordered flow: the positive and negative vorticities contained in the initial flow are separated and collected into two large scale vortical flows of opposite sign. The streamfunction ψ in these states reached asymptotically at relaxation from turbulence verifies the *sinh*-Poisson equation

$$\Delta\psi + \lambda\sinh\psi = 0 \tag{2}$$

where $\lambda > 0$ is a parameter. This equation is exactly integrable [15].

The discretized form of Eq.(1) has been extensively studied [16], [17], [18], [19]. We just review few elements of this theory, for further reference.

Consider the discretization of the vorticity field $\omega(x,y)$ in a set of $2N$ point-like vortices ω_i each carrying the elementary quantity ω_0 ($=$ const > 0) of vorticity which can be positive or negative $\omega_i = \pm\omega_0$. There are N vortices with the vorticity $+\omega_0$ and N vortices with the vorticity $-\omega_0$. The current position of a point-like vortex is (x_i, y_i) at the moment t. The vorticity is expressed as

$$\omega(x,y) = \sum_{i=1}^{2N} \omega_i a^2 \delta(x - x_i)\,\delta(y - y_i) \tag{3}$$

where a is the radius of an effective support of a smooth representation of the Dirac δ functions approximating the product of the two δ functions [16]. Then $\omega_i a^2$ approximates the *circulation* γ_i which is the integral of the vorticity over a small area around the point (x_i, y_i): $\gamma_i = \int d^2x\omega_i$ [19]. The formal solution of the equation $\Delta\psi = \omega$, connecting the vorticity and the streamfunction, can be obtained using the Green function for the Laplace operator

$$\Delta_{x,y} G(x,y;x',y') = \delta(x-x')\delta(y-y') \tag{4}$$

where (x', y') is a reference point in the plane. As shown in Ref.[16] $G(\mathbf{r};\mathbf{r}')$ can be approximated for a small compared to the space extension of the fluid, L, $a \ll L$, as the Green function of the Laplacian

$$G(\mathbf{r};\mathbf{r}') \approx \frac{1}{2\pi}\ln\left(\frac{|\mathbf{r}-\mathbf{r}'|}{L}\right) \tag{5}$$

where L is the length of the side of the square domain in plane. The solution of the equation $\Delta\psi = \omega$ is

$$\psi(\mathbf{r}) = \sum_{i=1}^{2N} \gamma_i \frac{1}{2\pi}\ln\left(\frac{|\mathbf{r}-\mathbf{r}_i|}{L}\right) \tag{6}$$

The velocity of the k-th point-vortex is $\mathbf{v}_k = -\left.\nabla\psi\right|_{\mathbf{r}=\mathbf{r}_k} \times \widehat{\mathbf{e}}_z$ and the equations of motion are

$$\frac{dx_k}{dt} = v_x^{(k)} = -\sum_{i=1, i\neq k}^{2N} \gamma_i \frac{1}{2\pi} \frac{y_k - y_i}{\left|\mathbf{r}_k - \mathbf{r}_i\right|^2} \tag{7}$$

$$\frac{dy_k}{dt} = v_y^{(k)} = \sum_{i=1, i\neq k}^{2N} \gamma_i \frac{1}{2\pi} \frac{x_k - x_i}{\left|\mathbf{r}_k - \mathbf{r}_i\right|^2}$$

The equations are purely kinematic (no inertia), can be derived from a Hamiltonian and the continuum limit of the discretization is mathematically equivalent with the fluid dynamics. The standard way of describing the discrete model is within a statistical approach [20], [16], [17], [21]. The elementary vortices are seen as elements of a system of interacting particles (a gas) that explores an ensemble of microscopic states leading to the macroscopic manifestation that is the fluid flow. The number of positive vortices in the state i is N_i^+ and the number of negative vortices in the state i is N_i^-. The total numbers of positive and respectively negative vortices are equal: $N^+ = \sum_i N_i^+ = \sum_i N_i^- = N^-$. This system has a statistical temperature that is negative when the energy is zero or positive [22], [23]. The energy of the discrete system of point-like vortices is $\mathcal{E} = \frac{1}{2}\sum_{ij} \omega\left(\mathbf{r}_i\right) G\left(\mathbf{r}_i, \mathbf{r}_j\right) \omega\left(\mathbf{r}_j\right)$ where $\omega\left(\mathbf{r}_i\right) = -\left(N_i^+ - N_i^-\right)$ is the vorticity. The probability of a state is calculated as a combinatorial expression

$$\mathcal{W} = \left\{\frac{N^+!}{\prod_i N_i^+!}\right\}\left\{\frac{N^-!}{\prod_i N_i^-!}\right\} \tag{8}$$

The *entropy* is the logarithm of this expression and by extremization one finds

$$\ln N_i^{\pm} + \alpha^{\pm} \pm \beta \sum_j G\left(\mathbf{r}_i, \mathbf{r}_j\right)\left(N_j^+ - N_j^-\right) = 0 \tag{9}$$

for $i = 1, N$, where $\alpha^{\pm}$ and β are Lagrange multipliers introduced to ensure $\sum N_i^+ = \sum N_i^- = N = \text{const}$ and conservation of the Energy $\mathcal{E}$. The solutions are written in terms of a continuous function $\psi\left(x, y\right)$

$$N_i^{\pm} = \exp\left[-\alpha^{\pm} \mp \beta\psi\left(x, y\right)\right] \tag{10}$$

implying $N_i^+ N_i^- = \text{const}$. From Eq.(9) and (10) the *sinh*-Poisson equation (2) is derived.

3. Field theoretical formulation of the continuum limit of the point-like vortex model

The model of point-like vortices captures the physics of the $2D$ Euler fluid in a new formulation: matter (density of point-like vortices), field (the Coulombian potential in Eq.(6)) and interaction. This suggests to formulate the continuum limit of the discrete point-like vortices as a field theory. The density of point-like vortices is represented by the matter field $\phi(x,y,t)$ and the potential of interaction by the gauge field $A_\mu(x,y,t)$, $\mu = 0,1,2$. The "matter" consists of the positive and negative vortices.The dynamics is $2D$ but we exploit the invariance to motion along the third axis to reveal the chiral nature of the elementary vortices. The positive vortices: (1) rotate anti-clockwise in plane: $\omega\widehat{\mathbf{e}}_z \sim \sigma$ spin is up; (2) move along the positive z axis: $\mathbf{p} = \widehat{\mathbf{e}}_z p_0$; (3) have positive chirality: $\chi = \sigma \cdot \mathbf{p}/|\mathbf{p}|$. The positive vortices can be represented as a point that runs along a positive helix, upward. In projection from the above the plane toward the plane we see a circle on which the point moves anti-clockwise.

The negative vortices: (1) rotate clockwise in plane: $(-\omega)\widehat{\mathbf{e}}_z \sim -\sigma$ spin is down; (2) move along the negative z axis: $-\mathbf{p} = \widehat{\mathbf{e}}_z(-p_0)$, along $-z$; (3) have positive chirality: $\chi = \sigma \cdot \mathbf{p}/|\mathbf{p}|$. The negative vortices can be represented as a point that runs along a positive helix, the same as above, but runs downward. In projection from the above the plane toward the plane we see a circle on which the point moves clockwise.

The positive vortices and the negative vortices have the same *chirality* and in a point where there is superposition of a positive and a negative elementary vortices the *chirality* is added. In particular, the vacuum consists of paired positive and negative vortices, with no motion of the fluid, which in physical variables means $\psi \equiv 0$, $\mathbf{v} \equiv \mathbf{0}$, $\omega \equiv 0$. In FT the vacuum consists of superposition of positive and negative vortices, which means: (1) zero spin, or zero *vorticity*; (2) zero momentum $\mathbf{p} = \mathbf{0}$; (3) $2\times$chiral charge. The Euler fluid at equilibrium ($\psi = 0$, $\mathbf{v} = \mathbf{0}$, $\omega = 0$) is in a vacuum with *broken chiral invariance.*

The $sl(2,\mathbf{C})$ Non-Abelian structure is necessary due to the *vortical* nature of the elementary object: the vorticity matter must be represented by a *mixed spinor*. The Lagrangian [24], [25]

$$\begin{aligned}\mathcal{L} = & -\kappa\varepsilon^{\mu\nu\rho}\mathrm{tr}\left(\left(\partial_\mu A_\nu\right)A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho\right) \\ & + i\mathrm{tr}\left(\phi^\dagger\left(D_0\phi\right)\right) - \frac{1}{2m}\mathrm{tr}\left(\left(D_k\phi\right)^\dagger\left(D^k\phi\right)\right) \\ & + \frac{1}{4m\kappa}\mathrm{tr}\left(\left[\phi^\dagger,\phi\right]^2\right)\end{aligned} \tag{11}$$

where $D_\mu = \partial_\mu + [A_\mu,]$ and κ, m are positive constants. The Euler - Lagrange equations for the action functional $\mathcal{S} = \int dxdydt\mathcal{L}$ are the equations of motion

$$\begin{aligned} iD_0\phi &= -\frac{1}{2m}D_kD^k\phi - g\left[\left[\phi,\phi^\dagger\right],\phi\right] \\ \kappa\varepsilon^{\mu\nu\rho}F_{\nu\rho} &= iJ^\mu \end{aligned} \tag{12}$$

where the current $J^\mu = \left(J^0, J^k\right)$

$$\begin{aligned} J^0 &= \left[\phi,\phi^\dagger\right] \\ J^k &= -\frac{i}{2m}\left(\left[\phi^\dagger,\left(D^k\phi\right)\right] - \left[\left(D^k\phi\right)^\dagger,\phi\right]\right) \end{aligned} \tag{13}$$

is covariantly conserved $D_\mu J^\mu = 0$. The energy density is

$$E = \frac{1}{2m}\mathrm{tr}\left(\left(D_k\phi\right)^\dagger\left(D^k\phi\right)\right) - \frac{g}{2}\mathrm{tr}\left(\left[\phi^\dagger,\phi\right]^2\right) \tag{14}$$

The Gauss constraint is the zero component of the second equation of motion

$$2\kappa F_{12} = iJ^0 = i\left[\phi, \phi^\dagger\right] \tag{15}$$

In the following we will use the combinations: $A_\pm \equiv A_x \pm iA_y$, $\partial/\partial z = \frac{1}{2}\left(\partial/\partial x - i\partial/\partial y\right)$, $\partial/\partial z^* = \frac{1}{2}\left(\partial/\partial x + i\partial/\partial y\right)$, and similar. Writing

$$\begin{aligned} \mathrm{tr}\left(\left(D_k\phi\right)^\dagger\left(D^k\phi\right)\right) &= \mathrm{tr}\left(\left(D_-\phi\right)^\dagger\left(D_-\phi\right)\right) - i\mathrm{tr}\left(\phi^\dagger\left[F_{12},\phi\right]\right) \\ &\quad - m\varepsilon^{ij}\partial_i\left[\phi^\dagger\left(D_j\phi\right) - \left(D_j\phi\right)^\dagger\phi\right] \end{aligned} \tag{16}$$

we replace in the expression Eq.(14) of the energy density and note that for smooth fields we can ignore the last term, which is evaluated at the boundary

$$E = \frac{1}{2m}\mathrm{tr}\left(\left(D_-\phi\right)^\dagger\left(D_-\phi\right)\right) \tag{17}$$

The states are *static* $\partial_0\phi = 0$ and minimize the energy ($E = 0$). Adding the Gauss constraint (after replacing $F_{12} = (i/2)\,F_{+-}$) we have a set of two equations for stationary states corresponding to the absolute minimum of the energy

$$\begin{aligned} D_-\phi &= 0 \\ F_{+-} &= \frac{1}{\kappa}\left[\phi,\phi^\dagger\right] \end{aligned} \tag{18}$$

From these equations the *sinh*-Poisson equation is derived [26], [27]. The states correspond to zero curvature in a formulation that involves the reduction to $2D$ from a four dimensional Self - Dual Yang Mills system, as shown in [26]. Therefore we will denote this state as Self - Dual (SD). The functions ϕ and A_μ are mixed spinors, elements of the algebra $sl\left(2,\mathbf{C}\right)$.

In order to solve this system and connect with variables of the real fluid, one can adopt the following algebraic ansatz [24],

$$\phi = \phi_1 E_+ + \phi_2 E_-\ ,\ \ \phi^\dagger = \phi_1^* E_- + \phi_2^* E_+ \tag{19}$$

and

$$A_- = aH\ ,\ \ A_+ = -a^* H \tag{20}$$

which is based on the three generators (E_+, H, E_-) of the Chevalley basis of $sl\left(2,\mathbf{C}\right)$. Explicitly: $E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. From the first equation of motion $D_-\phi = 0$ we obtain

$$\frac{\partial\phi_1}{\partial z} + a\phi_1 = 0 \tag{21}$$

$$\frac{\partial\phi_2}{\partial z} - a\phi_2 = 0 \tag{22}$$

The Gauss equation becomes

$$\frac{\partial a}{\partial x_+}+\frac{\partial a^*}{\partial x_-}=\frac{1}{k}\left(\rho_1-\rho_2\right) \tag{23}$$

where $\rho_{1,2}\equiv\left|\phi_{1,2}\right|^2$. Using Eqs.(21) and its complex conjugate the left hand side of Eq.(23) becomes

$$\frac{\partial a}{\partial x_+}+\frac{\partial a^*}{\partial x_-}=-2\frac{\partial^2}{\partial z\partial z^*}\ln\left(\left|\phi_1\right|^2\right)=-\frac{1}{2}\Delta\ln\left(\left|\phi_1\right|^2\right)$$

or

$$-\frac{1}{2}\Delta\ln\rho_1=\frac{1}{\kappa}\left(\rho_1-\rho_2\right) \tag{24}$$

Similarly, using Eq.(22) in Eq.(23) we obtain

$$\frac{\partial a}{\partial x_+}+\frac{\partial a^*}{\partial x_-}=2\frac{\partial^2}{\partial z\partial z^*}\ln\left(\left|\phi_2\right|^2\right)=\frac{1}{2}\Delta\ln\left(\left|\phi_2\right|^2\right)$$

or

$$\frac{1}{2}\Delta\ln\rho_2=\frac{1}{\kappa}\left(\rho_1-\rho_2\right) \tag{25}$$

The right hand side in Eqs.(24) and (25) is the same and if we substract the equations we obtain

$$\Delta\ln\rho_1+\Delta\ln\rho_2=0 \tag{26}$$

It results that $\rho_1=\rho_2^{-1}\equiv\rho$. Now we introduce a scalar function ψ, defined by $\rho=\exp\left(\psi\right)$ and the Eqs.(24) and (25) take the unique form

$$\Delta\ln\rho=-\frac{2}{\kappa}\left(\rho-\frac{1}{\rho}\right) \tag{27}$$

which is the *sinh*-Poisson equation (also known as the elliptic *sinh*-Gordon equation)

$$\Delta\psi+\frac{4}{\kappa}\sinh\psi=0 \tag{28}$$

The states identified by the FT are the absolute extrema of the action functional and are characterized by: (1) stationarity; (2) double periodicity, *i.e.* the function $\psi\left(x,y\right)$ must only be determined on a "fundamental" square in plane; (3) the total vorticity is zero; (4) the states verify Eq.(28). A more detailed discussion can be found in [27], [28], [29]. The field theoretical model is a reformulation of the system of point-like vortices. The parallel between the two formulations has led to the conclusion that the extremum of entropy (for negative statistical temperature) corresponds to the states of self-duality.

4. The surface in the Euclidean $\mathbf{E}^3$ space

It is interesting that the same equation governing the asymptotic states of the $2D$ Euler fluid is also the equation that identifies the surfaces in $\mathbf{E}^3$ that have Constant Mean Curvature (CMC). To make more clear the connection we review very briefly the theory of Constant Mean Curvature (CMC) surfaces, following [30].

A surface is a mapping from a domain in the plane (x, y) to the points $\mathbf{F} = (F_1, F_2, F_3) \in \mathbf{E}^3$. Equivalently, one can use complex variables $(x, y) \to (z, z^*)$ giving the conformal parametrization of the surface and the metric $ds^2 = \exp(u)(dx^2 + dy^2) = \exp(u)\, dzdz^*$ where $u(z, z^*)$ is a real function. The following normalizations are applied to the vectors in the tangent plane in the current point of the surface

$$\frac{\partial \mathbf{F}}{\partial z} \cdot \frac{\partial \mathbf{F}}{\partial z^*} = \frac{1}{2} \exp(u) \ , \ \frac{\partial \mathbf{F}}{\partial z} \cdot \frac{\partial \mathbf{F}}{\partial z} = 0 \ , \ \frac{\partial \mathbf{F}}{\partial z^*} \cdot \frac{\partial \mathbf{F}}{\partial z^*} = 0 \tag{29}$$

The vector normal to the tangent plane is $\mathbf{N} \sim \frac{\partial \mathbf{F}}{\partial z} \times \frac{\partial \mathbf{F}}{\partial z^*}$ and is normalized $\mathbf{N} \cdot \mathbf{N} = 1$. The three vectors define the *moving frame*

$$\sigma \equiv \left(\frac{\partial \mathbf{F}}{\partial z}, \frac{\partial \mathbf{F}}{\partial z^*}, \mathbf{N} \right)^T \tag{30}$$

The change of a vector of the moving frame at an infinitesimal displacement on the surface is expressed by Christoffel coefficients. More compactly, the change of the frame (*Gauss Weingarten* equations) is expressed in terms of two 3×3 matrices $(\mathcal{U}, \mathcal{V})$

$$\frac{\partial \sigma}{\partial z} = \mathcal{U}\sigma \ , \ \frac{\partial \sigma}{\partial z^*} = \mathcal{V}\sigma \tag{31}$$

The consistency condition is the equality of mixed derivatives (Gauss Codazzi equations) $\frac{\partial \mathcal{U}}{\partial z^*} - \frac{\partial \mathcal{V}}{\partial z} + [\mathcal{U}, \mathcal{V}] = 0$. The first form $I = d\mathbf{F} \cdot d\mathbf{F}$ and the second form $II = d\mathbf{F} \cdot \mathbf{N}$ of the surface are expressed by the formulas

$$I = \exp(u) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{32}$$

$$II = \begin{pmatrix} Q + Q^* + H \exp(u) & i(Q - Q^*) \\ i(Q - Q^*) & -(Q + Q^*) + H \exp(u) \end{pmatrix} \tag{33}$$

in terms of the projections of the second order derivatives of $\mathbf{F}$ on the normal versor

$$\frac{\partial^2 \mathbf{F}}{\partial z \partial z} \cdot \mathbf{N} = Q \ , \ \frac{\partial^2 \mathbf{F}}{\partial z \partial z^*} \cdot \mathbf{N} = \frac{1}{2} H \exp(u) \tag{34}$$

The *principal curvatures* κ_1 and κ_2 are eigenvalues of the matrix $(II) \cdot I^{-1}$ and the mean - , respectively the Gauss curvature are $H = \frac{1}{2}(\kappa_1 + \kappa_2)$ and $K = \kappa_1 \kappa_2 = H^2 - 4QQ^* \exp(-2u)$. The *Gauss-Codazzi* equations become

$$\begin{aligned} \frac{\partial^2 u}{\partial z \partial z^*} + \frac{1}{2} H^2 \exp(u) - 2QQ^* \exp(-u) &= 0 \\ \frac{\partial Q}{\partial z^*} &= \frac{1}{2} \frac{\partial H}{\partial z} \exp(u) \end{aligned} \tag{35}$$

If the mean curvature is constant: $H = 1/2$ the function Q is holomorphic and can be taken constant $\neq 0$ (assuming that there is no umbilic point). Taking $Q = 1/4$ the first equation becomes the *sinh*-Poisson equation $\Delta u + \sinh(u) = 0$. To make it coincide with (28) one chooses the constants: $H = 1/\sqrt{\kappa}$, $Q = 1/(2\sqrt{\kappa})$ where κ is the coefficient of the Chern-Simons term in (11). For any flow in the asymptotic regime of the Euler fluid, there is a corresponding CMC surface. From point-like vortices and from FT models we know that the solutions of sinh-Poisson for fluids must be doubly periodic, which means that the CMC surfaces must be tori. We identify $u \to \psi$ (the streamfunction) and $\rho = \exp \psi$ is the factor in the isothermal metric Eq.(32) of the surface [30]. The Gaussian curvature is $K = -\omega/(2\rho) = \kappa^{-1}(1 - 1/\rho^2)$ where ω is the vorticity. The integral on the surface of the Gaussian curvature is $\iint K dA = 2\pi\chi = 0$, where χ is the Euler characteristic of the torus. This was expected because $\iint K dA = \iint K \exp(\psi) \, dx dy = -\frac{1}{2}\iint \omega dx dy = 0$, since we know that the absolute extremum for both models SPV and FT finds zero total vorticity in the domain.

This purely geometric derivation of the *sinh*-Poisson equation seems far from the other two discussed above: the extremum of entropy for the statistical ensemble of point-like vortices and the extremum of the action functional in the field theoretical model. However the CMC surfaces can also be derived from an extremum condition. The mean curvature $H = (1/R_1 + 1/R_2)/2$ of a surface ($R_{1,2}$ are the radii along principal curvature lines) intervenes in the balance between the capillary force and the difference Δp of pressure on the two sides, the Laplace law: $\Delta p = 2\sigma H$ where $\sigma =$const. is the coefficient of surface tension. The CMC surface is “in mechanical equilibrium” since $H =$ constant means equal pressure Δp in all points. For surfaces the principle of extremum is connected with the energy of the capillary forces, $E = \sigma A$, where A is the area of the surface. It is clear that the “equilibrium” mentioned above corresponds to the minimum energy, *i.e.* of the area A. Then the *sinh*-Poisson equation appears again as being derived from a principle of extremum. Explicit analytical application of the minimization of surface at fixed volume can be done for the unduloids of Delaunay [31], well-known CMC surfaces.

Regarding the asymptotic states we note that to every flow ($u \equiv$ theta-function solution of sinh-Poisson equation) one can now associate a CMC surface with metric (32). The reverse direction, *i.e.* to derive the physical possibility of a stationary flow from particularities of the CMC surfaces (like embedding versus immersion, multiple ends, etc.) has been much less investigated.

Random perturbations on the surface, violating the CMC condition, corresponds to the regime that precedes the asymptotic ordered states of the fluid. The tendency of capillary forces to smooth out the perturbations and to install the CMC state, corresponds to the fluid being driven by inverse cascade to the coherent dipolar flow, and with the extremization of the FT action leading to the self-dual states. The local perturbations of the surface, by which it departs from CMC $H \neq$ const, are corrected by curvature flow, where the velocity of a point of the surface along the normal is proportional to the local mean curvature, a well known dynamics of interfaces in physical systems. In the model of point-like vortices the entropy decreases and the temperature is negative when the system evolves to organization [22]. Similarly, the entropy, defined for a line as $S = \alpha \int ds \; H(t) \ln H(t)$, decreases when the curvature $H(t)$ evolves according to the curvature flow [32].

For $H=1/\sqrt{\kappa}$ and $K=\kappa^{-1}\left(1-1/\rho^2\right)$ it results $\left(\sqrt{\kappa}\rho\right)^{-1}=\kappa_2-\kappa_1$. If an umbilic point ($\kappa_1=\kappa_2$) existed on the surface, this would correspond to a singular vorticity $\rho\rightarrow\infty$. Since the CMC surface associated to the fluid flow is a torus an umbilic point does not exist which means that a singular vorticity cannot exist in the asymptotic state of the flow. But in the regime that precedes it we may expect that a localized, high concentration of vorticity should correspond on the perturbed quasi-CMC surface to a position where there is approximate equality of the principal curvatures.

5. The chiral model and the uniton

The equations of self-duality of the FT model, (18) can be formulated as equations of the chiral model. The suggestion comes from the observation that the new potential defined by $\mathcal{A}_-=A_-+\sqrt{\frac{1}{\kappa}}\phi^\dagger$ and $\mathcal{A}_+=A_+-\sqrt{\frac{1}{\kappa}}\phi$ has zero curvature [24]

$$\mathcal{F}_{+-}=\partial_+\mathcal{A}_--\partial_-\mathcal{A}_++\left[\mathcal{A}_+,\mathcal{A}_-\right]=0 \tag{36}$$

if $A_\pm$ and ϕ verify the SD equations (18) (independently of the algebraic ansatz; if the algebraic ansatz is adopted the equation $\mathcal{F}_{+-}=0$ leads to the *sinh*-Poisson equation). Then the new potential is locally a gauge field $\mathcal{A}_-=g^{-1}\partial_-g$. Using g to transform the matter field ϕ

$$\chi=\sqrt{\frac{1}{\kappa}}g\phi g^{-1} \tag{37}$$

it can be checked [24] that the two SD (18) equations are verified if χ is the solution of

$$\partial_-\chi=\left[\chi^\dagger,\chi\right] \tag{38}$$

Then, expressing χ as

$$\chi\equiv\frac{1}{2}h^{-1}\partial_+h \tag{39}$$

one obtains the equation of the chiral model [33], [34]

$$\partial_+\left(h^{-1}\partial_-h\right)+\partial_-\left(h^{-1}\partial_+h\right)=0 \tag{40}$$

where h is a map from a domain in the Euclidean plane to the group $SL\left(2,\mathbf{C}\right)$. Until this point no special assumption is made on the algebraic content of h (hence of ϕ, through χ). Finding a solution of this equation provides us with a more general solution than that offered by the *sinh*-Poisson equation, since the latter is derived under the simplified algebraic ansatz, Eqs.(19 - 20). The solutions of the chiral model Eq.(40) are unitons, *i.e.* harmonic maps $\mathbf{R}^2\rightarrow SL\left(2,\mathbf{C}\right)$. For this group the construction of uniton is based on a single rational function $f\left(z\right)$ [35], [36], [24] and has the expression [37]

$$h=\frac{i}{1+\left|f\right|^2}\left(\begin{array}{cc}1-\left|f\right|^2 & f\\ f^* & -1+\left|f\right|^2\end{array}\right) \tag{41}$$

$$\chi = \frac{1}{2}h^{-1}\partial_+ h = \frac{f}{\left(1+|f|^2\right)^2} 2\frac{\partial f^*}{\partial z^*}\left(\begin{array}{cc} 1 & f \\ -\frac{1}{f} & -1 \end{array}\right) \tag{42}$$

All Chevalley generators are present. Calculating the matrix $\left[\chi, \chi^\dagger\right]$ it is shown [24] that this can be diagonalized by a unitary matrix g and using Eq.(37) one obtains the *covariant charge density*, Eq.(13)

$$\begin{aligned} J^0 = & \left[\phi, \phi^\dagger\right] = \kappa g^{-1}\left[\chi, \chi^\dagger\right] g \\ = & -\kappa \partial_+ \partial_- \ln\left(1+|f|^2\right)\left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right) \end{aligned} \tag{43}$$

The fact that the matter density $\left[\phi, \phi^\dagger\right]$ has only the Cartan component H may be seen as justifying the ansatz $A_\pm \sim H$. As we know, applying the Eqs.(19), (20) to the SD equations (18) and (27) it results

$$\left[\phi, \phi^\dagger\right] = \left(\rho - \frac{1}{\rho}\right) H = \left(-\frac{\kappa}{2}\Delta \ln \rho\right) H \tag{44}$$

and there is the correspondence $1 + |f|^2 = \exp(\psi/2) = \sqrt{\rho}$. Here we recall the matrix Φ that transforms the euclidean frame $(\widehat{\mathbf{e}}_1 + i\widehat{\mathbf{e}}_2, \widehat{\mathbf{e}}_1 - i\widehat{\mathbf{e}}_2, \widehat{\mathbf{e}}_3)$ into the moving frame $\left(\frac{\partial \mathbf{F}}{\partial z}, \frac{\partial \mathbf{F}}{\partial z^*}, \mathbf{N}\right)$ attached to the current point of the surface, by the formulas [38]

$$\begin{aligned} \frac{\partial F}{\partial z} = & -i \exp\left(\frac{u}{2}\right) \Phi^{-1} E_- \Phi \\ \frac{\partial F}{\partial z^*} = & -i \exp\left(\frac{u}{2}\right) \Phi^{-1} E_+ \Phi \end{aligned} \tag{45}$$

These expressions are used in the Gauss-Weingarten equations (31). The compatibility condition, the definitions Eq.(34) and the normalization $\det \Phi = \exp(u/2)$ determine explicitly Φ [39]. Comparing $1+|f|^2 = \exp(\psi/2)$ with $\det \Phi = \exp\left(\frac{\psi}{2}\right)$ we are suggested that $1 + |f|^2$ has for unitons the similar meaning as $\det(\Phi)$ for CMC surfaces, with Φ the *quaternion* which provides the mapping between (a) the euclidean fixed orthogonal frame, to (b) the moving frame attached to the current point $\mathbf{F} \equiv F$ of the surface. Indeed, the equation (40) of the chiral model is equivalent to a system (Weierstrass-Enneper) of equations defining CMC surfaces [40] . The uniton (41) of the chiral model is also obtained in terms of solutions of these equations for constant mean curvature surfaces [41].

The energy of the chiral model [24] [37] is given by the abelian charge of the FT

$$E[h] = \frac{1}{2}\int d^2x \ \mathrm{Tr}\left[\left(h^{-1}\partial_- h\right)\left(h^{-1}\partial_+ h\right)\right] = \frac{2}{\kappa} Q^A$$

The density of Q^A is $\mathrm{Tr}\left(\phi^\dagger \phi\right) = \rho + 1/\rho$. The fact that the energy of the chiral model and the abelian charge Q^A are integer multiples of some physical constant is connected in our case with the property of double periodicity of the absolute extremum of the FT action, as for entropy in SPV. From those models we know that for Eq.(2) one can only retain solutions expressed in terms of Riemann theta functions.

In summary, the chiral model with equation (40) is an equivalent form of the FT model at self-duality, with equations Eqs.(18) and is also equivalent to the constant mean curvature surface model, with the Weierstrass-Enneper equations for the matrix Φ.

6. A set of time-dependent unitons

The physical system in the regime preceding the static final state consists of few large-amplitude vortices moving slowly in a weak turbulent field. The unitons Eq.(41) constitute a possible representation of the quasi-coherent structures (lumps of energy) but they are static since they solve the SD equations (18) or equivalently Eq.(40). We should now leave the asymptotic state and use the equations of motion with time dependence (12) at least in its close proximity. One should use the procedure of Manton [42] as applied, for example, to the Abelian Higgs (AH) model of superconductivity. The exact solutions of the AH self-duality equations are static localized vortices (topological defects) parametrized by the positions of their centers. All possible sets of parameters constitutes a manifold, the moduli space of SD solutions. Assuming now that the parameters can be slow functions of time, the solutions are inserted in the Lagrangian. The resulting structure of the Lagrangian induces a metric on the manifold and the part which is quadratic in the time derivatives drives the motion of the vortices, as a mechanical kinetic energy. It is a geodesic flow on the manifold.

For this procedure to be applied to the SD or chiral solutions it has been first necessary to modify the uniton equation by extending it with time dependence [37]. The derivation operators in Eq.(39) are replaced $\partial_\pm \to \partial_\pm \pm i\partial_t$ and the equation which replaces (40) is

$$\left(\eta^{\mu\nu} + \varepsilon^{\mu\nu\rho} V_\rho\right) \partial_\nu \left(J^{-1} \partial_\mu J\right) = 0 \tag{46}$$

where $\eta^{\mu\nu} = diag\,(-1, 1, 1)$, $\varepsilon^{\mu\nu\rho}$ is the totally antisymmetric tensor and V_ρ is a fixed versor with the (t, x, y) components $(0, 1, 0)$ [43]. The solution of this new equation has the same form as the uniton Eq.(41) but now the rational function $f\,(z)$ depends on time. Taking

$$f\,(z) = \prod_{k=1}^{N} (z - p_k) \prod_{m=1}^{N} (z - q_m)^{-1} \tag{47}$$

the positions of zero's and poles of f are assumed to be functions of time with trajectory $\gamma\,(t) \equiv (p_k\,(t)\,, q_m\,(t))_{k,m=1,N}$. Inserting f in Eq.(41) the time variation in the new equation of the uniton J, Eq.(46), is calculated $\partial_t J = \frac{\delta J[\gamma(t)]}{\delta \gamma_i} \frac{d\gamma_i(t)}{dt}$, $i = 1, 2N$. The kinetic terms in the energy-momentum tensor $T = -\frac{1}{2} \int d^2x \mathrm{Tr}\,(J^{-1}\partial_t J)^2$ take the general form $h_{jk} \frac{d\gamma_j}{dt} \frac{d\gamma_k}{dt}$ and the motion is governed by the constants h_{jk}. Explicit calculation of the metric h_{jk} of the manifold of uniton solutions are provided in [44], [45].

This approach is probably the closest to our objective of studying the dynamics of structures near the static self-dual state. It will be necessary to introduce a random factor in the parameters of the unitons, *i.e.* in the positions of zeros and poles of the rational function f. Another possibility is to follow Dunajski and Manton [45] and consider that an external magnetic field (which we can take random) alters the geodesic motion in the manifold which represents the moduli space of uniton solutions.

7. Mapping between formulations

Returning to the connection between the chiral model equation (40) and the constant mean curvature equations for Φ one should investigate how the dynamics of unitons can be mapped to a similar time variation of the matrix Φ which defines the moving frame on the surface. This mapping can be formulated as follows.

Consider a solution $\psi(x, y)$ of the exactly integrable equation (2). It is the streamfunction of a stationary flow of the Euler fluid. Using the conformal metric (32) and the second form (33) with Q, H chosen to lead to the same form of sinh-Poisson equation, one can construct the constant mean curvature surface that corresponds to that flow. This also gives the expression of the matrix Φ, which connects the fixed Euclidean frame to the moving frame on the surface. All this is strictly limited to the asymptotic state. Now assume that the CMC surface is perturbed. Random protuberances of low amplitude are admitted and the exact CMC property is lost. But the CMC property is the result of a variational extremization of a functional: minimum area for fixed volume. An evolution similar to the effect of capillarity on an elastic membrane (foam) will try to return the surface to the CMC state, via curvature flow. For a single protuberance the curvature flow will dissipate the area such as to draw the local curvatures $\kappa_{1,2}$ to the values of the CMC. For a random ensemble of perturbations, the curvature flow may induce displacements and coalescences as part of the area dissipation. In any moment of such evolution one can calculate the matrix Φ since the frame in any point of the perturbed surface is known. Using Φ one can infer the function f *i.e.* the expression of the uniton (41). When it is represented in $\mathbf{R}^2$, the density of energy is a set of lumps [36]. The evolution will also include changes of topology: a function f as in (47) induces through (41) a mapping $\mathbf{R}^2 \rightarrow S^2$ with topological charge [34]. For the perturbation of the surfaces under the curvature flow the suppression of a perturbation is smooth but a smooth change in Φ can lead to changes of f across topological classes.

This program is analytically complicated but it offers a possible comparison with experiment. The rate of return of the perturbed surface to the CMC state through area dissipation governed by curvature flow will translate into a rate of change in the portrait of lumps in the density of energy of the uniton field. Their number should decrease, since the evolution should get closer to the dipolar structure, solution of sinh-Poisson equation. But this rate is known from experiments [46], [47].

Another development of this analytical programme which can lead to comparison with experiment and numerical simulation is connected with the statistics of high amplitude, localized vortices, in a turbulent fluid field. As mentioned above, high amplitude of vorticity concentration are associated with large magnitude of ρ (27). As explained above, the mapping FT to CMC surfaces suggests $(\sqrt{\kappa}\rho)^{-1} = \kappa_2 - \kappa_1$ which means that an approximative localization of high-ω peaks can be obtained from the localization of the umbilic points $\kappa_1 = \kappa_2$. This can only be approximative, *i.e.* $|\kappa_1 - \kappa_2| < \Lambda$ since in fact umbilic points cannot exist on the torus (*i.e.* the basic rectangle of the double periodic solutions of the sinh-Poisson equation). One can take Λ a fraction of H. Now, starting from a CMC surface we consider a statistical ensemble of its realizations under random perturbations. For Gaussian statistics of the perturbations of the surface it is possible to determine the average density of the randomly-occuring umbilic points [48]. This also provides the average density of the localized, high magnitude concentration of vorticity in a turbulent field.

8. Summary

The presence of convective structures in a turbulent field is recognized as a frequent experimental regime in fluids and plasmas. It is also a difficult theoretical problem. In this work we have examined a series of models which, placed together, can lay the basis for a systematic analytical investigation of this particular regime. The characteristic of a possible approach, as we have tried to propose, is the need to work in the close proximity of exactly integrable states, identified by variational methods (extremizing functionals), not by conservation equations. It is determined, in this way, a set of structures. They are known to have emerged in the late phase of turbulence relaxation and for this reason, can be approximately taken as existing even in the dynamical regime which precedes the final static state. The way to do this analytically requires however a complicated apparatus. The uniton solution appears to be a good direction of investigation. This is part of a series of models that, beyond their apparent differences, are strongly related: Euler fluid, point-like vortices, field theory of matter and Chern-Simons gauge field, constant mean curvature surfaces. As we have explained before, each model provides a certain realization of the regime preceding the final state: (1) a set of fluid vortices moving slowly in a weak turbulent field; (2) the matter and gauge fields of FT evolve in close proximity of the self-duality state; (3) a set of unitons move on the manifold of moduli space of solutions of the chiral model, with random perturbations relative to the geodesic trajectory; (4) a constant mean curvature surface perturbed by random deformations returns through local curvature flows to the CMC state.

The chiral model, strongly related to constant mean curvature surfaces and to the self-dual equations of the FT, offers the structures we need (unitons) and their dynamic in time. Practical applications are possible and in some cases the result can be confronted with observations.

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