Exact periodic solutions of the Liouville equation and the "snake" of density in JET

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Abstract

Several processes of anomalous pinching of the density are possibly a manifestation of thermodynamic constraints: stationary entropy, turbulent equipartition or extremum of the action functional for an equivalent discrete system. The anlytical models suggest the particular role of the Liouville equation. Since the general solutions are less useful, we present the general procedure to obtain exact solutions of this equation. We show that the solution on periodic domain exhibits characteristics qualitatively similar with the density perturbation observed in the "snake" phenomenon in JET. The procedure can be applied to other physical processes described by the Liouville equation: the coalescence instability of a chain of magnetic islands, the vortical structures of the Ion Temperature Gradient instability, the Kelvin-Helmholtz instability of sheared velocity profiles, etc.

The anomalies of the density evolution

1. *snake* in JET: persistent perturbations of density on rational q surfaces (Weller *et al.* 1987);
2. large inward density pinch (Weisen *et al.* 2004);
3. stationary impurity accumulation in the plasma centre(Scavino *et al.* 2004);

Turbulent equipartition involves the whole volume of plasma. However a model should be applicable to localised density perturbations.

Models implying discretized forms of the system lead to *sinh*-Poisson and to Liouville equations

(*Montgomery, Jackiw, Spineanu and Vlad*)

The stationary states of ideal fluids and plasmas

- ideal fluid two-dimensional Euler equation. This is

$$\frac{d\boldsymbol{\omega}}{dt} = 0 \longrightarrow \left[\left(-\boldsymbol{\nabla}_{\perp}\psi \times \widehat{\mathbf{n}}\right) \cdot \boldsymbol{\nabla}_{\perp}\right]\left(-\boldsymbol{\nabla}_{\perp}^{2}\psi\right) = 0$$

- the Hasegawa-Mima-Charney equation
- the ion-convective cells equation
- the Flierl-Petviashvili equation
- the nonlinear drift-Alfvén equation

A fluid topological quantity : the **kinematic helicity**.

The equations:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\, \mathbf{v} = -\frac{1}{\rho}\boldsymbol{\nabla} p$$

For isentropic fluids p is afunction of only the density ρ: $-\frac{1}{\rho}\boldsymbol{\nabla} p = -\boldsymbol{\nabla}\frac{dV}{d\rho}$ and the

Hamiltonian and the equations of motions are

$$H = \int dr \left(\frac{1}{2} \rho \mathbf{v}^2 + V(\rho) \right)$$

$$\dot{\rho} = \{H, \rho\}$$

$$\dot{\mathbf{v}} = \{H, \mathbf{v}\}$$

if the nonvanishing brakets are taken to be

$$\left\{ v^i(\mathbf{r}), \rho(\mathbf{r}') \right\} = \partial_i \delta(\mathbf{r} - \mathbf{r}')$$

$$\left\{ v^i(\mathbf{r}), v^j(\mathbf{r}') \right\} = -\frac{\omega_{ij}(\mathbf{r})}{\rho(\mathbf{r})} \delta(\mathbf{r} - \mathbf{r}')$$

We want to construct the **canonical** 1-**form** that leads to this symplectic structure. If the velocity is *irrotational*,

$$\mathbf{v} = \boldsymbol{\nabla} \theta$$

and it is sufficient to postulate

$$\{\theta(\mathbf{r}), \rho(\mathbf{r}')\} = \delta(\mathbf{r} - \mathbf{r}')$$

and the Lagrangian is

$$L_{irrotational} = \int dr \theta \dot{\rho} - H$$

If the velocity is rotational (there is $\boldsymbol{\omega} \neq 0$) there is a problem. There exists a quantity which *commutes* with the variables $(\rho, \mathbf{v})$ (a **Casimir**) which makes that the *kernel* of the algebra is nonempty and the algebra is degenerate. This quantity is the kinematic helicity

$$h \equiv \frac{1}{2} \int d^3 r \, \mathbf{v} \cdot \boldsymbol{\omega}$$

In order to neutralize h it is necessary to adopt the *Clebsch* representation of the velocity:

$$\mathbf{v} = \boldsymbol{\nabla} \theta + \alpha \boldsymbol{\nabla} \beta$$

This gives $\boldsymbol{\omega} = \boldsymbol{\nabla} \alpha \times \boldsymbol{\nabla} \beta$, $\mathbf{v} \cdot \boldsymbol{\omega} = \boldsymbol{\nabla} \theta \cdot (\boldsymbol{\nabla} \alpha \times \boldsymbol{\nabla} \beta) = \boldsymbol{\nabla} \cdot (\theta \boldsymbol{\nabla} \alpha \times \boldsymbol{\nabla} \beta)$ and the helicity is a *surface integral*

$$h = \frac{1}{2} \int d\mathbf{S} \cdot \theta \left(\boldsymbol{\nabla} \alpha \times \boldsymbol{\nabla} \beta \right)$$

Equivalence with discrete models

The physical models supporting these equations can be *mapped* on a discrete model of point-like vortices moving in plane with velocity derived from a potential of mutual interaction.

Basic models:

- logarithmic potential
- Bessel function K_0 potential

Ideal fluid in $2D$ space (Euler eq.)

System of interacting finite-mass particles in plane

A system of particles in the plane interacting through a potential. The Hamiltonian is

$$H = \sum_{s=1}^{N} \frac{1}{2} m_s \mathbf{v}_s^2$$

where

$$m_s \mathbf{v}_s = \mathbf{p}_s - e_s \mathbf{A} \left(\mathbf{r}_s | \mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_N \right)$$

the potential at the point $\mathbf{r}_s$

$$\mathbf{A}\left(\mathbf{r}_s | \mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_N\right) \equiv \left(a_s^i\left(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_N\right)\right)_{i=1,2}$$

$$a_s^i\left(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_N\right) = \frac{1}{2\pi\kappa}\varepsilon^{ij}\sum_{q\neq s}^{N} e_q \frac{r_s^j - r_q^j}{\left|\mathbf{r}_s - \mathbf{r}_q\right|^2}$$

The vector potential $\mathbf{A}_s$ is the *curl* of the Green function of the Laplacian

$$\frac{1}{2\pi}\varepsilon^{ij}\frac{r^j}{r^2} = \varepsilon^{ij}\partial_j \frac{1}{2\pi}\ln r$$

The Green function of the $2D$ Laplacian

$$\boldsymbol{\nabla}^2 \frac{1}{2\pi}\ln r = \delta^2\left(r\right)$$

Embedding this into a field theory

- separate the matter degrees of freedom
- Consider the interaction potential as a *free* field = new degree of freedom of the system, and find the Lagrangian which can give this potential.
- Couple the matter and the field by an interaction term in the Lagrangian

According to Jackiw and Pi the field theory Lagrangian

$$L = L_{matter} + L_{CS} + L_{interaction}$$

with

$$L_{matter} = \sum_{s=1}^{N} \frac{1}{2} m_s \mathbf{v}_s^2$$

The static self-dual solutions

There is the identity

$$|\mathbf{D}\Psi|^2 = |(D_1 \pm iD_2)\Psi|^2 \pm m\boldsymbol{\nabla} \times \mathbf{j} \pm eB\rho$$

Then the *energy density* is

$$H = \frac{1}{2m} |(D_1 \pm iD_2)\Psi|^2 \pm \frac{1}{2}\boldsymbol{\nabla} \times \mathbf{j} - \left(\frac{g}{2} \pm \frac{e^2}{2m\kappa}\right)\rho^2$$

Taking the particular relation

$$g = \mp \frac{e^2}{m\kappa}$$

and considering that the space integral of $\boldsymbol{\nabla} \times \mathbf{j}$ vanishes,

$$H = \frac{1}{2m} \int d^2r \ |(D_1 \pm iD_2)\Psi|^2$$

This is non-negative and attains its minimum, zero, when Ψ satisfies

$$D_1\Psi \pm iD_2\Psi = 0$$

or

$$\mathbf{D}\Psi = i\mathbf{D}\times\Psi$$

which is the self-duality condition.

Then decomposing again Ψ in the phase and amplitude parts,

$$\mathbf{A} = \boldsymbol{\nabla}\omega \pm \frac{1}{2e}\boldsymbol{\nabla}\times \ln\rho$$

Introducing in the relation derived from Chern-Simons

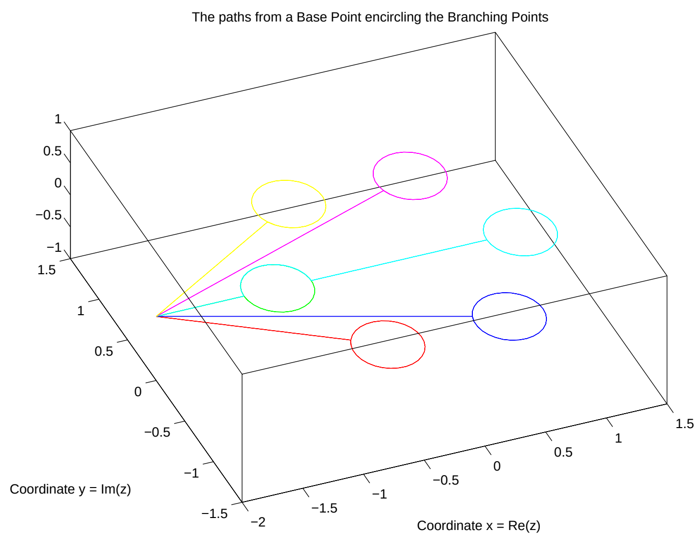
$$B = \boldsymbol{\nabla} \times \mathbf{A} = -\frac{e}{\kappa}\rho$$

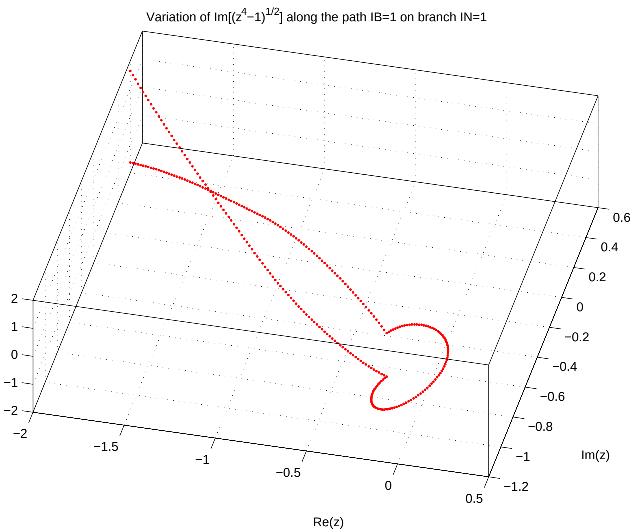
we have

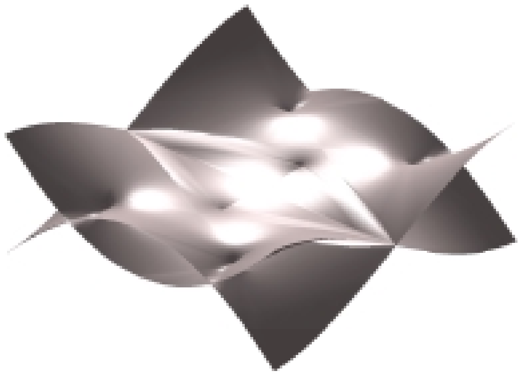
$$\boldsymbol{\nabla}^2 \ln\rho = \pm 2\frac{e^2}{\kappa}\rho$$

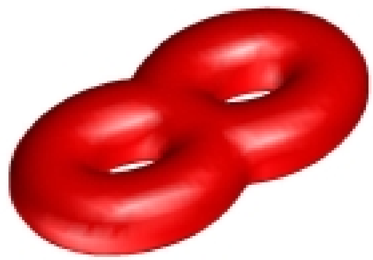
which is the Liouville equation.

Six branching points: a genus 2 Riemann curve









The Lax operators for the *sinh*-Poisson eq.

The first equation can be written

$$\begin{pmatrix} \frac{\lambda^2}{16p}A - p & -i\frac{\partial}{\partial x} - \frac{1}{4}\left(\frac{\partial u}{\partial y} + i\frac{\partial u}{\partial x}\right) \\ i\frac{\partial}{\partial x} - \frac{1}{4}\left(\frac{\partial u}{\partial y} + i\frac{\partial u}{\partial x}\right) & \frac{\lambda^2}{16p}B - p \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0$$

and the second equation is

$$\begin{pmatrix} -\frac{\lambda^2}{16p}A - p & -\frac{\partial}{\partial y} - \frac{1}{4}\left(\frac{\partial u}{\partial y} + i\frac{\partial u}{\partial x}\right) \\ \frac{\partial}{\partial y} - \frac{1}{4}\left(\frac{\partial u}{\partial y} + i\frac{\partial u}{\partial x}\right) & -\frac{\lambda^2}{16p}B - p \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0$$

The compatibility condition is

$$\begin{aligned} & - \left[\frac{i}{4}\Delta u - i\frac{\lambda^2}{8}(A-B)\right]^2 \\ & + \left(\frac{\lambda^2}{16p}\right)^2 \left(\frac{\partial u}{\partial y} + i\frac{\partial u}{\partial x}\right)^2 \left[\left(A + \frac{dA}{du}\right)\left(B - \frac{dB}{du}\right)\right] \\ = \; & 0 \end{aligned}$$

The exact solution of the *sinh*-Poisson equation in terms of Riemann *theta* function

$$\phi(x,y) = 2\ln\left(\frac{\Theta\left(\mathbf{l}+\frac{1}{2}\mathbf{1}\right)}{\Theta(\mathbf{l})}\right)$$

where $\mathbf{l} = \mathbf{k}_x x + \mathbf{k}_y y + \mathbf{l}_0$, $\mathbf{l}_0$ is a vector of constants, initial phases, and

$$k_{x,j} \equiv (-1)^N \frac{C_{jN}}{8\sqrt{Q}} + 2C_{j1}$$

$$k_{y,j} \equiv i(-1)^N \frac{C_{jN}}{8\sqrt{Q}} - 2iC_{j1}$$

The physical content of the problem is in the square matrix C whose dimension is half the number of eigenvalues in the main spectrum. The matrix C is obtained from integrals of a basis of differential one-forms defined on the hyperelliptic Riemann surface along the basis of closed paths (cycles). These integrals can be converted into integrals along closed paths on the plane of the spectral variable, around cuts or crossing these cuts. The geometrical aspect of this conversion is numerically complicated due to the jumps of the phases of the complex integrand at crossing the cuts. However, the symmetries of the main spectrum allows us to use general forms of the matrix

$$\begin{aligned} C_{ij} &= 16^{N-2j+1} C^*_{i,N-j+1}, \; j \leq N/2 \\ C_{ij} &= C^*_{N-i+1,j} \end{aligned}$$

A particular choice of the entries of C (which obeys the symmetries) corresponds, physically, to a particular form of the boundary condition assumed for ϕ, on a linear section of the periodic domain.

Limiting process for obtaining the solution of the Liouville equation

The solution of the Liouville equation can be obtained from that for the *sinh*-Poisson equation in a certain limit. This limit has been translated into a particular distribution of the functions of the auxiliary spectrum (Tracy *et al.*). For any (x, y) there are three classes according to the positions relative to the inversion circle, which is given by $|E|^2 = \lambda^4/256$, E being the spectral variable. First, one notes that the discrete points of the *main spectrum* are situated in certain positions around this circle: (1) there are N inversion pairs, (E_j, E_{N+j}), in the set E_1, E_2, ..., E_{2N} with: $E_{N+j} = \lambda^4/\left(256 E^*_j\right)$ for $j = 1, N$. (2) there are M pairs E_{2N+1}, ..., E_{2N+2M} such that $E_{2N+k} = \lambda^2 \alpha_k/16$, $E_{2N+M+k} = \left(\lambda^2/16\right)/\alpha^*_k$, with α_k independent of λ. From each pair of the eigenvalues

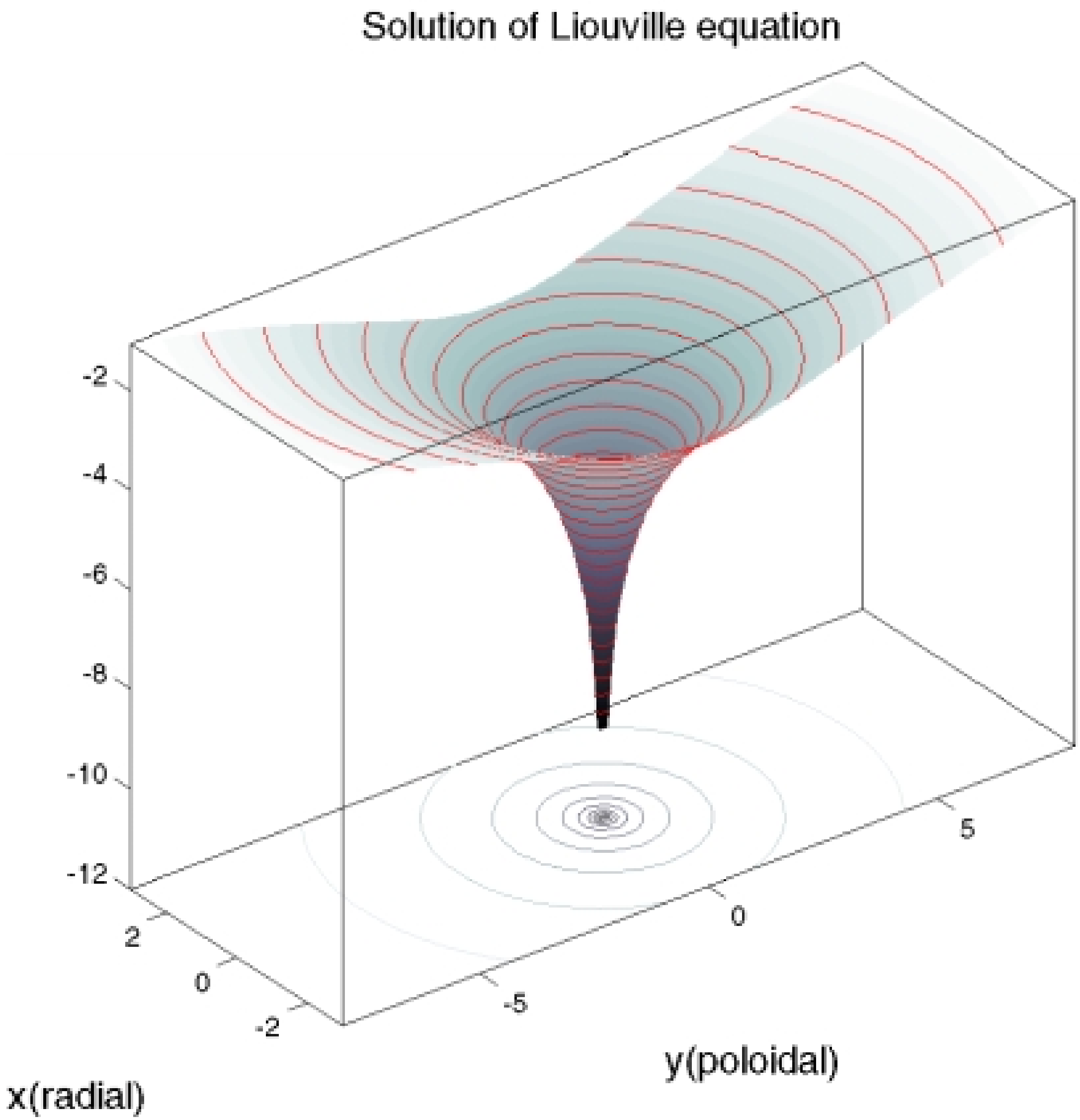


Figure 1: The streamfunction ϕ solution of the Liouville equation.

of the set (2) (situated near the inversion circle) there are M auxiliary functions, which scale as λ^2. The rest of the points of the auxiliary spectrum are devided into two classes. The first contains the auxiliary functions which are outside the inversion circle and are independent of λ. The second are defined inside the inversion circle and are scaled as λ^4. In this way, $\lambda^2 \exp(-\phi)$ is independent of λ. This classifications cannot be directly employed, but they suggest a particular choice for the points of the main spectrum. These studies, mainly numerical, are still in progress.

A conclusion can be drawn at this stage: the solution exhibits a localised perturbation on the poloidal direction, while the helical symmetry is still that given of the q of the surface. This solution is solitonic and therefore is stable (we still have to clarify the effect of the limiting procedure on the set of invariants when passing from *sinh*-Poisson to Liouville). The fact that this kind of solution is an attractor comes from the general property of the plasma state: the self-duality (leading to the Liouville equation) corresponds to the extremum of the action of the system of discrete elements.

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