Coherent flows of ideal $2D$ fluids and Constant Mean Curvature surfaces

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When can-we say that a system is highly organized?
We only have a descriptive definition of the coherency, but one concept seems to be the most un-equivocal indicator of organization:

the state of the system is a topological mapping.

An example: the nonlinear $O(3)$ model (plane nematic liquid crystals)

In every point of the plane $x^\mu = (x, y)$ there is a vector $\phi = \left(\phi^1, \phi^2, \phi^3\right)$ of length 1

$$\phi \cdot \phi - 1 = 0$$

The tip of the vector is a point on a sphere S^2 (called space of internal symmetry).

Taking the condition that ϕ *is the same on a circle of very large radius in the plane*, the infinite distant "boundary" can be replaced by a point: the plane is compactified to a sphere S^2. The field ϕ

represents a map:

(the plane $\mathbf{R}^2$ compactified)$\rightarrow$(the space of internal symmetry)

$$S^2 \xrightarrow{\phi} S^2$$

The field has a topological nature. Any realization of the field ϕ is a map which cover the target sphere (internal space) with the basis sphere (the compactified $\mathbf{R}^2$ space) once, twice, ..., **an integer number of times**.

For such systems, there is a functional (action) that can be reduced to the form

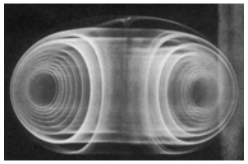
$$\mathcal{S} = \int d^2r \left[(\cdots)^2 + (\cdots)^2 \right] + n \cdot k$$

and the extremum is clearly the vanishing of the squared terms. The action is bounded from below by the topological term, it is an absolute minimum. These states are called *Self-Dual*.

Self-Duality : a differential form in a fiber space is equal to its Hodge dual. $F = *F$

A kind of flux is equal to another kind of flux. An example: Faraday law (the time variation of the magnetic flux through a surface is equal to $(-)$ the integral of the electric field along the boundary curve).

Everything in the world that shows coherent organization
is derived from a structure with the property of Self-Duality.



Quasi-coherent structures are observed in $2D$ fluids (in oceans and in laboratory experiments)

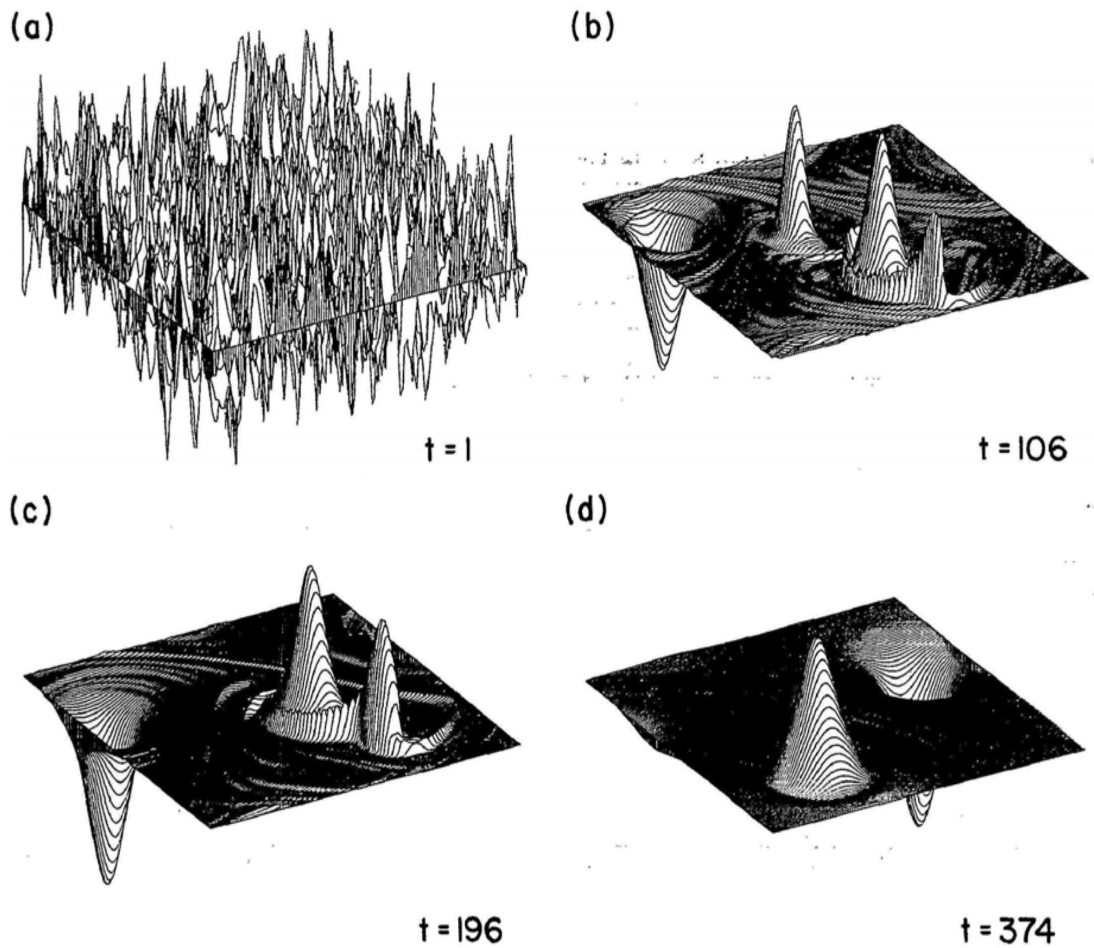
Is water related to the Self-Duality?

Yes, it is. We just have to change the perspective.

All about water

Season 1: $2D$

Coherent structures in fluids and plasmas (numerical)



Numerical simulations of the Euler equation.

D. Montgomery, W.H. Matthaeus, D. Martinez, S. Oughton, Phys. Fluids A4 (1992) 3.

Compare the two approaches

Conservation eqs.

$$\begin{aligned} \frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) &= 0 \\ mn\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\mathbf{v} &= -\nabla p - \nabla\cdot\pi + \mathbf{F} \\ \frac{3}{2}n\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)T &= -\nabla\cdot\mathbf{q} - p\left(\nabla\cdot\mathbf{v}\right) - \pi : \nabla\mathbf{v} + Q \end{aligned}$$

Valid for : coffee, ocean, sun.

Lagrangian

$$\begin{aligned} \mathcal{L}\left(x^{\mu}, \phi^{\nu}, \partial_{\rho}\phi^{\nu}\right) &\rightarrow \mathcal{S} = \int dx\,dt\,\mathcal{L} \\ \frac{\partial}{\partial x^{\mu}}\frac{\delta\mathcal{L}}{\delta\left(\frac{\partial\phi^{\nu}}{\partial x^{\mu}}\right)} - \frac{\delta\mathcal{L}}{\delta\phi^{\nu}} &= 0 \end{aligned}$$

Valid for : a single system.

Just give the initial state.

Lagrangians are preferable. But, how to find a Lagrangian ? See Phys.Rev.

The discrete models

We remember that there is a discrete model for the $2D$ ideal fluid. It carries a fundamental reformulation: *matter, field, interaction*.

An equivalent discrete model for the Euler equation in $2D$

$$\frac{dr_k^i}{dt} = \varepsilon^{ij} \frac{\partial}{\partial r_k^j} \sum_{n=1, n\neq k}^{N} \omega_n G\left(\mathbf{r}_k - \mathbf{r}_n\right) \ , \ i,j = 1,2 \ , \ k = 1,N \tag{1}$$

the Green function of the Laplacian

$$G\left(\mathbf{r}, \mathbf{r}'\right) \approx -\frac{1}{2\pi} \ln\left(\frac{|\mathbf{r} - \mathbf{r}'|}{L}\right) \tag{2}$$

Few incomfortable observations on the system of point-like vortices.

The third axis is implicitely present.

The vorticity is a vector, implicitely involves the z direction.

Anti-vortices are necessary

In the equations of the discrete set of point-like vortices in plane there is

NO intrinsic representation of the fact that they represent *vortices*. The information that the equations refer to the motion of point-like *vortices* (and NOT charges) must be added, as a supplementary theoretical information. It is NOT embedded in the set of equation, it is simply added: we *know* that the equations refer to point-like vortices.

This justifies the extension of the model: we need to implement somehow the information that the elementary objects are *vortices*.

Note: The point-like vortices are similar to spins:

- Just one magnitude, two projections
- Not two in the same state (here: position)

But: *no flip*, no virtual states, etc. Classical spinors: representation of the Lorentz group.

Then we need to introduce another set of vortices. They will have opposite spin, they come from future and propagate backward in time, as if they had negative energy. They are antiparticles.

The ensemble of the point-like vortices : forward and backward in time are grouped into a single theoretical object, a Weyl (mixed) spinor

$$x^{\alpha\dot{\beta}}$$

and this is equivalent with the matrices of $sl\left(2,\mathbf{C}\right)$.

This is the explanation of the introduction of the non-Abelian model.

The system moves along z with an arbitrary speed.

Is better to take it non-zero. There is a momentum $\mathbf{p}$ along z.

Now: Back to continuum within the point-like vortices model:

- the Lorentz motion $\rightarrow$ Chern Simons term
- density of point-like vortices $\rightarrow$ field Ψ
- vortex nature of the discrete objects $\rightarrow$ all fields are matrices

There are two physical quantities:

- spin (vorticity)
- chirality $\sigma \cdot \mathbf{p}/|\mathbf{p}|$: what?

The water Lagrangian

2D Euler fluid: Non-Abelian $SU(2)$, Chern-Simons, 4^{th} order

$$\begin{aligned}\mathcal{L} &= -\varepsilon^{\mu\nu\rho} Tr\left(\partial_\mu A_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho\right) + \\ & iTr\left(\Psi^\dagger D_0 \Psi\right) - \frac{1}{2} Tr\left(\left(D_i\Psi\right)^\dagger D_i \Psi\right) + \frac{1}{4} Tr\left(\left[\Psi^\dagger, \Psi\right]\right)^2\end{aligned} \tag{3}$$

where

$$D_\mu \Psi = \partial_\mu \Psi + [A_\mu, \Psi]$$

The equations of motion are

$$iD_0\Psi = -\frac{1}{2}\mathbf{D}^2\Psi - \frac{1}{2}\left[\left[\Psi, \Psi^\dagger\right], \Psi\right] \tag{4}$$

$$F_{\mu\nu} = -\frac{i}{2}\varepsilon_{\mu\nu\rho} J^\rho \tag{5}$$

The Hamiltonian density is

$$\mathcal{H} = \frac{1}{2} Tr \left((D_i \Psi)^\dagger (D_i \Psi) \right) - \frac{1}{4} Tr \left(\left[\Psi^\dagger, \Psi \right]^2 \right) \tag{6}$$

Using the notation $D_\pm \equiv D_1 \pm i D_2$

$$\begin{aligned} Tr \left((D_i \Psi)^\dagger (D_i \Psi) \right) &= Tr \left((D_- \Psi)^\dagger (D_- \Psi) \right) + \\ & \frac{1}{2} Tr \left(\Psi^\dagger \left[\left[\Psi, \Psi^\dagger \right], \Psi \right] \right) \end{aligned}$$

Then the energy density is

$$\mathcal{H} = \frac{1}{2} Tr \left((D_- \Psi)^\dagger (D_- \Psi) \right) \geq 0 \tag{7}$$

and the Bogomol'nyi inequality is saturated at *self-duality*

$$D_- \Psi = 0 \tag{8}$$

$$\partial_+ A_- - \partial_- A_+ + [A_+, A_-] = \left[\Psi, \Psi^\dagger \right] \tag{9}$$

The *static* solutions of the *self-duality* equations

The algebraic *ansatz*:

$$\begin{aligned} [E_+, E_-] &= H \\ [H, E_\pm] &= \pm 2 E_\pm \\ \mathrm{tr}\,(E_+ E_-) &= 1 \\ \mathrm{tr}\left(H^2\right) &= 2 \end{aligned} \tag{10}$$

taking

$$\psi = \psi_1 E_+ + \psi_2 E_- \tag{11}$$

and

$$\begin{aligned} A_x &= \frac{1}{2}\left(a - a^*\right) H \\ A_y &= \frac{1}{2i}\left(a + a^*\right) H \end{aligned} \tag{12}$$

The gauge field tensor

$$F_{+-} = \left(-\partial_+ a^* - \partial_- a\right) H$$

and from the first self-duality equation

$$\frac{\partial \psi_1}{\partial x} - i\frac{\partial \psi_1}{\partial y} - 2\psi_1 a^* = 0 \tag{13}$$

$$\frac{\partial \psi_2}{\partial x} - i\frac{\partial \psi_2}{\partial y} + 2\psi_2 a^* = 0 \tag{14}$$

and their complex conjugate from $\left(D_- \psi\right)^\dagger = 0$.

Notation : $\rho_1 \equiv \left|\psi_1\right|^2$, $\rho_2 \equiv \left|\psi_2\right|^2$

$$\Delta \ln(\rho_1 \rho_2) = 0 \tag{15}$$

$$\Delta \ln \rho_1 + 2(\rho_1 - \rho_1^{-1}) = 0 \tag{16}$$

We then have

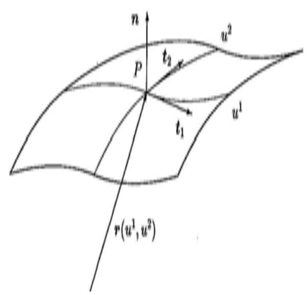
$$\Delta\psi + \gamma \sinh(\beta\psi) = 0. \tag{17}$$

The Field Theoretical model for the Euler fluid works. Now we dispose of a new framework besides $(\psi, \mathbf{v}, \omega)$

What to do next:

- try to understand things that we could not understand in $(\psi, \mathbf{v}, \omega)$
- look for applications

Strange : the Constant Mean Curvature surfaces verify the same equation, *sinh*-Poisson



The points of the surface $\mathcal{F}$ are described by vectors F with components $\mathbf{F} \equiv (F_1, F_2, F_3)$, $F_i(x,y) = F_i(z,\overline{z})$ where $z = x + iy$. The metric Ω is

$$\Omega = 4\rho(x,y)\left(dx^2 + dy^2\right) = 4\exp(\psi)\, dz d\overline{z}$$

The vectors $\frac{\partial \mathbf{F}}{\partial z}$ and $\frac{\partial \mathbf{F}}{\partial \overline{z}}$ are tangents to the surface. With these vectors one can define the *normal* to the surface

$$\mathbf{N} = \frac{\frac{\partial \mathbf{F}}{\partial z} \times \frac{\partial \mathbf{F}}{\partial \overline{z}}}{\left|\frac{\partial \mathbf{F}}{\partial z} \times \frac{\partial \mathbf{F}}{\partial \overline{z}}\right|} \quad , \quad \frac{\partial \mathbf{F}}{\partial z} \cdot \mathbf{N} = 0 \quad , \quad \frac{\partial \mathbf{F}}{\partial \overline{z}} \cdot \mathbf{N} = 0$$

One defines a triplet of vectors

$$\sigma \equiv \begin{pmatrix} \frac{\partial \mathbf{F}}{\partial z} \\ \frac{\partial \mathbf{F}}{\partial \overline{z}} \\ \mathbf{N} \end{pmatrix}$$

and the displacement along the independent directions given by z and $\overline{z}$ on the surface of the trihedral of vectors σ induces the following modifications

$$\begin{aligned} \frac{\partial \sigma}{\partial z} &= \mathcal{U}\sigma \\ \frac{\partial \sigma}{\partial \overline{z}} &= \mathcal{V}\sigma \end{aligned}$$

where

$$\mathcal{U} = \begin{pmatrix} \frac{\partial \psi}{\partial z} & 0 & Q \\ 0 & 0 & B \\ -\frac{\exp(-\psi)}{2} B & -\frac{\exp(-\psi)}{2} Q & 0 \end{pmatrix}$$

$$\mathcal{V} = \begin{pmatrix} 0 & 0 & B \\ 0 & \frac{\partial \psi}{\partial \overline{z}} & \overline{Q} \\ -\frac{\exp(-\psi)}{2} \overline{Q} & -\frac{\exp(-\psi)}{2} B & 0 \end{pmatrix}$$

The new variables are defined

$$Q = \frac{\partial^2 \mathbf{F}}{\partial z \partial z} \cdot \mathbf{N} \quad B = \frac{\partial^2 \mathbf{F}}{\partial z \partial \overline{z}} \cdot \mathbf{N}$$

The first quadratic form of the surface is

$$I \equiv d\mathbf{F} \cdot d\mathbf{F} = [4 \exp(u)]\, dx^2 + [4 \exp(u)]\, dy^2$$

The second differential form of the surface is

$$II \equiv -d\mathbf{F}\cdot d\mathbf{N} = Q dz dz + 2B dz d\overline{z} + \overline{Q} d\overline{z} d\overline{z}$$

The *principal curvatures* κ_1 and κ_2 are the eigenvalues of the operator II relative to the operator I.

With the principal curvatures one can define:

The *mean curvature*:

$$H \equiv \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{1}{2}\mathrm{tr}\left[(II)(I)^{-1}\right] = \frac{1}{2}B\exp(-u)$$

The *Gaussian curvature*:

$$K \equiv \kappa_1\kappa_2 = \det\left[(II)(I)^{-1}\right] = \frac{1}{4}\left(B^2 - Q\overline{Q}\right)\exp(-2u)$$

The equation of compatibility **Gauss Petersen Codazzi** after displacement of the triplet σ is

$$\frac{\partial^2 \psi}{\partial z \partial \overline{z}} + \frac{1}{2} B^2 \exp(-\psi) - \frac{1}{2} Q \overline{Q} \exp(-\psi) = 0$$

The *constant mean curvature* surfaces are defined as H =const. Taking $H = \frac{1}{2}$,$B = \exp(\psi)$.

$$\frac{\partial^2 \psi}{\partial z \partial \overline{z}} + \frac{1}{2} \exp(\psi) - \frac{1}{2} Q \overline{Q} \exp(-\psi) = 0$$

and the module of the holomorphic function Q can be taken 1. Then

$$\Delta \psi + 4 \sinh(\psi) = 0$$

Every flow in asymptotic relaxation of the Euler fluid corresponds to a Constant Mean Curvature surface, and reciprocal.

Does anyone has an idea what to do with this conclusion ?

Now it is the time for Field Theory

The conformal metric as

$$ds^2 = 4\exp(\psi)\left(dx^2 + dy^2\right)$$

and obtains

$$\left(\kappa_1 - \kappa_2\right)^2 = Q\overline{Q}\exp(-2\psi)$$

$$\Delta\psi + 4\sinh(\psi) = 0$$

we obtain

$$\begin{aligned} \kappa_1 - \kappa_2 &= \exp(-\psi) \\ \kappa_1 + \kappa_2 &= 2H = 1 \end{aligned}$$

then

$$\begin{aligned}\kappa_1 &= \frac{1+\exp(-\psi)}{2} \\ \kappa_2 &= \frac{1-\exp(-\psi)}{2}\end{aligned}$$

the identification

$$\begin{aligned}\rho_2 &\rightarrow \kappa_1-\kappa_2 \\ \rho_1 &\rightarrow \frac{(\kappa_1+\kappa_2)^2}{\kappa_1-\kappa_2}=\frac{1}{\kappa_1-\kappa_2} \quad \text{at SD}\end{aligned}$$

and

$$\omega = -\frac{2}{\kappa}(\rho_1-\rho_2) \quad \text{at SD}$$

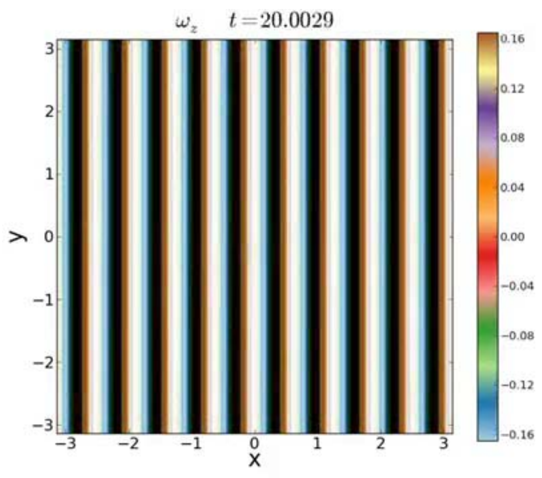
Fluid	$\leftrightarrow$	Delaunay surfaces
asymptotic flow sinh-Poisson		CMC sinh-Poisson
extremum of entropy at constant E_{total} and ω_{total}		minimum area for constant volume
ψ as label of the streamlines		$\rho = \exp(\psi)$ length in the tangent plane
streamline (closed)		$v \in [0, 2\pi)$ circle of invariance

Constant Mean Curvature Surfaces

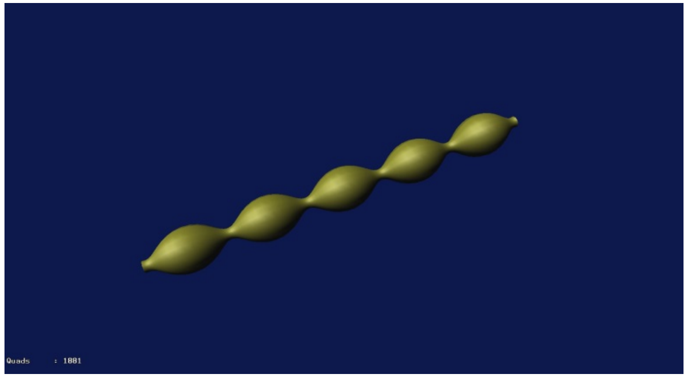
The only CMC surface which is compact and embedded is the sphere.
The others need to extend to infinity.
One example is the *Delaunay unduloid*.
[Of course there are also immersed surfaces – with self-intersections]

A possible correspondence :

Kolmogorov flow



unduloid



Realizability of the stationary $2D$ flows of the Euler equation derived from the connection with the Constant Mean Curvature surfaces

Solutions $\psi(x,y)$ of the sinh-Poisson eq. $\Delta\psi + \sinh\psi = 0 \rightarrow$

$\rightarrow$ CMC surfaces $\mathbf{F}$ corresponding to the function $\psi(x,y) \rightarrow$

$$\rightarrow \text{surfaces } \mathbf{F} \text{ are } \begin{cases} \text{embedded (sphere)} \\ \text{immersed} = \text{self-intersected} \\ \text{immersed periodic, with edges} \end{cases}$$

$\rightarrow$flows are stable only for periodic or doubly periodic surfaces

The single positive vortex in a region that covers all the plane is NOT a stable solution. The solution, even periodic in plane, consisting of only *positive* vorticity cannot be stable.

Only solutions that are periodic and consist of vortices with alternate signs are stable.

Stability of the cylinder and of the unduloids

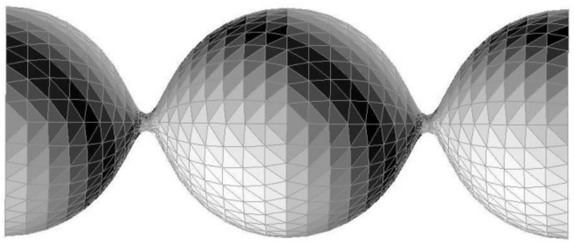


Figure 6: Delaunay surfaces: An embedded unduloid on the left, and on the right a nodoid cut open to display its self-intersections.

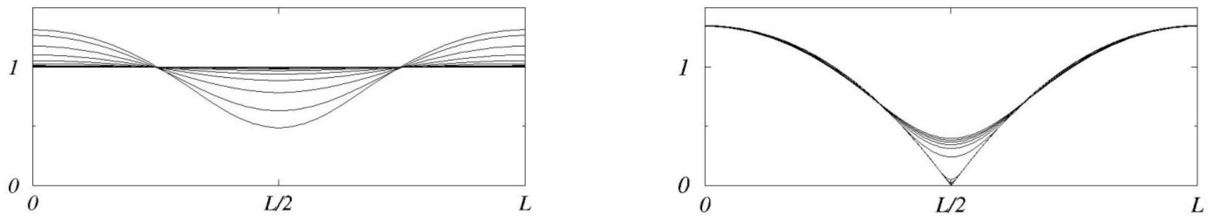
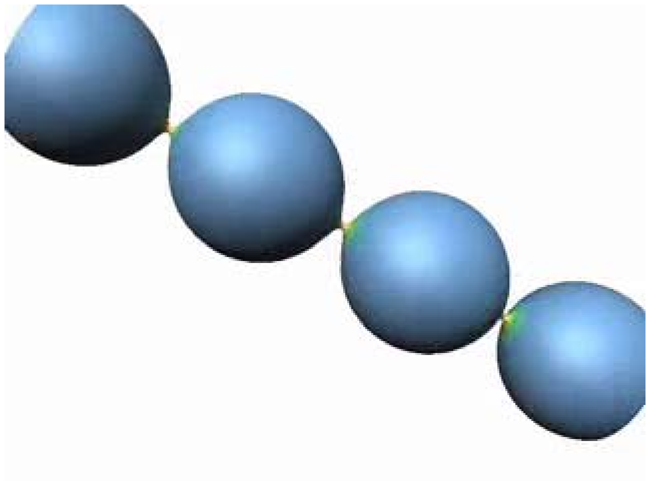


Figure 6: Evolution of the perturbed unduloid starting with initial data (3.24). In the first case, $\epsilon = 0.05$ and the perturbation is 'towards' the cylinder; we observe relaxation to it in infinite time. In the second case, $\epsilon = -0.05$ the perturbation is 'away' from the cylinder and we observe finite time pinch-off. This particular example is for the unduloid with period $L \approx 5.2 < 2\pi$ however the same qualitative results were seen for all perturbed unduloids.

Limiting case

Neck size (its radius) goes to zero, the *unduloid* becomes a chain of tangent spheres



Merging of small scale structures: random encounters or effective interaction ?

Large scale structures are created by processes of encounters and merging of small scale structures. The Field Theory can account for the interaction between vortices, close to SD:

- geodesic flow of vortices (Manton): point-like vortices rotate one around the other
- close to Self-Duality the energy is lowered by vortices approaching (Regge, for ANO)

The FT equations are Topology-preserving motions which drive the system closer to Self-Duality.

The reconnections change the topology and reset the data for the FT evolutions.

Vortex mergings and surface smoothing

- The connection between

  1.capillarity-induced surface smoothing

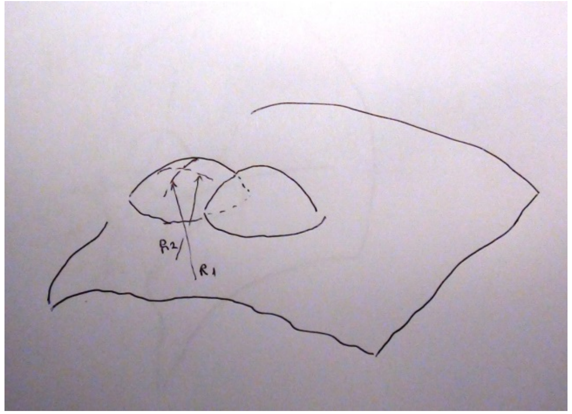
  2.vortex mergings in relaxation

- The smoothing of the surface by capillarity is *mapped* through the complicated map: fluid $\leftrightarrow$ surface to the vortex merging. Then one should not look for an *interaction* between vortices.
- coalescence of saddle cuasi-umbilic points on the surface corresponds to merging of negative vortices; they may exist in the initial state as perturbation of the *neck*, with main variation along the circle transversal to the symmetry axis of the *perturbed unduloid*, evolving towards CMC state
- coalescence of positive protuberances having the character of

cuasi-umbilic points of the surface corresponds to merging of positive vortices.

- coalescence of saddle points with positive protuberances (locally spherical protuberances) does not take place. Correspondingly the merging of a positive and of a negative vortices is not seen in fluids. There may be *annihilation* however? Indeed annihilation exists for the Abelian - Higgs vortices.

Perturbations of the surface in close proximity of the Constant Mean Curvature and the vortex mergings in fluids



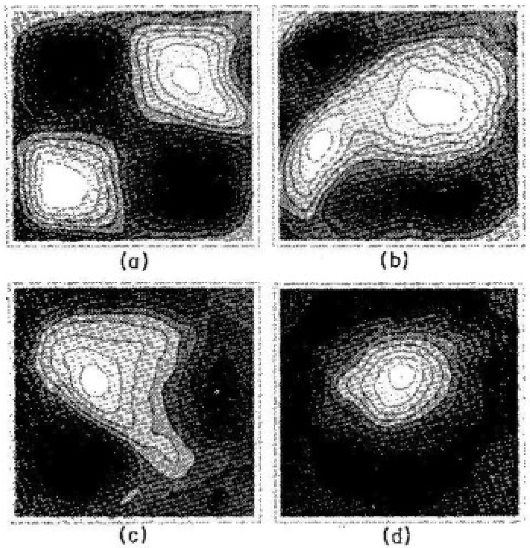
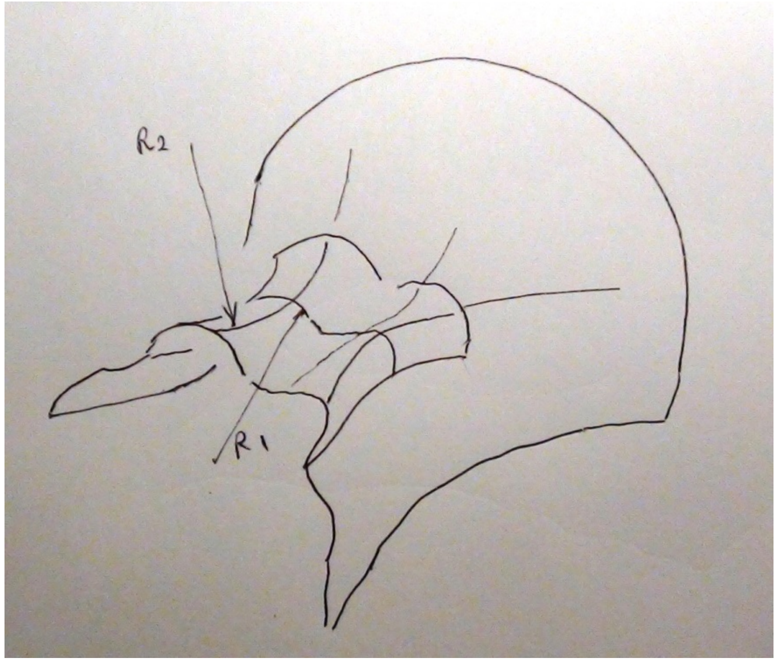


FIG. 1. Evolution of the vorticity field for a system of four counterrotating vortices, enclosed in a square box: (a) $t=0$ s, (b) $t=2.3$ s, (c) $t=8.2$ s, and (d) $t=28$ s.

Tabeling



Still thinking that the elementary point-like vortices are of this world ?
(*i.e.* they are like a stick with an arrow)

Try to produce a *positive* physical vorticity in a point, using exclusively *positive* elementary vortices.
It is impossible, you need *negative* vortices too.

What says the Field Theory in alliance with the Surface Theory:

there is no possibility that in a point of the fluid the vorticity to be calculated on only the base of one kind of vortices (positive or negative): both must be present in every point of the fluid. This is because if in one point we would have $\rho_2 = 0$ *then in that point we would have singular* ρ_1 *equivalently singular vorticity and correspondingly in CMC an umbilic point. There is a theorem about the fact that the CMC surfaces cannot have umbilic points.*

There will never be order in $(3+1)D$: the Chern-Simons term

In the $(2+1)\,D$ Abelian case:

$$\mathcal{L} = \frac{\kappa}{2}\varepsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho$$

$$\mathcal{L} = \frac{\kappa}{2}\frac{\partial \mathbf{A}}{\partial t}\times \mathbf{A} - \kappa A^0 B$$

This is the density of the *helicity* in $3D$ it is: $\mathbf{A}\cdot\mathbf{B}$ or $\mathbf{v}\cdot\omega$. In the $(2+1)\,D$ Non-Abelian, CS term is

$$\mathcal{L} = \kappa\varepsilon^{\mu\nu\rho}\mathrm{tr}\left(\left(\partial_\mu A_\nu\right)A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho\right)$$

It is *first order* in the time derivative: no real dynamics.

Basic property: we cannot write such a term in $(3+1)\,D$: the indices do not match. The CS Lagrangian can be written in any *odd* dimension, for example in $(4+1)\,D$:

$$\varepsilon^{\mu\nu\rho\sigma\lambda} A_\mu \left(\partial_\nu A_\rho\right) \left(\partial_\sigma A_\lambda\right)$$

Without CS there is no Self-Duality. Then there is no coherent structure of the flow.

Conclusions

We have started from fluid models in $2D$, for which discrete models are available.

We have provided a field theoretical formulation of the continuum limit of the discrete models. The evolution of the system toward the extrema of the action is the origin of the self-organization. The extrema are obtained at *self-duality*.

Wide space of investigation:

- flow stability described by CMC surfaces,
- turbulence of unitons
- contour dynamics as section of Riemann surfaces (solutions of FT)

By the way: there is an equivalent discrete model for the plasma in strong magnetic field and for the planetary atmosphere, in $2D$

The equations of motion for the vortex ω_k at (x_k, y_k) under the effect of the others are

$$\begin{aligned} -2\pi\omega_k \frac{dx_k}{dt} &= \frac{\partial W}{\partial y_k} \\ -2\pi\omega_k \frac{dy_k}{dt} &= -\frac{\partial W}{\partial x_k} \end{aligned}$$

where

$$W = \pi \sum_{\substack{i=1 \\ i \neq j}}^{N} \sum_{j=1}^{N} \omega_i \omega_j K_0 \left(m \left| \mathbf{r}_i - \mathbf{r}_j \right| \right)$$

Physical model $\rightarrow$ point-like vortices $\rightarrow$ field theory.

The Lagrangian of $2D$ *plasma in strong magnetic field: Non-Abelian* $SU(2)$, *Chern-Simons*, 6^{th} *order*

- gauge field, with "potential" A^{μ}, ($\mu = 0, 1, 2$ for (t, x, y)) described by the Chern-Simons Lagrangean;
- matter ("Higgs" or "scalar") field ϕ described by the covariant kinematic Lagrangean (*i.e.* covariant derivatives, implementing the minimal coupling of the gauge and matter fields)
- matter-field self-interaction given by a potential $V\left(\phi, \phi^{\dagger}\right)$ with 6^{th} power of ϕ;
- the matter and gauge fields belong to the *adjoint* representation of the algebra $SU(2)$

$$\begin{aligned}\mathcal{L} &= -\kappa\varepsilon^{\mu\nu\rho}\mathrm{tr}\left(\partial_\mu A_\nu A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho\right) \\ &\quad -\mathrm{tr}\left[\left(D^\mu\phi\right)^\dagger\left(D_\mu\phi\right)\right] \\ &\quad -V\left(\phi,\phi^\dagger\right)\end{aligned} \tag{18}$$

Sixth order potential

$$V\left(\phi,\phi^\dagger\right) = \frac{1}{4\kappa^2}\mathrm{tr}\left[\left(\left[\left[\phi,\phi^\dagger\right],\phi\right] - v^2\phi\right)^\dagger\left(\left[\left[\phi,\phi^\dagger\right],\phi\right] - v^2\phi\right)\right]. \tag{19}$$

The Euler Lagrange equations are

$$D_\mu D^\mu \phi = \frac{\partial V}{\partial \phi^\dagger} \tag{20}$$

$$-\kappa\varepsilon^{\nu\mu\rho}F_{\mu\rho} = iJ^\nu \tag{21}$$

The energy can be written as a sum of squares. The *self-duality* eqs.

$$\begin{aligned} D_{-}\phi &= 0 \\ F_{+-} &= \pm\frac{1}{\kappa^2}\left[v^2\phi-\left[\left[\phi,\phi^{\dagger}\right],\phi\right],\phi^{\dagger}\right] \end{aligned} \tag{22}$$

The algebraic *ansatz* : in the Chevalley basis

$$\begin{aligned} \left[E_{+},E_{-}\right] &= H \\ \left[H,E_{\pm}\right] &= \pm 2E_{\pm} \\ \mathrm{tr}\left(E_{+}E_{-}\right) &= 1 \\ \mathrm{tr}\left(H^2\right) &= 2 \end{aligned} \tag{23}$$

The fields

$$\phi=\phi_1 E_{+}+\phi_2 E_{-}$$

$$A_{+}=aH, A_{-}=-a^{*}H$$

Equations for the components of the density of vorticity (here for $'+'$)

$$-\frac{1}{2}\Delta \ln \rho_1 = -\frac{1}{\kappa^2}(\rho_1 - \rho_2)\left[2(\rho_1+\rho_2) - v^2\right] \tag{24}$$

$$-\frac{1}{2}\Delta \ln \rho_2 = \frac{1}{\kappa^2}(\rho_1 - \rho_2)\left[2(\rho_1+\rho_2) - v^2\right] \tag{25}$$

$$\Delta \ln(\rho_1\rho_2) = 0$$

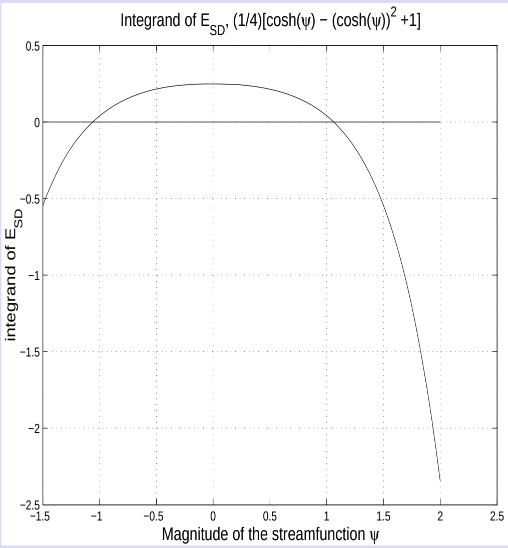
introduce a single variable

$$\rho \equiv \frac{\rho_1}{v^2/4} = \frac{v^2/4}{\rho_2} \tag{26}$$

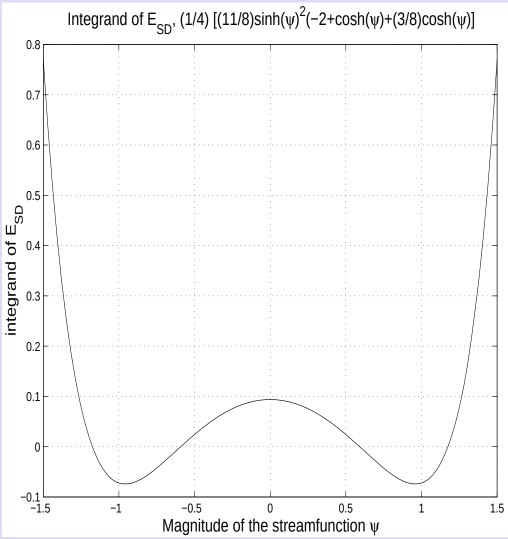
and obtain

$$-\frac{1}{2}\Delta \ln \rho = -\frac{1}{4}\left(\frac{v^2}{\kappa}\right)^2\left(\rho - \frac{1}{\rho}\right)\left[\frac{1}{2}\left(\rho + \frac{1}{\rho}\right) - 1\right] \tag{27}$$

The energy at Self-Duality for two choices of the Bogomolnyi form for the action functional



$$\Delta\psi - \sinh\psi\,(\cosh\psi - 1) = 0$$



$$\Delta\psi + \frac{1}{2}\sinh\psi\,(\cosh\psi - 1) = 0$$

This simplest form of the equation governing the stationary states of the CHM eq.

$$\Delta\psi + \frac{1}{2}\sinh\psi\,(\cosh\psi - 1) = 0$$

The 'mass of the photon' is

$$m = \frac{v^2}{\kappa} = \frac{1}{\rho_s}$$

$$\begin{aligned} \kappa &\equiv c_s \\ v^2 &\equiv \Omega_{ci} \end{aligned}$$

Formulation in terms of a curvature

SD is a geometrico-algebraic property of a fiber space : a differential form is equal to its Hodge dual.

For this model there is no clear geometric structure. However:

Define the two "potential-like" fields

$$\begin{aligned} \mathcal{A}_+ &= A_+ - \lambda\phi \\ \mathcal{A}_- &= A_- + \lambda\phi^\dagger \end{aligned}$$

and calculate the "curvature-like" fields

$$K_\pm \equiv \partial_\pm \mathcal{A}_\mp - \partial_\mp \mathcal{A}_\pm + [\mathcal{A}_\pm, \mathcal{A}_\mp]$$

We then have

$$\begin{aligned}
& \operatorname{tr}\{K_+K_-\} \\
= \; & -2\left[(\partial_+ a^* + \partial_- a) + \lambda^2 (\rho_1 - \rho_2)\right]^2 \\
& -\lambda^2 \left|(\partial_+ \phi_2^* + \partial_- \phi_1) + 2(a\phi_2^* - a^* \phi_1)\right|^2
\end{aligned}$$

or

$$-\operatorname{tr}\{K_+K_-\} \geq 0$$

since it is a sum of squares and the equality with zero is precisely the SD equations.

The self-duality indeed appears as a condition of a flat connection. A non-zero curvature means that the Euler fluid is *not* at stationarity.

The energy close to stationarity (or: self-duality)

We can use the expression of the energy, after applying the Bogomolnyi procedure,

$$E = \frac{1}{2m} \mathrm{tr} \left(\left(D_{-}\phi \right)^{\dagger} \left(D_{-}\phi \right) \right)$$

The energy becomes

$$E = \frac{1}{2m} \left(\rho_1 \left| \frac{1}{2\rho_1} \frac{\partial \rho_1}{\partial x_-} + i \frac{\partial \chi}{\partial x_-} - 2a^* \right|^2 + \rho_2 \left| \frac{1}{2\rho_2} \frac{\partial \rho_2}{\partial x_-} + i \frac{\partial \eta}{\partial x_-} + 2a^* \right|^2 \right)$$

and, if we take

$$\begin{aligned} \rho_1 &= \frac{1}{\rho_2} = \rho = \exp\left(\psi \right) \\ \chi &= -\eta \end{aligned}$$

we have

$$E = \frac{1}{2m}\left[\exp\left(\psi\right) + \exp\left(-\psi\right)\right]\left|\frac{1}{2}\frac{\partial\psi}{\partial x_-} + i\frac{\partial\chi}{\partial x_-} - 2a^*\right|^2$$

This form of the energy shows in what consists the approach to the stationarity and the formation of structure:

1. a constant ψ on the equilines combines its radial variation with that of of the angle χ;

2. the potentials a and a^* become velocities and they contain the derivatives along the equilines of the angle χ.

The expression of the FT current

The formula for the FT current

$$
\begin{aligned}
J^0 &= \left[\Psi^\dagger, \Psi\right] \\
J^i &= -\frac{i}{2}\left(\left[\Psi^\dagger, D_i\Psi\right] - \left[\left(D_i\Psi\right)^\dagger, \Psi\right]\right)
\end{aligned}
$$

We have

$$
\begin{aligned}
J^x &= \frac{1}{2}\left[2i(a-a^*)\left(\rho_1+\rho_2\right) - i\frac{\partial}{\partial x}\left(\rho_1-\rho_2\right)\right] H \\
J^y &= \frac{1}{2}\left[2(a+a^*)\left(\rho_1+\rho_2\right) - i\frac{\partial}{\partial y}\left(\rho_1-\rho_2\right)\right] H \\
J^0 &= \left(\rho_1-\rho_2\right) H
\end{aligned}
$$

or

$$J_+ = \frac{1}{2} i \left(\rho_1 + \rho_2\right) \partial_+ \left[\psi - \left(2i\chi\right)\right] - \frac{1}{2} i \partial_+ \left(\rho_1 - \rho_2\right)$$

$$J_- = -\frac{1}{2} i \left(\rho_1 + \rho_2\right) \partial_- \left[\psi + \left(2i\chi\right)\right] - \frac{1}{2} i \partial_- \left(\rho_1 - \rho_2\right)$$

at SELF-DUALITY we have

$$\omega = -\sinh \psi$$

and it results

$$\begin{aligned} J_+ &= \frac{1}{2} i \left(\rho_1 + \rho_2\right) \partial_+ \left[\psi - \left(2i\chi\right)\right] - \frac{1}{2} i \partial_+ \omega \\ J_- &= -\frac{1}{2} i \left(\rho_1 + \rho_2\right) \partial_- \left[\psi + \left(2i\chi\right)\right] - \frac{1}{2} i \partial_- \omega \end{aligned}$$

Is-there any pinch of vorticity?

The equations of motion of the FT model

The equation resulting from E_+.

$$
\begin{aligned}
& i\frac{\partial \phi_1}{\partial t} - 2ib\phi_1 \qquad (28) \\
= \quad & -\frac{1}{2}\frac{\partial^2 \phi_1}{\partial x^2} + \frac{1}{2}\left[\frac{\partial (a - a^*)}{\partial x}\phi_2 + (a - a^*)\frac{\partial \phi_2}{\partial x}\right] \\
& -\frac{1}{2}\frac{\partial \phi_1}{\partial x}(a - a^*) - \frac{1}{2}(a - a^*)^2 \phi_1 \\
& -\frac{1}{2}\frac{\partial^2 \phi_2}{\partial y^2} + \frac{1}{2i}\left[\frac{\partial (a + a^*)}{\partial y}\phi_2 + (a + a^*)\frac{\partial \phi_2}{\partial y}\right] \\
& -\frac{1}{2}\frac{\partial \phi_2}{\partial y}\left(-\frac{1}{i}\right)(a + a^*) + \frac{1}{2}(a + a^*)^2 \phi_2 \\
& -(\rho_1 - \rho_2)\phi_1
\end{aligned}
$$

The equation resulting from E_-.

$$
\begin{aligned}
& i\frac{\partial \phi_2}{\partial t}+2ib\phi_2 \qquad (29)\\
= \quad & -\frac{1}{2}\frac{\partial^2\phi_2}{\partial x^2}+\frac{1}{2}\left[\frac{\partial\left(a-a^*\right)}{\partial x}\phi_2+\left(a-a^*\right)\frac{\partial\phi_2}{\partial x}\right]\\
& -\frac{1}{2}\frac{\partial\phi_2}{\partial x}\left(a-a^*\right)+\frac{1}{2}\left(a-a^*\right)^2\phi_2\\
& -\frac{1}{2}\frac{\partial^2\phi_2}{\partial y^2}+\frac{1}{2i}\left[\frac{\partial\left(a+a^*\right)}{\partial y}\phi_2+\left(a+a^*\right)\frac{\partial\phi_2}{\partial y}\right]\\
& +\frac{1}{2i}\frac{\partial\phi_2}{\partial y}\left(a+a^*\right)+\frac{1}{2}\left(a+a^*\right)^2\phi_2\\
& +\left(\rho_1-\rho_2\right)\phi_2
\end{aligned}
$$

Compare with Liouville (non-Abelian) case. Where is the dynamics?

Abelian-dominated dynamics

The last Lagrangian

In certain cases the model collapses to an Abelian structure, where (ϕ, A^{μ}) are complex scalar functions

$$\mathcal{L} = (D^{\mu}\phi)^{*}(D_{\mu}\phi) + \frac{1}{4}\kappa\varepsilon^{\mu\nu\rho}A_{\mu}F_{\nu\rho} - V\left(|\phi|^{2}\right)$$

where

$$D_{\mu}\phi = \frac{\partial\phi}{\partial x^{\mu}} + ieA_{\mu}\phi$$

and

$$V\left(|\phi|^{2}\right) = \frac{e^{2}}{\kappa^{2}}|\phi|^{2}\left(|\phi|^{2} - v^{2}\right)^{2}$$

with metric

$$g^{\mu\nu} = (1, -1, -1)$$

The equations of motion

$$\begin{aligned} D^{\mu} D_{\mu} \phi &= -\frac{\partial V}{\partial \phi^{*}} \\ \frac{1}{2} \varepsilon^{\mu\nu\rho} F_{\nu\rho} &= J^{\rho} \end{aligned}$$

where

$$J^{\mu} = ie \left[\phi^{*} \left(D^{\mu} \phi \right) - \left(D^{\mu} \phi \right)^{*} \phi \right]$$

From the second equation of motion $B = -\frac{e}{\kappa} \rho$ one finds

$$A^{0} = \frac{\kappa}{2e^{2}} \frac{B}{|\phi|^{2}} - \frac{1}{e} \frac{\partial}{\partial t} \left[\text{phase of } (\phi) \right]$$

In a field theory one can obtain the energy-momentum tensor by writing the action with the explicit presence of the metric $g^{\mu\nu}$

followed by variation of the action to this metric.

$$\begin{aligned} T_{\mu\nu} &= (D_\mu\phi)^* (D_\nu\phi) + (D_\mu\phi)(D_\nu\phi)^* \\ &\quad -g_{\mu\nu}\left[(D_\lambda\phi)^*(D_\lambda\phi) - V\left(|\phi|^2\right)\right] \end{aligned}$$

The energy is the *time-time* (00) component of this tensor

$$\begin{aligned} E &= \int d^2r \left[(D_0\phi)^*(D_0\phi) + (D_k\phi)^*(D_k\phi) + V\left(|\phi|^2\right)\right] \\ &= \int d^2r \left[\left(\frac{\partial|\phi|}{\partial t}\right)^2 + \frac{\kappa^2}{4e^2}\frac{B}{|\phi|^2} + (D_k\phi)^*(D_k\phi) + V\left(|\phi|^2\right)\right] \end{aligned}$$

The second term imposes that B and $|\phi|^2$ vanish in the same points. Then the magnetic flux lies in a ring around the zeros of $|\phi|^2$.

The *SELF-DUALITY*

The energy is transformed similar to the Bogomolnyi form

$$
\begin{aligned}
E &= \int d^2r \left[\left| \left(D_x \pm i D_y \right) \phi \right|^2 \right. \\
&\quad \left. + \left| \frac{\kappa}{2e} \phi^{-1} B \pm \frac{e^2}{\kappa} \phi^* \left(|\phi|^2 - v^2 \right) \right|^2 + \left(\frac{\partial |\phi|}{\partial t} \right)^2 \right] \\
&\quad \pm e v^2 \Phi + \frac{1}{2} \int_{r=\infty} \mathbf{dl} \cdot \mathbf{J}
\end{aligned}
$$

Restrict to the states

1. static ($\partial/\partial t \equiv 0$);
2. the current goes to zero at infinity such that the last integral is zero.

Then the energy consists of a sum of squared terms plus an additional term that has a *topological* nature, proportional with the total magnetic flux through the area.

Taking to zero the squared terms we get

$$\begin{aligned}(D_x \pm iD_y)\,\phi &= 0 \\ eB &= \mp \frac{m^2}{2}\frac{|\phi|^2}{v^2}\left(1 - \frac{|\phi|^2}{v^2}\right)\end{aligned}$$

The mass parameter is

$$m \equiv 2e^2 \frac{v^2}{\kappa}$$

These are the equations of self-duality and the energy in this case is *bounded from below* by the flux

$$E \geq ev^2\,|\Phi|$$

The equation for the *ring-type* vortex

The first of the two SD equations can be written

$$eA^k = \pm\varepsilon^{kj}\partial_j \ln|\phi| + \partial^k \left[\text{phase of } \phi\right]$$

Replacing the potential in the second SD equation we get

$$\Delta \ln\left(|\phi|^2\right) - m^2 \frac{|\phi|^2}{v^2}\left(\frac{|\phi|^2}{v^2} - 1\right) = 0$$

equation that is valid in points where $|\phi| \neq 0$. For these points there is an additional term, a Dirac δ coming from taking the rotational operator applied on the term containing the phase of ϕ.

$$\Delta\psi = \exp(\psi)\left[\exp(\psi) - 1\right] + 4\pi\sum_{j=1}^{N}\delta\left(\mathbf{x} - \mathbf{x}_j\right)$$

The return of the topological constraint

At infinity ($|\phi| \simeq v$) the covariant derivative term goes to 0

$$D^k \phi \to 0 \text{ at } r \to \infty \qquad\qquad \partial_k \phi + ieA_k \phi \to 0$$

$$\int\limits_{r=\infty} \mathbf{dl} \cdot \nabla \ln (\phi) = i \int d\,(\text{phase of } \phi) = 2\pi i n \tag{30}$$

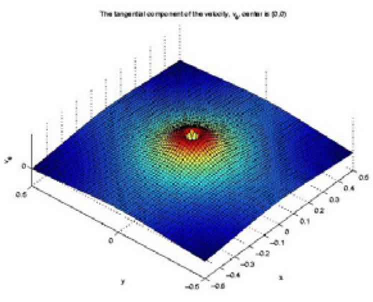
The flux is

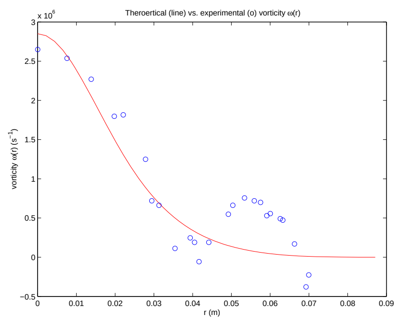
$$\Phi = \int d^2 r \,(\nabla \times \mathbf{A}) = \frac{2\pi}{e} n$$

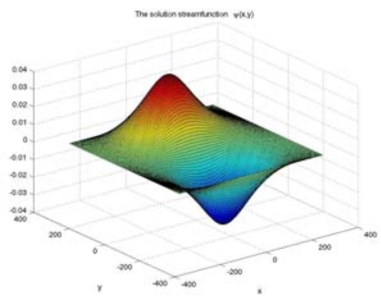
The magnetic flux is discrete, *integer* multiple of a physical quantity. The topological constraint is ensured by a mapping from the circle at infinity into the circle representing the space of the internal phase of the field ϕ in the asymptotic region, $S^1 \to S^1$ classified according to the first homotopy group,

$$\pi_1 \left(S^1\right) = \mathbf{Z}$$

Various applications







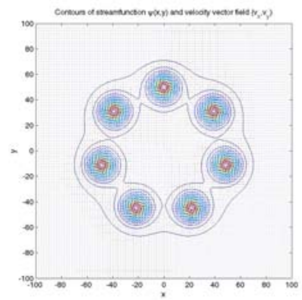
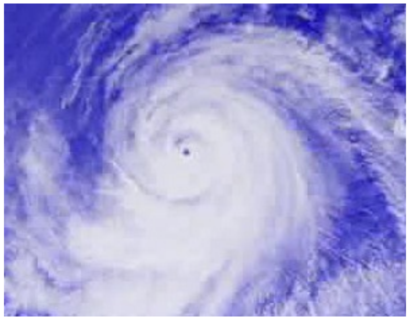
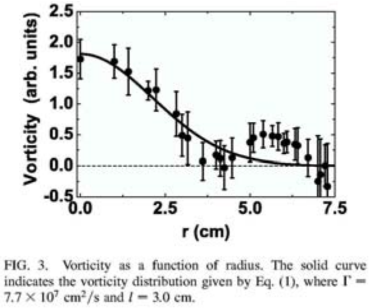
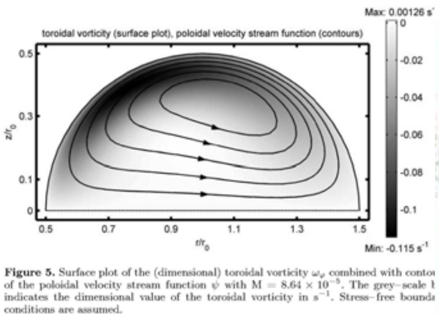
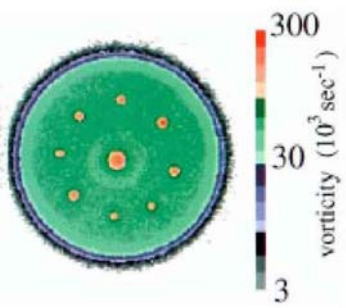


Figure 1: The atmospheric vortex, the plasma vortex, the flows in tokamak,the crystal of vortices in non-neutral plasma.

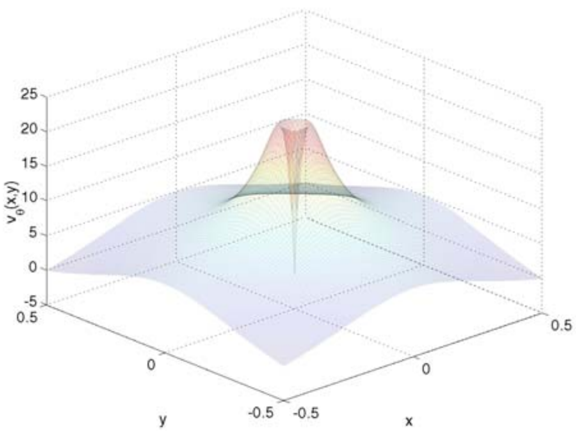








The tropical cyclone



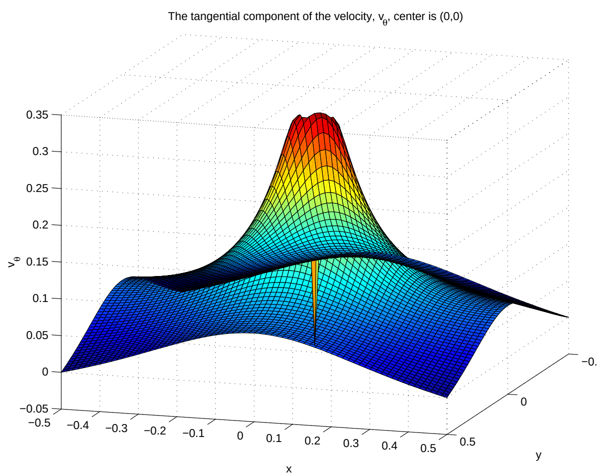
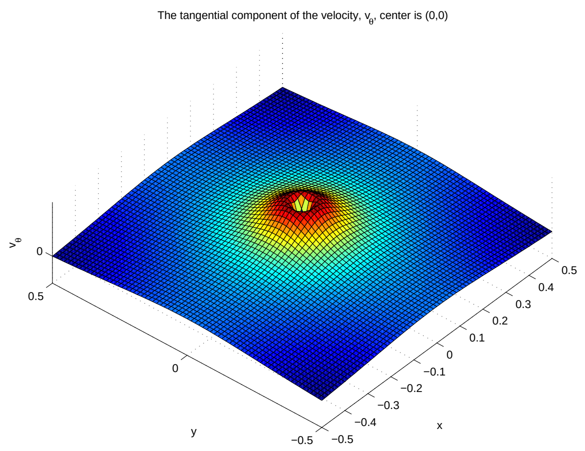


Figure 2: The tangential component of the velocity, $v_\theta(x, y)$

This is an atmospheric vortex.

The tropical cyclone , comparisons



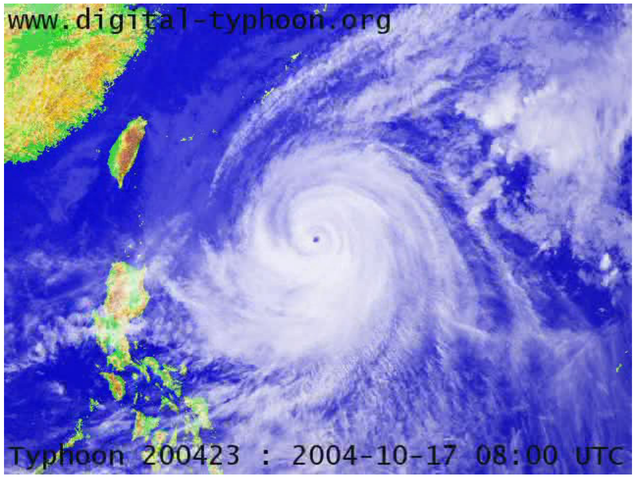
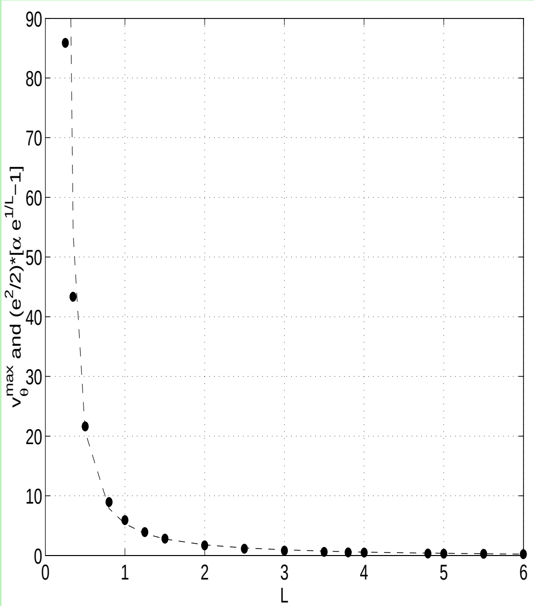


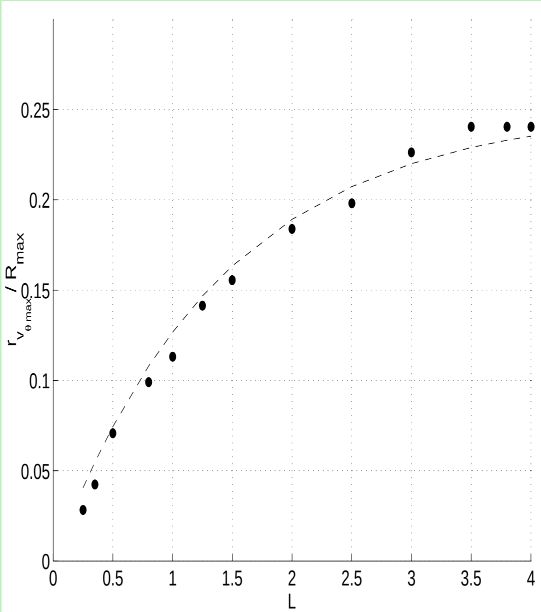
Figure 3: The solution and the image from a satelite.

The solution reproduces the *eye* radius, the radial extension and the vorticity magnitude.

Scaling relationships between main parameters of the tropical cyclone
eye-wall radius, maximum tangential wind, maximum radial extension

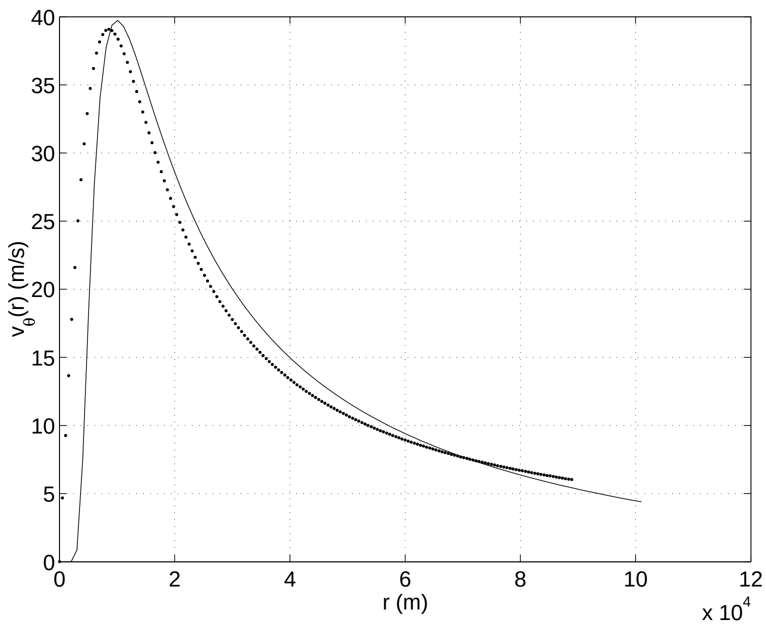


$$v_\theta^{\max}(L) \simeq \frac{e^2}{2}\left[\alpha \exp\left(\frac{\sqrt{2}}{R_{\max}}\right) - 1\right]$$



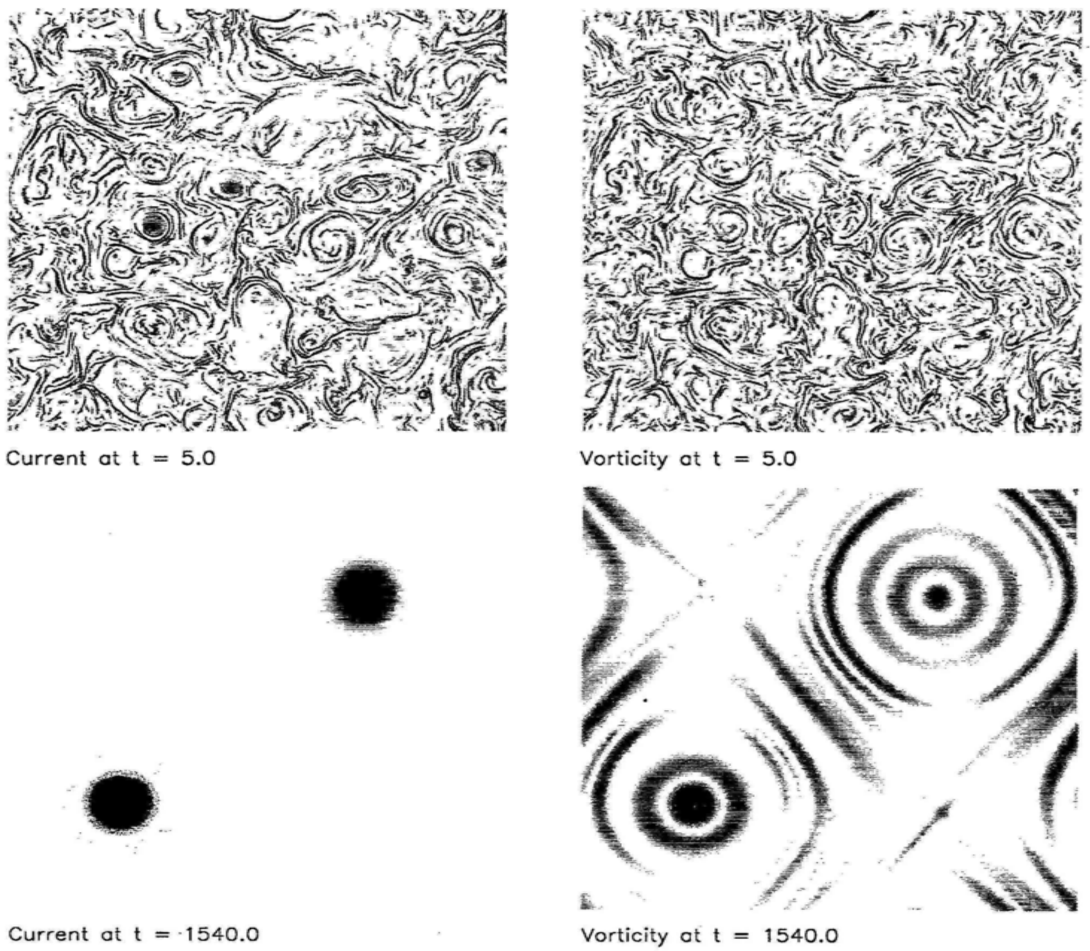
$$\frac{r_{v_\theta^{max}}}{R_{max}} = \frac{1}{4}\left[1 - \exp\left(-\frac{R_{max}}{2}\right)\right]$$

Profile of the azimuthal wind velocity $v_\theta(r)$



Comparison between the Holland's empirical model for v_θ (continuous line) and our result (dotted line).

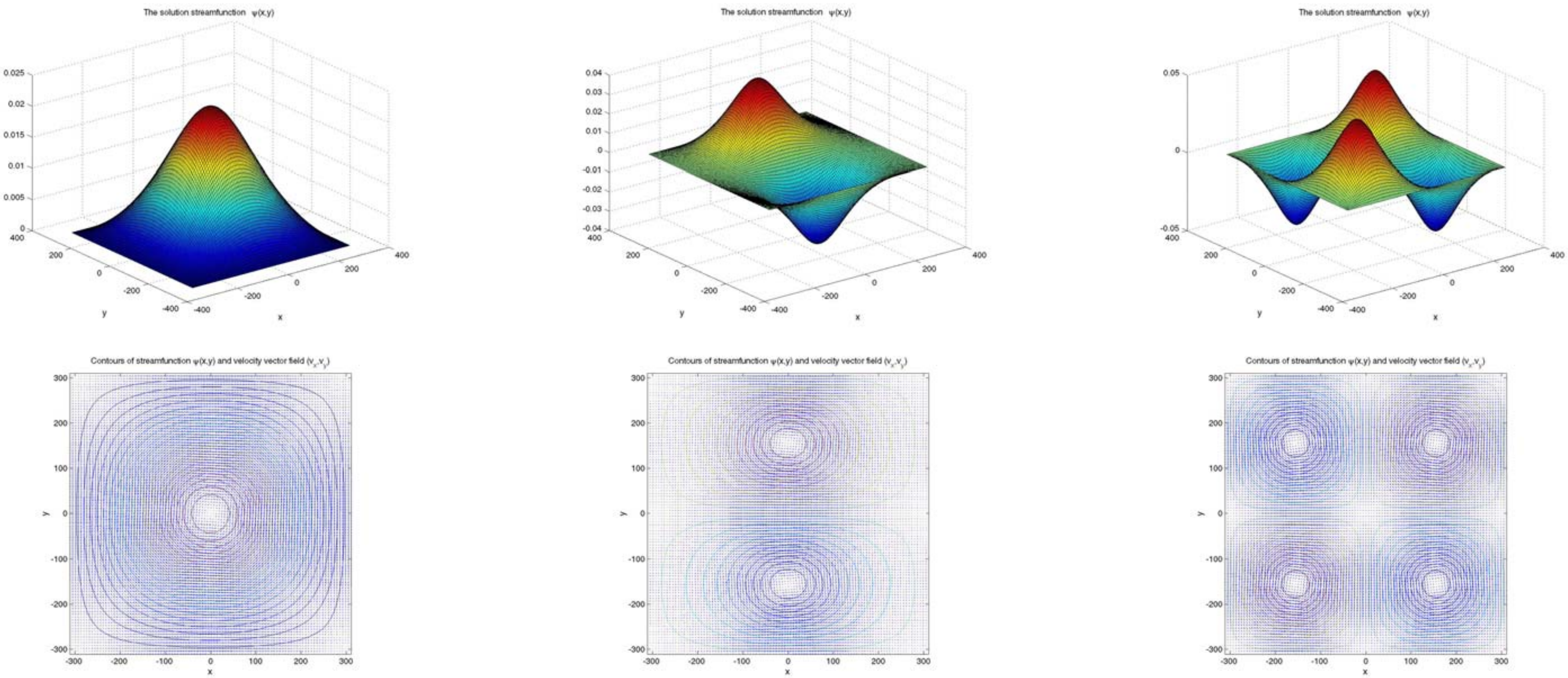
Coherent structures in fluids and plasmas (numerical 3)



R. Kinney, J.C. McWilliams, T. Tajima
Phys. Plasmas 2 (1995) 3623.

Numerical simulations of the MHD equations.

Tokamak plasma. Solution for $L = 307$: mono- and multipolar vortex



The plasma vortex : comparison of our results with the experiment

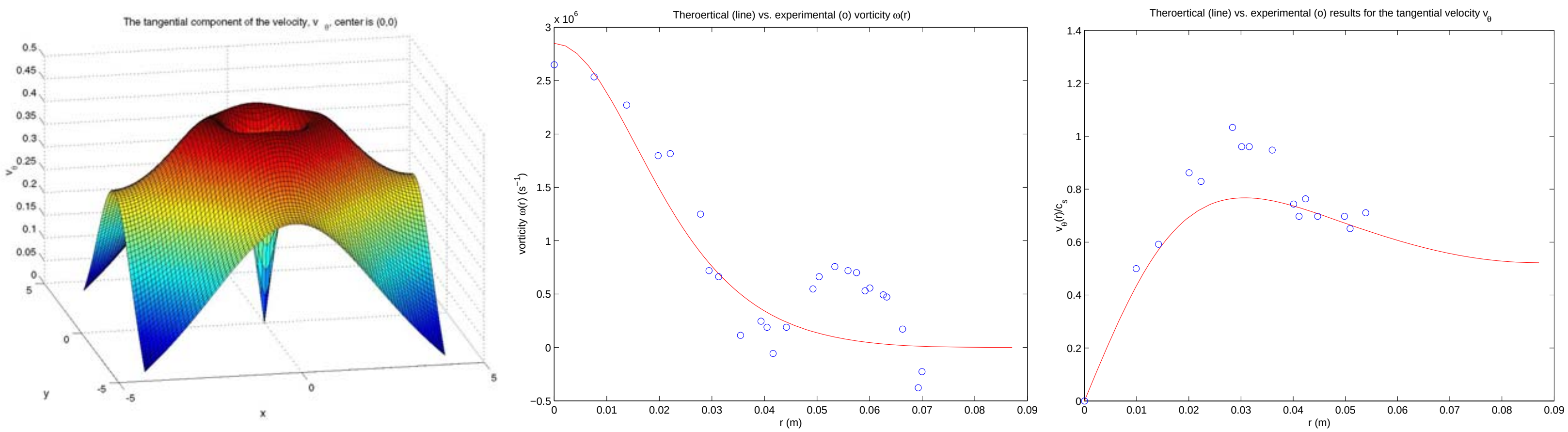


Figure 4: The calculated vortex and comparison with experiment.

Comparison of our vortex solution with experiment.

The crystals of plasma vortices

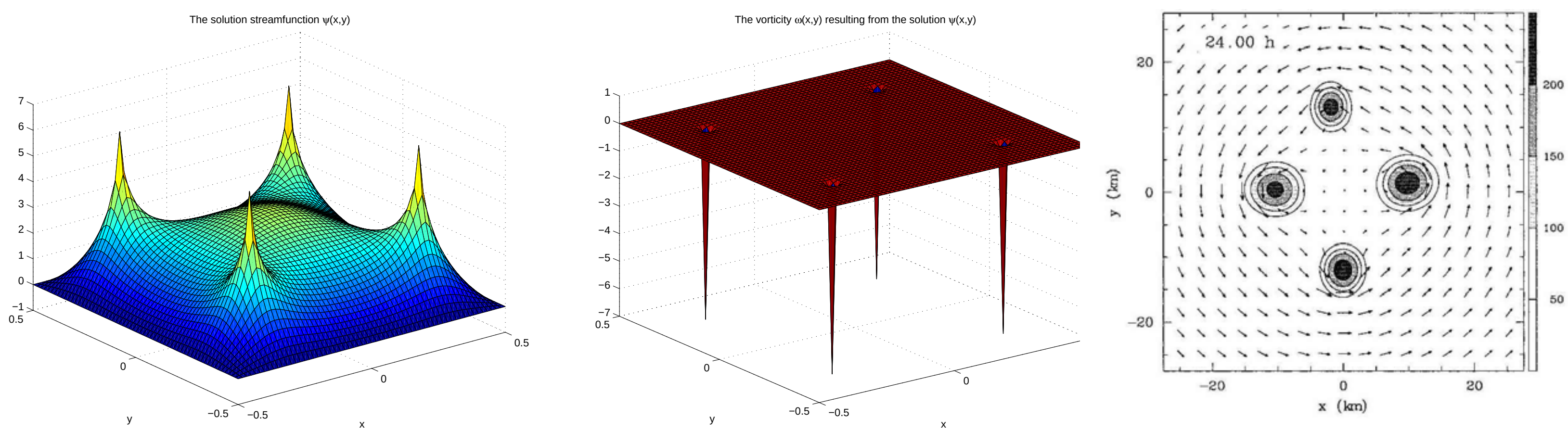


Figure 5: The crystals of plasma vortices.

Comparisons of crystal-type solutions with experiment.

Vortex crystals in non-neutral plasma

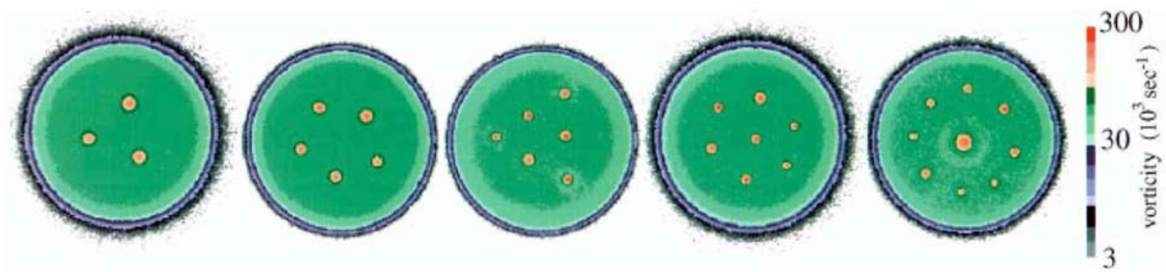
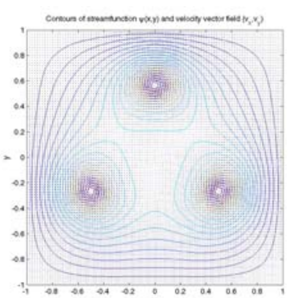
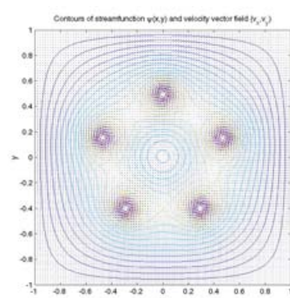
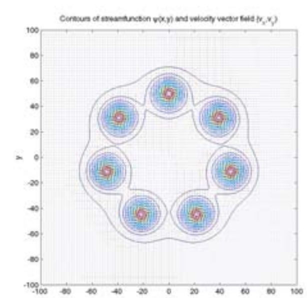


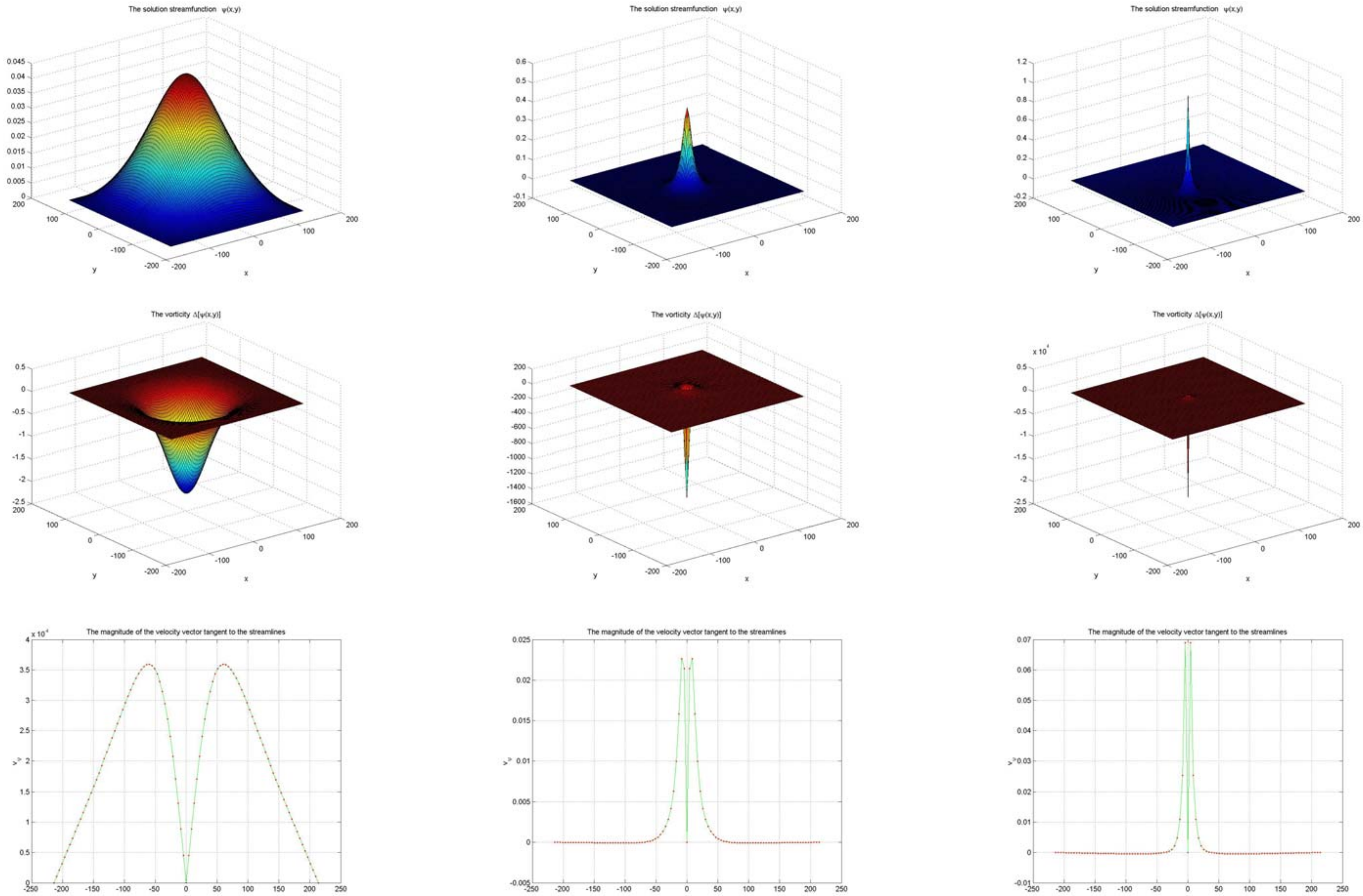
FIG. 1. Vortex crystals observed in magnetized electron columns (Ref. 8). The color map is logarithmic. This figure shows vortex crystals with (from left to right) $M=3$, 5, 6, 7, and 9 intense vortices immersed in lower vorticity backgrounds. In a vortex crystal equilibrium, the entire vorticity distribution $\zeta(r,\theta)$ is stationary in a rotating frame; i.e., ζ is a function of the variable $-\psi+\frac{1}{2}\Omega r^2$, where ψ is the stream function and Ω is the frequency of the rotating frame.



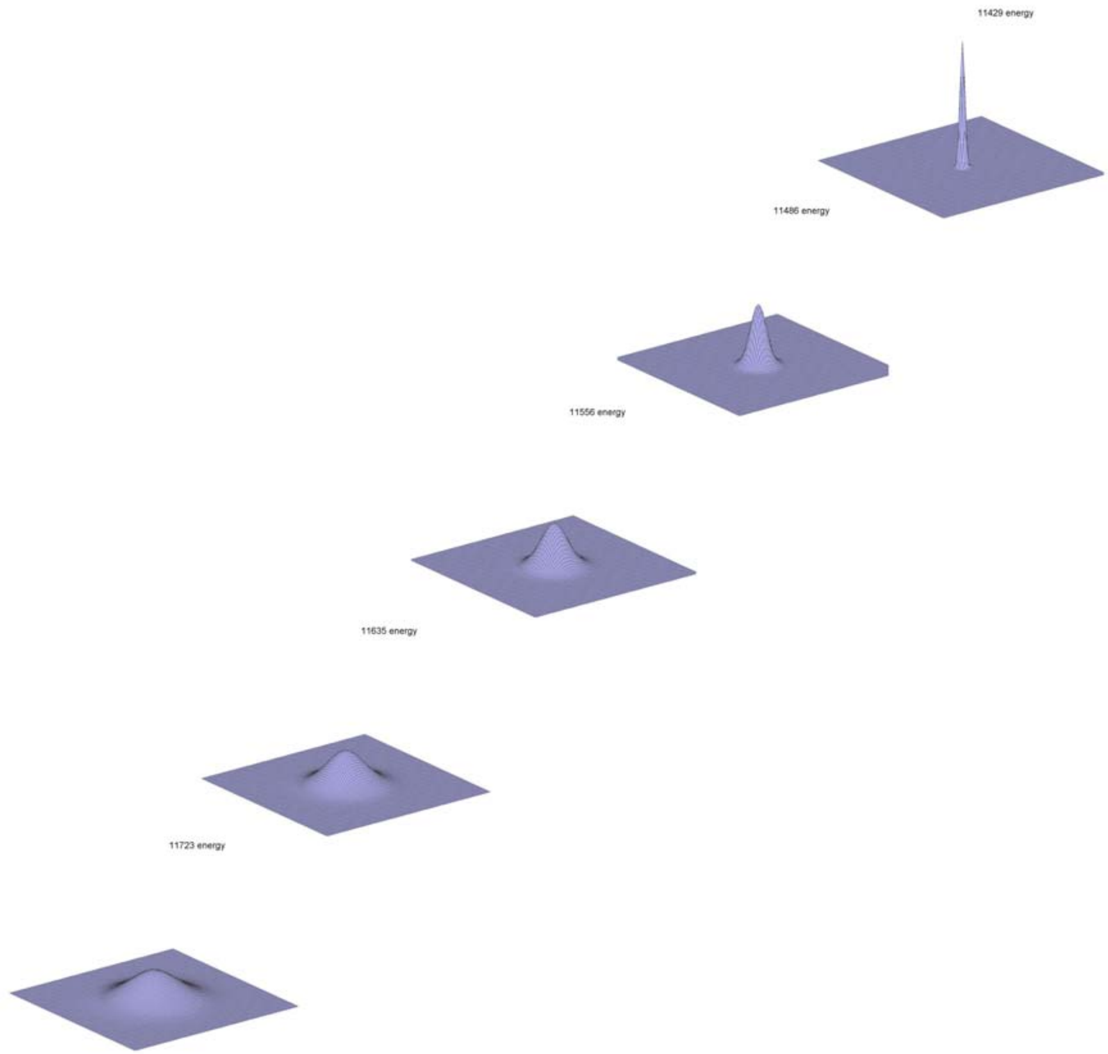




Comparison of our vortex solution with experiment.


The solution streamfunction ψ(x,y)
The solution streamfunction ψ(x,y)
The solution streamfunction ψ(x,y)
The vorticity Δ[ψ(x,y)]
The vorticity Δ[ψ(x,y)]
The vorticity Δ[ψ(x,y)]
The magnitude of the velocity vector tangent to the streamlines
The magnitude of the velocity vector tangent to the streamlines
The magnitude of the velocity vector tangent to the streamlines

Peaked profiles have lower energy



Numerical solution starting with *sech4/3*

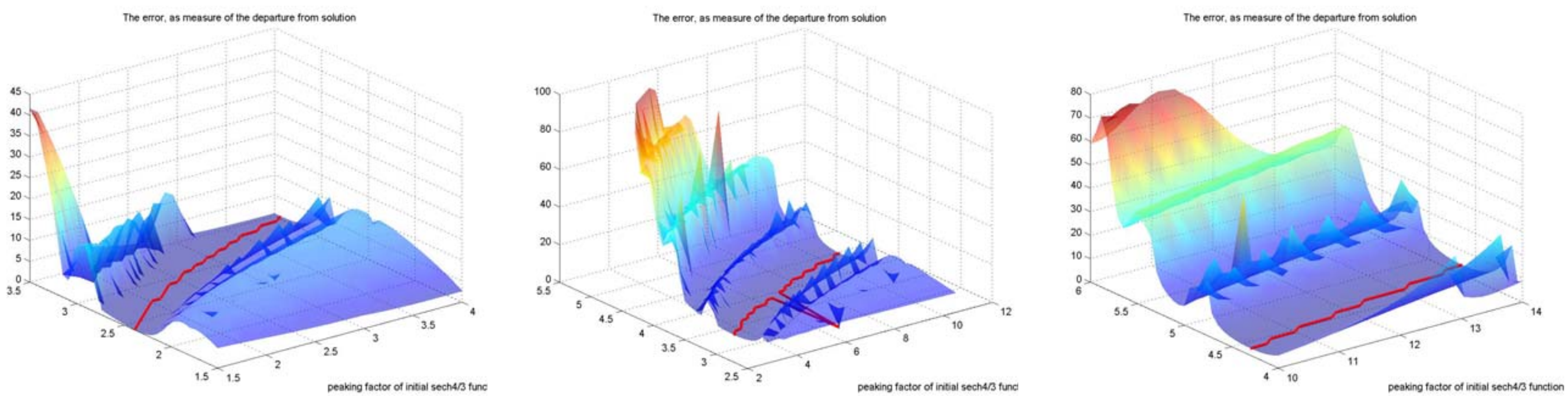
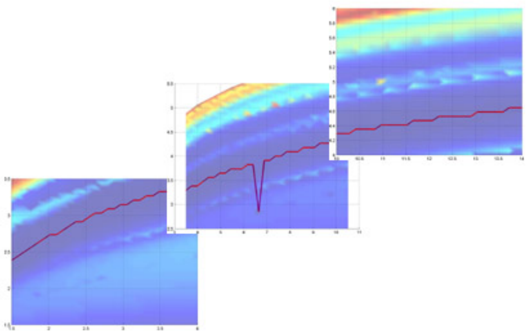


Figure 6: Three intervals on the (peaking factor, amplitude) parameter space.

Very weak variation of the *error functional* along the path (line of minimum error relative to the exact solution).

Radial integration



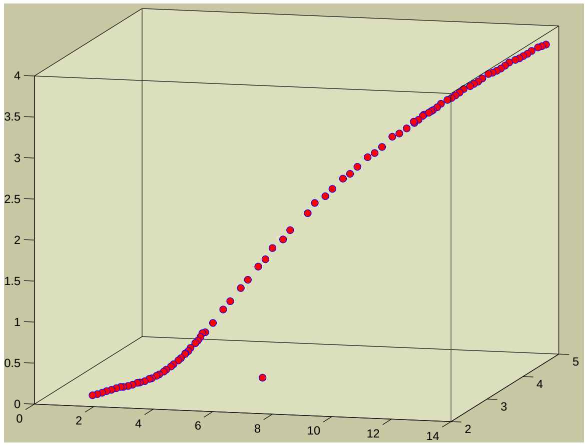


Figure 7: The functional error $\int d^2 r(\omega + nl)^2$.

String of quasi-solutions.

Along the string of quasi-solutions the vortices are more and more concentrated

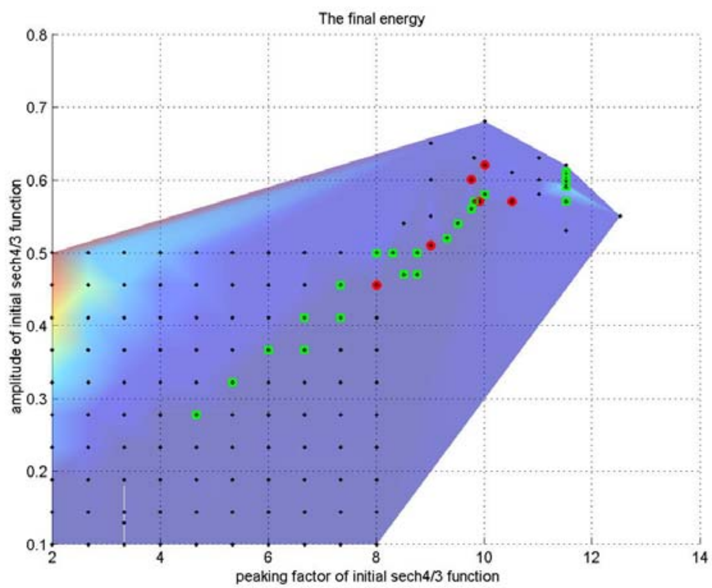


Figure 8: Green points: smooth, but progressively more peaked vortices; red: quasi-singular vortices.

The energies $\mathcal{E}_{final}$ and the vorticities Ω_{final} are only slightly different. We conclude that the system can drift along this path, under the action of even a small external drive.

Why we substitute ρ with $\exp(\psi)$

The paper on **Bosonization of three dimensional non-abelian fermion field theories** by **Bralic, Fradkin, Schaposnik**.

The initial self-interacting massive fermionic $SU(N)$ theory in Euclidean $2+1=3$ space

$$\mathcal{L} = \overline{\psi}\left(i\partial\!\!\!/ + m\right)\psi - \frac{g^2}{2} j^{a\mu} j_\mu^a$$

NOTE

This is precisely the Lagrangian for the *Thirring* model, for which it is possible to demonstrate the **quantum** equivalence with the *sine-Gordon* model. See **Ketov**.

The model is here *Abelian*.

The action is

$$I_T[\psi] = \int d^2x \left[\overline{\psi}\gamma^\mu\partial_\mu\psi - m_F\overline{\psi}\psi - \frac{g}{2}\left(\overline{\psi}\gamma^\mu\psi\right)^2\right]$$

In order to show the equivalence the following substitution is made

$$\psi_\pm = \exp\left\{\frac{2\pi}{i\beta}\int_{-\infty}^{x} dx' \frac{\partial\phi\left(x'\right)}{\partial t} \mp \frac{i\beta}{2}\phi\left(x\right)\right\}$$

where

$$\psi \equiv \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$$

Note that ψ are spinors and ϕ are bosons.

The **equivalence** will now consist of the following statement:

The functions $\psi_\pm$ satisfy the Thirring equations of motion provided the function ϕ satisfies the sine-Gordon equation.

And viceversa.

This allows to demonstrate the equivalence between the correlation functions of the two models.

Between the coupling constant of the two theories there is the following relation

$$\frac{\beta^2}{4\pi} = \frac{1}{1+g/\pi}$$

which shows that the strong coupling of the *Thirring* (fermions) model is mapped onto the weak coupling of the *sine-Gordon* (kinks and anti-kinks) model.

The mesons of the SG theory are the fermion-antifermion bound states of the Thirring theory.

The quantum bosonisation is done on the basis of the substitution shown above, but taking the *normal-ordered* form of the exponential.

$$\psi_{\pm} = C_{\pm} : \exp\left[A_{\pm}(x)\right] :$$

where

$$A_{\pm}(x) = \frac{2\pi m}{i\sqrt{\lambda}}\left(\int_{-\infty}^{x} dx' \frac{\partial\phi(x')}{\partial t}\right) \mp \frac{i\sqrt{\lambda}}{2m}\phi(x)$$

This implies the relations

$$\begin{aligned}\frac{m_0^2 m^2}{\lambda}\cos\left(\frac{\sqrt{\lambda}}{m}\phi\right) &= -m_F\overline{\psi}\psi \\ -\frac{\sqrt{\lambda}}{2\pi m}\varepsilon^{\mu\nu}\partial_\nu\phi &= \overline{\psi}\gamma^\mu\psi\end{aligned}$$

We make the following **Remark**: We see that the density of spinors (or point-like vortices) $\overline{\psi}\psi$ is expressed as the cos function of the scalar field of the SG model. This looks very similar to what we have in our, more complex, model. In our model the density of vorticity (which represents the continuum limit of the density of point-like vortices) is

$$\phi^\dagger\phi = \rho_1 - \rho_2$$

and the two functions are

$$\begin{aligned}\rho_1 &\equiv |\phi_+|^2 \\ \rho_2 &\equiv |\phi_-|^2\end{aligned}$$

We can introduce scalar streamfunctions for each of these densities, since they are associated with a sign of helicity

$$\rho_{1,2} = \exp\left(\psi_{1,2}\right)$$

Then the total density of vorticity should be written

$$\begin{aligned}\phi^{\dagger}\phi &= \rho_1 - \rho_2 \\ &= \exp\left(\psi_1\right) - \exp\left(\psi_2\right)\end{aligned}$$

But we know that at self-duality

$$\Delta \ln \rho_1 + \Delta \ln \rho_2 = 0$$

or

$$\Delta\psi_1 + \Delta\psi_2 = 0$$

If we do not consider any background flow, then one possible solution of this equation is

$$\psi_1 = -\psi_2$$

and this gives the form of the density of vorticity

$$\begin{aligned}\phi^{\dagger}\phi &= \exp(\psi_1) - \exp(\psi_2) \\ &= 2\sinh\psi\end{aligned}$$

We conclude that our theory is an extended form of the equivalence between the fermion system in plane (like the *Thirring* model) and the *Sinh-Gordon* model in plane.

Then, using the equivalences shown in the *Thirring-sine-Gordon* case, we can identify the function ϕ from their equation (the *sine-Gordon* variable) with the streamfunction ψ of our fluid, but multiplied with i.

And the current of fermions in their case $\overline{\psi}\gamma^{\mu}\psi$, which is proved to be expressed as a rotational of the SG function ϕ, appears in our case as follows: the current of point-like vortices is equal with the velocity since their ϕ is our streamfunction ψ and their rotational of the SG's ϕ is our rotational of ψ, or the physical velocity.

We can say that we assist at a typical scenario of equivalence between the system of point-like vortices and the system of *sinh-Gordon* streamfunction field, in a more extended, including *Non-Abelian* form.

The simplified result of the classical equivalence: *Thirring/sine-Gordon* was that the density of vorticity is *cos* of a bosonic field.

We do not need the bosonization, *i.e.* the substitution of the fermionic variable with the exponential of the bosonic variable. However *this can be a demonstration of the adequacy of the substitution*

$$\rho \equiv \exp(\psi)$$

we do at the end of the calculation: we do that since we have in mind the equivalence Thirring/sine-Gordon and the possibility to interpret our introduction of the streamfunction ψ as a similar relationship between the fermionic and bosonic fields.

System of interacting particles in plane

A system of particles in the plane interacting through a potential. The Hamiltonian is

$$H=\sum_{s=1}^{N}\frac{1}{2}m_s\mathbf{v}_s^2$$

where

$$m_s\mathbf{v}_s=\mathbf{p}_s-e_s\mathbf{A}\left(\mathbf{r}_s|\mathbf{r}_1,\mathbf{r}_2,...,\mathbf{r}_N\right)$$

the potential at the point $\mathbf{r}_s$

$$\mathbf{A}\left(\mathbf{r}_s|\mathbf{r}_1,\mathbf{r}_2,...,\mathbf{r}_N\right)\equiv\left(a_s^i\left(\mathbf{r}_1,\mathbf{r}_2,...,\mathbf{r}_N\right)\right)_{i=1,2}$$

$$a_s^i\left(\mathbf{r}_1,\mathbf{r}_2,...,\mathbf{r}_N\right)=\frac{1}{2\pi\kappa}\varepsilon^{ij}\sum_{q\neq s}^{N}e_q\frac{r_s^j-r_q^j}{\left|\mathbf{r}_s-\mathbf{r}_q\right|^2}$$

The vector potential $\mathbf{A}_s$ is the *curl* of the Green function of the Laplacian

$$\frac{1}{2\pi}\varepsilon^{ij}\frac{r^j}{r^2}=\varepsilon^{ij}\partial_j\frac{1}{2\pi}\ln r \qquad \nabla^2\frac{1}{2\pi}\ln r=\delta^2\left(r\right)$$

The continuum limit is a classical field theory

- separate the matter degrees of freedom
- Consider the interaction potential as a *free* field = new degree of freedom of the system, and find the Lagrangian which can give this potential.
- Couple the matter and the field by an interaction term in the Lagrangian

According to Jackiw and Pi the field theory Lagrangian

$$L = L_{matter} + L_{CS} + L_{interaction}$$

with

$$L_{matter} = \sum_{s=1}^{N} \frac{1}{2} m_s \mathbf{v}_s^2$$

The Chern-Simons part of the Lagrangian

$$\begin{aligned} L_{CS} &= \frac{\kappa}{2}\int d^2r\,\varepsilon^{\alpha\beta\gamma}\partial_\alpha A_\beta A_\gamma \\ &= \frac{\kappa}{2}\int d^2r\,\frac{\partial \mathbf{A}}{\partial t}\times\mathbf{A} - \int d^2r\,A^0 B \end{aligned}$$

where

$$x^\mu = (ct, \mathbf{r})$$

$$\mathbf{B} = \nabla\times\mathbf{A}$$

$$\mathbf{E} = -\nabla A^0 - \frac{\partial \mathbf{A}}{\partial t}$$

The interaction Lagrangian is

$$L_{int} = \sum_{s=1}^{N} e_s \mathbf{v}_s\cdot\mathbf{A}\left(t,\mathbf{r}_s\right) - \sum_{s=1}^{N} e_s A^0\left(t,\mathbf{r}_s\right)$$

Define the current

$$v^{\mu} = (c, \mathbf{v}_s)$$

$$j^{\mu}(t, \mathbf{r}) = \sum_{s=1}^{N} e_s v_s^{\mu} \delta(\mathbf{r} - \mathbf{r}_s)$$

the interaction Lagrangian can be written

$$\begin{aligned} L_{int} &= -\int d^2r \; A_{\mu} j^{\mu} \\ &= \int d^2r \; \mathbf{A} \cdot \mathbf{j} - \int d^2r \; A^0 \rho \end{aligned}$$

The current at the continuum limit

$$j^{\mu} = (\rho, \mathbf{j})$$

with

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0$$

Two steps to get the Hamiltonian form

1. *Eliminate the gauge-field variables in favor of the matter variables, by using the gauge-field equations of motion.*

The **equations of motion of the gauge field** are

$$\frac{\kappa}{2}\varepsilon^{\alpha\beta\gamma}F_{\alpha\beta} = j^{\mu} \tag{31}$$

$$B = -\frac{1}{\kappa}\rho$$

$$E^{i} = \frac{1}{\kappa}\varepsilon^{ij}j^{j}$$

2. *Define the canonical momenta.*

But not yet.

It is time to find the field that will represent the continuum limit of the density of discrete points

The right choice : a complex scalar field Φ.

Remember now that the momentum is the generator of the space translations which means that it has the form : $\partial/\partial x$.

(No subversive quantum activities)

Define the momenta as **covariant derivatives**

$$\begin{aligned} \mathbf{\Pi}(\mathbf{r}) &\equiv [\nabla - ie\mathbf{A}(\mathbf{r})]\,\Psi(\mathbf{r}) \\ &= \mathbf{D}\Psi(\mathbf{r}) \end{aligned}$$

and the conjugate

$$\mathbf{\Pi}^{\dagger} \equiv (\mathbf{D}\Psi)^{\dagger}$$

The number density operator is

$$\rho = \Psi^{\dagger}\Psi$$

The **potential** $\mathbf{A}(\mathbf{r})$ is constructed such as to solve the Chern-Simons relation between the field $\mathbf{B} = \nabla \times \mathbf{A}$ and the charge density $e\rho$:

$$B = -\frac{e}{\kappa}\rho$$

The **potential** is then

$$\mathbf{A}(\mathbf{r}) = \nabla \times \frac{e}{\kappa} \int d^2 r' \; \mathbf{G}(\mathbf{r} - \mathbf{r}') \, \rho(\mathbf{r}')$$

where $\mathbf{G}(\mathbf{r} - \mathbf{r}')$ is the Green function of the Laplaceian in plane. The *curl* of the Green function is

$$\nabla \times \mathbf{G}(\mathbf{r} - \mathbf{r}') = -\frac{1}{2\pi} \nabla \theta (\mathbf{r} - \mathbf{r}')$$

where

$$\tan \theta (\mathbf{r} - \mathbf{r}') = \frac{y - y'}{x - x'}$$

and θ is multivalued.

The Hamiltonian

$$H = \int d^2r \; \mathcal{H}$$

is

$$\mathcal{H} = \frac{1}{2m} \left(\mathbf{D}\Psi \right)^* \left(\mathbf{D}\Psi \right) - \frac{g}{2} \left(\Psi^* \Psi \right)^2$$

with the equation of motion

$$i \frac{\partial \Psi \left(\mathbf{r}, t \right)}{\partial t} = -\frac{1}{2m} \mathbf{D}^2 \Psi \left(\mathbf{r}, t \right) + eA^0 \left(\mathbf{r}, t \right) - g\rho \left(\mathbf{r}, t \right) \Psi \left(\mathbf{r}, t \right) \qquad (32)$$

The potential is related to the density ρ and to the current $\mathbf{j}$:

$$\mathbf{A} \left(\mathbf{r}, t \right) = \nabla \times \frac{e}{\kappa} \int d^2r \; \mathbf{G} \left(\mathbf{r} - \mathbf{r}' \right) \rho \left(\mathbf{r}', t \right) + \text{ gauge term}$$

$$A^0 \left(\mathbf{r}, t \right) = -\nabla \times \frac{e}{\kappa} \int d^2r \; \mathbf{G} \left(\mathbf{r} - \mathbf{r}' \right) \mathbf{j} \left(\mathbf{r}', t \right) + \text{ gauge term}$$

Write Ψ as amplitude and phase $\Psi = \rho^{1/2} \exp\left(ie\chi\right)$ and inserting this expression into the equation of motion derived from the Hamiltonian the imaginary part gives the **equation of continuity**

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0$$

and the real part gives:

$$\begin{aligned} \nabla^2 \ln \rho &= 4m\left(eA^0 - g\rho\right) \\ &\quad +2\left(e\mathbf{A} - \frac{1}{2}\nabla\times \ln \rho\right)\left(e\mathbf{A} + \frac{1}{2}\nabla\times \ln\rho\right) \end{aligned}$$

The static self-dual solutions

All starts from the identity (Bogomolnyi)

$$\left|\mathbf{D}\Psi\right|^2 = \left|\left(D_1 \pm iD_2\right)\Psi\right|^2 \pm m\nabla\times\mathbf{j} \pm eB\rho$$

Then the *energy density* is

$$H = \frac{1}{2m}\left|\left(D_1 \pm iD_2\right)\Psi\right|^2 \pm \frac{1}{2}\nabla\times\mathbf{j} - \left(\frac{g}{2} \pm \frac{e^2}{2m\kappa}\right)\rho^2$$

Taking the particular relation

$$g = \mp\frac{e^2}{m\kappa}$$

and considering that the space integral of $\nabla\times\mathbf{j}$ vanishes,

$$H = \frac{1}{2m}\int d^2r\ \left|\left(D_1 \pm iD_2\right)\Psi\right|^2$$

This is non-negative and attains its minimum, zero, when Ψ

satisfies

$$D_1\Psi \pm iD_2\Psi = 0$$

or

$$\mathbf{D}\Psi = i\mathbf{D}\times\Psi$$

which is the self-duality condition.

Then decomposing again Ψ in the phase and amplitude parts,

$$\mathbf{A} = \nabla\chi \pm \frac{1}{2e}\nabla\times \ln\rho$$

Introducing in the relation derived from Chern-Simons

$$B = \nabla\times\mathbf{A} = -\frac{e}{\kappa}\rho$$

we have

$$\nabla^2 \ln\rho = \pm 2\frac{e^2}{\kappa}\rho$$

which is the Liouville equation.