Field theoretical formulation of the asymptotic relaxation states of ideal fluids

F. Spineanu and M. Vlad
National Institute of Laser, Plasma and Radiation Physics
Magurele, Bucharest 077125, Romania

September 18, 2014

Abstract

The ideal incompressible fluid in two dimensions (Euler fluid) evolves at relaxation from turbulent states to highly coherent states of flow. For the case of double spatial periodicity and zero total vorticity it is known that the streamfunction verifies the *sinh*-Poisson equation. These exceptional states can only be identified in a description based on the extremum of an action functional. Starting from the discrete model of interacting point-like vortices it was possible to write a Lagrangian in terms of a matter function and a gauge potential. They provide a dual representation of the same physical object, the vorticity. This classical field theory identifies the stationary, coherent, states of the $2D$ Euler fluid as derived from the self-duality. We first provide a more detailed analysis of this model, including a comparison with the approach based on the statistical physics of point-like vortices. The second main objective is the study of the dynamics in close proximity of the stationary self-dual state, *i.e.* before the system has reached the absolute extremum of the action functional. Finally, limitations and possible extensions of this field theoretical model for the $2D$ fluids model are discussed and some possible applications are mentioned.

Contents

Keywords: Ideal fluid, coherent structures, field theory, self - duality

1 Introduction

The ideal (non-dissipative) incompressible fluid in two - dimensions, which we will shortly call $2D$ Euler fluid, can be described by three related functions $(\psi, \mathbf{v}, \omega)$. The streamfunction is a scalar field $\psi(x, y, t)$ from which the velocity vector field $\mathbf{v}(x, y, t)$ is derived: from the incompressibility $\nabla \cdot \mathbf{v} = 0$, one can write $\mathbf{v} = \nabla\chi - \nabla\psi \times \widehat{\mathbf{e}}_z$ where χ is a harmonic function, $\Delta\chi = 0$, $\widehat{\mathbf{e}}_z$ is the versor perpendicular on the plane of the motion and the operators ∇ and Δ are restricted to $2D$. Applying the rotational operator one obtains the vorticity $\omega\widehat{\mathbf{e}}_z = \nabla \times \mathbf{v} = \Delta\psi\widehat{\mathbf{e}}_z$ and the Euler equation is

$$\frac{d\omega}{dt} = \frac{\partial}{\partial t}\Delta\psi + \left[\left(-\nabla\psi \times \widehat{\mathbf{e}}_z\right) \cdot \nabla\right] \Delta\psi = 0. \tag{1}$$

The velocity vector field $\mathbf{v}(x, y, t)$ has the fundamental quality that it can be measured in physical fluids, offering a direct connection with the experiments and observations. It is then natural that almost all studies on the fluid dynamics are expressed in terms of these three functions and any more abstract description must finally return to them.

It is known that in $2D$ there is inverse cascade, *i.e.* there is flow of energy in the spectrum from small spatial scales towards the large spatial scales. The numerical simulation of the $2D$ Euler fluid in a box with doubly periodic boundary conditions fully confirms this behavior. Adding just a small viscosity and starting from a state of turbulence, the fluid evolves to a state of highly ordered flow: the positive and negative vorticities contained in the initial flow are separated and collected into two large scale vortical flows of opposite sign. Fully convincing pictures of the asymptotic states are shown in Ref. [1] and [2]. The motion is stationary for a long time, being finally dissipated by the friction associated to the small viscosity. It has been found that the streamfunction ψ in these states reached asymptotically at relaxation from turbulence verify the *sinh*-Poisson equation

$$\Delta\psi + \lambda \sinh\psi = 0 \tag{2}$$

where $\lambda > 0$ is a parameter. The significance of this fact is very deep and can be appreciated by the following considerations. If we want to find the stationary solution of Eq.(1) we take $\partial\psi/\partial t = 0$ and look for the solutions of

$$[(-\nabla\psi \times \widehat{\mathbf{e}}_z) \cdot \nabla] \Delta\psi = 0 \tag{3}$$

It is obvious (and widely adopted) that we can solve this equation by taking the vorticity to be an arbitrary function F of the streamfunction: $\omega = \Delta\psi = F(\psi)$. Equivalently this is a recognition of the fact that Eq.(3) has an indefinitely large space of solutions. However the nature does not confirm this: the fluid left to evolve from a turbulent initial state will end up by taking one of the functions $\psi(x, y)$ that verify Eq.(2), *i.e.* it goes precisely towards a tiny subset within the whole function space that seemingly was at its disposal. This dramatically underlines the contrast: while $\omega = F(\psi)$ with arbitrary F is a result of the conservation law $d\omega/dt = 0$, the strict evolution towards solutions ψ of Eq.(2) suggests that there are exceptional states and they should be chosen by some variational principle that is expected to apply to this system.

The equation (2) is exactly integrable [3]. Since in general the coherent structures and the integrability are connected with self-duality [4], one may be interested to identify the analytical framework where the coherent structures of the stationary $2D$ Euler fluid flow appear as a consequence of the self-duality (SD). We note that, at least at first sight, the classical formulation in terms of $(\psi, \mathbf{v}, \omega)$ does not appear to be adequate to express the property of self-duality.

Although the accumulation of results on the dynamics of the $2D$ Euler fluid is immense, there is an obstacle if we want to exploit it in order to construct a formulation that exhibits the connection "coherent flow" - "self duality". The classical formulation uses the conservation laws as dynamical equations. The zero-divergence of the velocity field is equivalent to the continuity equation, *i.e.* the conservation of the fluid mass. The conservation of the momentum is the zero-dissipation version of the Navier-Stokes equation, which, after applying the operator $\nabla\times$, becomes Eq.(1). Further, commonly used are the conservation of the energy, of angular momentum, etc. If there is a change of one of the variables of which the state of the system depends, the conservation laws show how the other variables must change such that certain quantities (mass, momentum, energy, etc.) remain invariant. The conservation laws cannot identify exceptional states. For this we need a functional of the state of the fluid and a variational principle able to identify the evolutions toward particular, exceptional states, like those given by Eq.(2). In other words, we need a description of the fluid motion in terms of the density of a Lagrangian, whose integral over space-time is an action functional. The dynamical equations would then be derived as Euler-Lagrange variational equations, by extremizing the action. Summarizing, we currently use the conservation laws as dynamical equations, which is formally not correct: the *dynamical* equations are by definition the Euler-Lagrange equations obtained from functional variation of an action functional. The difficult problem is, of course, to find the Lagrangian. The Lagrangian must be inferred from basic physical facts about the system, and it is not satisfactory to simply find a functional (like a minimizer) or Lyapunov-type.

Finding the adequate Lagrangian for the two-dimensional Euler fluid is however possible. The reason is the existence of a model consisting of a discrete version of the physical dynamics expressed by Eq.(1): a set of point-like vortices interacting in plane by a potential generated by themselves. The interaction is long - range (Coulombian) and the equations of motion are a discrete version of the advection of the elementary vortices by the velocity field produced by themselves. It is well established (and will be reminded below) that the set of discrete, point-like, vortices can be treated as a statistical ensemble with the result that at maximum entropy the Eq.(2) is obtained. Several other applications of the discrete model have led to interesting results but in general the model is difficult to be used directly. From the point of view of what we are looking for, *i.e.* a Lagrangian for the Euler fluid, the discrete model is however extremely suggestive [5]. Instead of $(\psi, \mathbf{v}, \omega)$ it uses *matter* (density of point-like vortices), *field* (corresponding to the potential generated by the discrete vortices) and *interaction*. This means that returning to the continuum limit but preserving this structure, we can formulate a classical field theory. This shift is a conceptual change and some inferrence is still needed in order to write the Lagrangian functional. Following the suggestion of the point-like vortex model two fields are involved, a field $\phi(x, y, t)$ representing the *matter* and a vector field $A^{\mu}(x, y, t)$ with $\mu = 0, x, y$, representing the *gauge potential*. The vortical nature of the elementary objects can be reproduced by a classical spin-like quantity. It is convenient to represent the negative vortices as positive vortices having backward time propagation, *i.e.* the positive and negative vortices behave as particles and antiparticles. The matter ϕ will be represented by a mixed spinor of the type $x^{\alpha\dot{\beta}}$, a 2×2 matrix with complex entries, with distinct spinorial transformations on its two indices (this is the reason of the dot on the index β). Accordingly the potential is a complex 2×2 matrix, an element of $sl(2, \mathbf{C})$. The Lorentz-type motion of the elementary vortices is represented by the Chern-Simons term in the Lagrangian. A nonlinear self-interaction of the matter field cancels, via Gauss constraint, the part of the kinetic energy which is due to the interaction between the rotational of the potential (the magnetic field) and the matter density. The extremum of the action corresponds to self-duality and the states are stationary with the streamfunction obeying Eq.(2). This shows that the coherent flows reached by the Euler fluid at relaxation belong to the same exceptional family of soliton or instanton-like solutions, a purely nonlinear effect. This represents also an analytical derivation of the Eq.(2), alternative to the statistical analysis.

A full framework for the description of the $2D$ Euler fluid is built in this way, using the powerful field theoretical (FT) formalism and ready to benefit from its achievements in the physics of vortices (Bose-Einstein condensate, superconductivity, topological field theories like $O(n)$, cosmic strings, etc.). Naturally there are limitations too: one still has to include dissipation and the change of topology of flows by breaking and reconnection of streamlines, study the isotopological dynamical aspects (*i.e.* between reconnection events), and adapt the formalism to various boundary conditions, etc. In the present work we focus on the $2D$ Euler fluid evolving in a box with boundary conditions leading to double periodicity. This is known to evolve asymptotically to solutions of Eq.(2) and, in the FT, exhibit the property of self-duality. We attach most importance to this fact since it has become more and more clear that all known coherent structures are connected with self-duality [4].

The states identified by the FT as extrema of the action functional are characterized by: (1) stationarity; (2) double periodicity, *i.e.* the function $\psi(x, y)$ must only be determined on a "fundamental" square in plane; (3) the total vorticity is zero; (4) the states verify Eq.(2). The self-dual states are the absolute minimum of the energy but in order them to be attained the system must have access to the class of configurations defined by these symmetry conditions : zero total vorticity in the field and double spatial periodicity. In the non-dissipative fluid these conditions are fixed at the initial state and the SD state cannot be reached in general. This means that here a large class of fluid asymptotic states will not be examined. The relevance of all these, for the physics of fluids, is an interesting subject, which we will not discuss here.

In Ref.[5] we have presented the derivation of the *sinh*-Poisson equation in a field theoretical model for the $2D$ Euler fluid. The objective of the present work is the study of the time evolution in close proximity of the SD state, for a system that asymptotically reaches the SD state. We derive the specific form taken by the equations of motion in this regime, the current of "matter" and the equations for the magnitudes of the positive and negative parts of the matter field that combines into a single physical variable, the vorticity. These equations are similar but not identical to equations of continuity and generalize the equations of the Abelian model [6].

The SD state depends on the equality between two parameters that enter the expression of the Lagrangian. Since we are interested in the states that are close, - but not exactly at SD, we suggest (in a qualitative discussion) that it may be possible to include situations where these two parameters are not equal but evolve slowly toward equality. It arises a possible reflection in the theory of the events of dissipative reconnection of streamlines and increase of the topological order of the flow, toward SD. Now there is attraction between *mesoscopic* vortices. (We use this name for the few vortices remaining in the late phase, which have already concentrated a large part of the initial vorticity; they move slowly in plane and their encounters and mergings is the last phase of the evolution toward the final, fully organized, state). The FT suggests the interpretation that an excess of "helicity" is removed at each reconnection until identity is reached between two different contributions to the energy: the FT energy is exactly zero at SD since the energy is only due to the motion of the centers of the *mesoscopic* vortices, which stop at SD, while the motion of the fluid on streamlines has zero energy. We suggest that a FT formalism similar with the *baryogenesis* but in reversed direction, *i.e.* decrease of Chern-Simons topological number, may provide an analytical description. Since the term of the Lagrangian that is so decreased becomes at SD the square vorticity, it seems that there is compatibility with the known decay of the enstrophy during vorticity self-organization in weakly dissipative fluids.

2 The model of interacting point-like vortices

The physical quantities describing the two-dimensional fluid dynamics are ψ ≡streamfunction, $\mathbf{v}$≡velocity, $\omega\widehat{\mathbf{e}}_z$ =vorticity, which are related by

$$\mathbf{v} = -\nabla\psi \times \widehat{\mathbf{e}}_z \quad , \quad \omega = \Delta\psi \tag{4}$$

and are solutions of the Euler equation (1). The discretized form of this equation has been extensively studied [7], [8], [9], [10]. The continuum limit of the discretization is matematically equivalent with the fluid dynamics. We just review few elements of this theory, for further reference.

Consider the discretization of the vorticity field $\omega(x,y)$ in a set of $2N$ point-like vortices ω_i each carrying the elementary quantity ω_0 ($=$ const > 0) of vorticity which can be positive or negative $\omega_i = \pm\omega_0$. There are N vortices with the vorticity $+\omega_0$ and N vortices with the vorticity $-\omega_0$. The current position of a point-like vortex is (x_i, y_i) at the moment t. The vorticity is expressed as

$$\omega(x,y) = \sum_{i=1}^{2N} \omega_i a^2 \delta(x - x_i)\,\delta(y - y_i) \tag{5}$$

where a is the radius of an effective support of a smooth representation of the Dirac δ functions approximating the product of the two δ functions [7]. Instead of $\omega_i a^2$ we can use the *circulation* γ_i which is the integral of the vorticity over a small area around the point (x_i, y_i): $\gamma_i = \int d^2x\omega_i$ [10]. The formal solution of the equation $\Delta\psi = \omega$, connecting the vorticity and the streamfunction, can be obtained using the Green function for the Laplace operator

$$\Delta_{x,y} G(x, y; x', y') = \delta(x - x')\delta(y - y') \tag{6}$$

where (x', y') is a reference point in the plane. As shown in Ref.[7] $G(\mathbf{r}; \mathbf{r}')$ can be approximated for a small compared to the space extension of the fluid, L, $a \ll L$, as the Green function of the Laplacian

$$G(\mathbf{r}; \mathbf{r}') \approx \frac{1}{2\pi} \ln\left(\frac{|\mathbf{r} - \mathbf{r}'|}{L}\right) \tag{7}$$

where L is the length of the side of the square domain. The solution of the equation $\Delta\psi = \omega$ is obtained using the Green function, using the circulation $\gamma_i = \omega_i a^2$,

$$\psi(\mathbf{r}) = \sum_{i=1}^{2N} \gamma_i \frac{1}{2\pi} \ln\left(\frac{|\mathbf{r} - \mathbf{r}_i|}{L}\right) \tag{8}$$

The velocity of the k-th point-vortex is $\mathbf{v}_k = -\left.\nabla\psi\right|_{\mathbf{r}=\mathbf{r}_k} \times \widehat{\mathbf{e}}_z$ and the equations of motion are

$$\begin{aligned} \frac{dx_k}{dt} &= v_x^{(k)} = -\sum_{i=1, i\neq k}^{2N} \gamma_i \frac{1}{2\pi} \frac{y_k - y_i}{|\mathbf{r}_k - \mathbf{r}_i|^2} \\ \frac{dy_k}{dt} &= v_y^{(k)} = \sum_{i=1, i\neq k}^{2N} \gamma_i \frac{1}{2\pi} \frac{x_k - x_i}{|\mathbf{r}_k - \mathbf{r}_i|^2} \end{aligned} \tag{9}$$

The equations can be derived from a Hamiltonian

$$H = \frac{1}{2\pi} \sum_{\substack{i=1 \\ i<j}}^{2N} \sum_{j=1}^{2N} \gamma_i \ln\left(\frac{|\mathbf{r}_i - \mathbf{r}_j|}{L}\right) \gamma_j \tag{10}$$

The standard way of describing the discrete model is within a statistical approach [11], [7], [8], [12], [13]. The elementary vortices are seen as elements of a system of interacting particles (like a gas) that explore an ensemble of microscopic states leading to the macroscopic manifestation that is the fluid flow. The number of positive vortices in the state i is N_i^+ and the number of negative vortices in the state i is N_i^-. The total numbers of positive and respectively negative vortices are equal: $N^+ = \sum_i N_i^+ = \sum_i N_i^- = N^-$. This system has a statistical temperature that is negative when the energy is zero or positive [14]. The energy of the discrete system of point-like vortices is $\mathcal{E} = \frac{1}{2}\sum_{ij} \omega(\mathbf{r}_i) G(\mathbf{r}_i, \mathbf{r}_j) \omega(\mathbf{r}_j)$ where $\omega(\mathbf{r}_i) = -\left(N_i^+ - N_i^-\right)$ is the vorticity. The probability of a state is calculated as a combinatorial expression

$$\mathcal{W} = \left\{ \frac{N^+!}{\prod_i N_i^+!} \right\} \left\{ \frac{N^-!}{\prod_i N_i^-!} \right\} \tag{11}$$

The *entropy* is the logarithm of this expression and by extremization one finds

$$\ln N_i^\pm + \alpha^\pm \pm \beta \sum_j G(\mathbf{r}_i, \mathbf{r}_j) \left(N_j^+ - N_j^-\right) = 0 \tag{12}$$

for $i = 1, N$, where $\alpha^\pm$ and β are Lagrange multipliers introduced to ensure $\sum N_i^+ = \sum N_i^- = N =$ const and conservation of the Energy $\mathcal{E}$. The solutions are written in terms of a continuous function $\psi(x, y)$

$$N_i^\pm = \exp\left[-\alpha^\pm \mp \beta\psi(x, y)\right]$$

implying $N_i^+ N_i^- =$ const, and this leads to the *sinh*-Poisson equation (2). The statistical approach has had to face particular problems: the system has finite phase space; there is no thermodynamic limit; there is no ergodicity; the temperature is negative; the entropy extremum is counter-intuitive, leading to maximum order; the final state of the system is not a *statistical* equilibrium but consists of non-fluctuating positions of the elementary vortices, composing a solution of (2). However the statistical approach succeeds to derive Eq.(2), is fully confirmed and generates successful exploration of similar problems. Since the field theoretical approach is different in an essential way it appears that the statistical approach has identified, in its specific way, the self-duality. Few aspects of the statistical approach will be discussed below in connection with FT formulation.

3 Field theoretical formulation of the continuum limit of the point-like vortex model

The physical vortical flow is represented by the Lorentz-type motion of the discrete set of point-like, massless, vortices. We note however that nowhere in the formulation (9) is made explicit the fact that we are dealing with *vortices*. The same equations describe a system of guiding-centers [15], point-like *charges* [6] or currents [16]. The information that it is question of vortices, *i.e.* objects that have the nature of vectors, must be supplemented to the system (9). We then also note that the third axis (z) although irrelevant for the plane motion, is implicitely present in the model.

In the basic model (Kraichnan and Montgomery [7], which will be taken as the reference model) it is assumed that the elementary vortices have equal magnitudes of vorticity ω_0 and, for periodicity, the numbers of positive and of negative vortices are equal, N. This N is invariant, *i.e.* there are no flip and/or annihilations. Physical vorticity ω in a point (x, y) is obtained by placing together n elementary vortices, $\omega \approx n\omega_0$ in an infinitesimal area around (x, y). The model does not allow building up higher similar objects *i.e.* $\pm 2\omega_0$, $\pm 3\omega_0$, etc. are not allowed as independent objects. In this representation the physical vorticity comes from the density of elementary vortices, *i.e.* like-sign vortices are not superposed one to the other, similar to the Pauli exclusion principle for fermion particles.

Therefore we have two types of elementary objects, carrying $+\omega_0$ and respectively $-\omega_0$ vorticity. The elementary vortices are similar to massless particles carrying half-integer spin but with fixed, unchangeable, projection along the transversal axis. The interaction between the two types of elementary vortices only affects their positions in plane, without changing their spin and projection.

Taking a fixed vorticity $\pm\omega_0$ for an elementary vortex there is no needs of an assumption on how this vorticity has been created, for example there is no need to imagine the presence of a fluid between the vortices. The model of point-like vortices fully replaces the model based on the physical variables $(\psi, \mathbf{v}, \omega)$. However, for theoretical purposes we can imagine that around each elementary vortex there is a fluid rotating such as to create the elementary vorticity ω_0. Obviously there is no unique way of prescribing such a velocity. The difference between the positive $(+\omega_0)$ and negative $(-\omega_0)$ vortices is the direction of the associated vortical flow in the plane: we take *anti-clockwise* for $+\omega_0$ and *clockwise* for $-\omega_0$. We note a particular property which is revealed by the representation based on the virtual-fluid rotation: the negative vortex can be obtained from a positive vortex by reversing the direction of time, since this leads to the reverse of the sense of rotation of the virtual fluid. Moreover, this ensures the invariance of the theory to time reversal transformation, if the total numbers of positive and negative-vorticity elements are equal.

As explained by Kraichnan and Montgomery [7] the elementary vortical structure in $3D$ is a vortex ring. A $3D$ ring of infinitesimal cross section intersects a plane that contains the center of the ring and is transversal on the plane of the ring, in two points. Close to these points the $3D$ ring is approximately reduced to two elementary linear vortices, perpendicular on the plane and with opposite vorticity. We add to this picture the observation that an axial flow in a $3D$ ring vortex becomes after reduction to $2D$ a flow perpendicular on the plane in positive z direction for one elementary vortex and in the negative z direction for its pair with opposite vorticity. The particularity of the $2D$ Euler fluid, that has been transfered to the discrete system, is the invariance against displacements on the z-axis, which can be defined locally arbitrarily. We will consider, without restricting generality, that the positive vortices $(+\omega_0)$ have a momentum $\mathbf{p} = p_0 \widehat{\mathbf{e}}_z$ and in agreement with the picture that represents a pair of opposite vortices as resulting from a $3D$ ring with axial flow, the negative vortices have a motion $\mathbf{p} = -p_0 \widehat{\mathbf{e}}_z$. The filaments can have a translation along the irrelevant axis (z) with arbitrary momentum, p_0. Again we note that the time inversion leaves invariant the system, with the positive vortices mapped onto negative ones. This will make the negative vortices actually to be defined as anti-vortices, similar to the anti-particles.

In the case of the point-like vortices for the Euler equation the positive energy vortices propagating forward in time are the usual physical point-like vortices. The time reflection vortices are propagating backward in time but they can be considered physical vortices with opposite charge (*i.e.* the vorticity $\omega_0 \to -\omega_0$) and propagates forward in time. They are simply physical point-like vortices with opposite vorticity.

With relation to the chiral analogy, we have "right-handed" and respectively "left-handed" vortices.

The two elements of the flow are positive and negative elementary vortices (point-like). The positive vortices: (1) rotate anti-clockwise in plane: $\omega \widehat{\mathbf{e}}_z \sim \sigma$ spin is up; (2) move along the positive z axis: $\mathbf{p} = \widehat{\mathbf{e}}_z p_0$; (3) have positive chirality: $\chi = \frac{\sigma \cdot \mathbf{p}}{|\mathbf{p}|}$. The positive vortices can be represented as a point that runs along a positive helix, upward. In projection from the above the plane toward the plane we see a circle on which the point moves anti-clockwise.

The negative vortices: (1) rotate clockwise in plane: $(-\omega) \widehat{\mathbf{e}}_z \sim -\sigma$ spin is down; (2) move along the negative z axis: $-\mathbf{p} = \widehat{\mathbf{e}}_z (-p_0)$, along $-z$; (3)

have positive chirality: $\chi = \frac{\sigma \cdot \mathbf{p}}{|\mathbf{p}|}$. The negative vortices can be represented as a point that runs along a positive helix, the same as above, but runs downward. In projection from the above the plane toward the plane we see a circle on which the point moves clockwise.

The positive vortices and the negative vortices have the same *chirality* and in a point where there is superposition of a positive and a negative elementary vortices the *chirality* is added. In particular, the vacuum consists of paired positive and negative vortices, with no motion of the fluid, which in physical variables means $\psi \equiv 0$, $\mathbf{v} \equiv \mathbf{0}$, $\omega \equiv 0$. In FT the vacuum consists of superposition of positive and negative vortices, which means: (1) zero spin, or zero *vorticity*; (2) zero momentum $\mathbf{p} = \mathbf{0}$; (3) 2×chiral charge. The Euler fluid at equilibrium ($\psi = 0$, $\mathbf{v} = \mathbf{0}$, $\omega = 0$) is in a vacuum with *broken chiral invariance*.

We can now return to comment the result of the statistical analysis based on the maximum entropy for the system of point-like vortices. The results was $N_i^+ N_i^-$ =const. This means that in order to reproduce a positive physical vorticity $|\omega|$ in a point we cannot simply take only positive elementary vortices $|\omega| = N_i^+ \omega_0$. We must also take a certain amount of negative point-like vortices $\left(N_i^-\right)$ in the same differentially small area of the discretization and obtain the physical vorticity $|\omega|$ as the difference between the two contributions, $|\omega| = \left|N_i^+ - N_i^-\right|$. None of them can ever be exactly zero, $N_i^{\pm} \neq 0$. This was the first indication that the elementary vortices are not as simple as pieces of vorticity. The zero vorticity does not mean absence of $N_i^{\pm}$. Both these numbers *must* remain non-zero but they are now equal and implicitly there is mutual annihilation of their virtual flows and of their z-momenta. This corresponds to *pairing* of vortices with anti-vortices. In a fermionic picture of the discrete system we have that the spin is zero but the chiral number is 2. The discrete system is an example, in the classical world, of the spontaneous breaking of chiral symmetry.

The energy at the continuum limit of the model of discrete point-like vortices is, according to Eq.(10)

$$E = \frac{1}{2\pi} \int d^2x d^2x' \omega\left(\mathbf{x}\right) \ln\left(\frac{|\mathbf{x} - \mathbf{x}'|}{L}\right) \omega\left(\mathbf{x}'\right) \tag{13}$$

We now have a problem that is similar to that mentioned above, about the nature of point-like objects (vortices or charges or currents). This time the problem arises because the same expression of energy can be written for a Coulombian gas of charges of density $\rho\left(\mathbf{x}\right)$ in plane, by replacing $\omega\left(\mathbf{x}\right) \rightarrow \rho\left(\mathbf{x}\right)$. In this case however the interaction leads to motion that is along the line separating the charges. The direction of the relative motion of two interacting point-like objects (charges) is given by the gradient of the scalar function $\psi(\mathbf{x}) = \int d^2x' \ln(|\mathbf{x}-\mathbf{x}'|/L)\,\rho(\mathbf{x}')$. The same scalar potential is introduced for the point-like vortices (8). But there, two interacting vortices will move in directions that are perpendicular on the line that connects them, *i.e.* they tend to rotate one around the other. Then, for the system of point-like vortices, the Hamiltonian must be *supplemented* with the prescription that the velocity is like that of a charge in a magnetic field (or geostrophic)

$$\mathbf{v} = -\nabla\psi \times \widehat{\mathbf{e}}_z \tag{14}$$

or, equivalently, of the $E \times B$-type.

The preceding observations prove to be essential when we go to the FT formulation: first, the fact that the equations of motion for the point-like objects refer to *vortices* (not charges, currents, etc.) imposes to consider the non-Abelian representation of the objects, finally leading to mixed spinors. Second, the fact that the Hamiltonian must be supplemented with the prescription that the motion is purely kinematic (*i.e.* we derive directly the velocities from ψ as in Eq.(14)) and the velocity is $E \times B$ - type requires to adopt the Chern-Simons (CS) term in the Lagrangian of the system. The CS term supports the *vortical* content of the dynamics. To close the discussion about the contrast between the interacting vortices ($\mathbf{v} \sim -\nabla\psi \times \widehat{\mathbf{e}}_z$) and the interacting charges in plane (force $\sim \nabla\psi$), we note that the Lagrangian for the latter system does not include CS term and the asymptotic limit is the Landau-Ginzburg equation [17]. For the point-like vortices we must include CS term and the asymptotic equation is *sinh*-Poisson.

The FT model for the system of charges in plane moving according to the Eqs.(9) was formulated by Jackiw and Pi [6] having in view the application to the Fractional Quantum Hall Effect. The classical part has identified the absolute extremum of the action as stationary self-dual states, solutions of the Liouville equation. The non-Abelian extension of this model has been introduced and discussed by Dunne *et al.* for a gauge algebra $su(N)$, with N arbitrary [18], [19]. The $sl(2,\mathbf{C})$ Non-Abelian structure is necessary due to the *vortical* nature of the elementary object. Due to the extension of the space of particles (elementary vortices) with anti-particles (anti-vortices) requested by the parity invariance, the vorticity matter will need to be represented by a *mixed spinor*. By contrast, Jackiw and Pi obtain Liouville equation in the model of *scalar charges* evolving in plane.

The Lagrangian [19]

$$\mathcal{L} = -\kappa\varepsilon^{\mu\nu\rho}\mathrm{tr}\left(\left(\partial_\mu A_\nu\right)A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho\right) + i\mathrm{tr}\left(\phi^\dagger\left(D_0\phi\right)\right) - \frac{1}{2m}\mathrm{tr}\left(\left(D_k\phi\right)^\dagger\left(D^k\phi\right)\right) - V\left(|\phi|\right) \tag{15}$$

where $D_\mu = \partial_\mu + [A_\mu,]$ and κ, m are positive constants. The matter self-interaction potential is

$$V\left(|\phi|\right) = -\frac{g}{2}\mathrm{tr}\left(\left[\phi^\dagger,\phi\right]^2\right) \tag{16}$$

The Euler - Lagrange equations for the action functional $\mathcal{S} = \int dxdydt\mathcal{L}$ are the equations of motion

$$iD_0\phi = -\frac{1}{2m}D_kD^k\phi - g\left[\left[\phi,\phi^\dagger\right],\phi\right] \tag{17}$$

$$\kappa\varepsilon^{\mu\nu\rho}F_{\nu\rho} = iJ^\mu \tag{18}$$

where the current

$$J^0 = \left[\phi,\phi^\dagger\right] \tag{19}$$

$$J^k = -\frac{i}{2m}\left(\left[\phi^\dagger,\left(D^k\phi\right)\right] - \left[\left(D^k\phi\right)^\dagger,\phi\right]\right) \tag{20}$$

is covariantly conserved $D_\mu J^\mu = 0$. The energy density is

$$E = \frac{1}{2m}\mathrm{tr}\left(\left(D_k\phi\right)^\dagger\left(D^k\phi\right)\right) - \frac{g}{2}\mathrm{tr}\left(\left[\phi^\dagger,\phi\right]^2\right) \tag{21}$$

The Gauss law is the zero component of the second equation of motion

$$2\kappa F_{12} = iJ^0 = i\left[\phi,\phi^\dagger\right] \tag{22}$$

In the following we will use convenient combinations of variables: $A_\pm \equiv A_x \pm iA_y$, $\partial/\partial z = \frac{1}{2}\left(\partial/\partial x - i\partial/\partial y\right)$, $\partial/\partial z^* = \frac{1}{2}\left(\partial/\partial x + i\partial/\partial y\right)$, and similar. Writting

$$\mathrm{tr}\left(\left(D_k\phi\right)^\dagger\left(D^k\phi\right)\right) = \mathrm{tr}\left(\left(D_-\phi\right)^\dagger\left(D_-\phi\right)\right) - i\mathrm{tr}\left(\phi^\dagger\left[F_{12},\phi\right]\right) - m\varepsilon^{ij}\partial_i\left[\phi^\dagger\left(D_j\phi\right) - \left(D_j\phi\right)^\dagger\phi\right] \tag{23}$$

we replace in the expression of the energy density and note that for smooth fields we can ignore the last term, which is evaluated at the boundary

$$E = \frac{1}{2m}\mathrm{tr}\left((D_{-}\phi)^{\dagger} (D_{-}\phi) \right) + \left(-\frac{g}{2} + \frac{1}{4m\kappa} \right) \mathrm{tr}\left(\left[\phi^{\dagger}, \phi \right]^{2} \right) \tag{24}$$

For $\kappa = |\kappa|$ the choice of the constants

$$g = \frac{1}{2m\kappa} > 0 \tag{25}$$

permits to obtain the absolute minimum of the action (the SD states) and it will be adopted below. Later we will discuss the effect of not adopting Eq.(25). The states are *stationary* $\partial_0 \phi = 0$ and minimise the energy ($E = 0$). Adding the Gauss constraint we have a set of two equations for stationary states corresponding to the absolute minimum of the energy

$$D_{-}\phi = 0 \tag{26}$$

$$F_{12} = \frac{i}{2\kappa} \left[\phi, \phi^{\dagger} \right] \tag{27}$$

From these equations the *sinh*-Poisson equation is derived [18]. The states correspond to zero curvature in a formulation that involves the two dimensional reduction from a four dimensional Self - Dual Yang Mills system, as shown in [18]. Therefore we will denote this state as Self - Dual (SD). The functions ϕ and A_{μ} are mixed spinors, elements of the algebra $sl\,(2, \mathbf{C})$. Adopting the algebraic ansatz,

$$\phi = \phi_1 E_{+} + \phi_2 E_{-} \ , \ \phi^{\dagger} = \phi_1^{*} E_{-} + \phi_2^{*} E_{+} \tag{28}$$

and

$$A_{-} = aH \ , \ A_{+} = -a^{*} H \tag{29}$$

which is based on the three generators (E_{+}, H, E_{-}) of the Chevalley basis, the Gauss equation becomes

$$\frac{\partial a}{\partial x_{+}} + \frac{\partial a^{*}}{\partial x_{-}} = \frac{1}{k} (\rho_1 - \rho_2) \tag{30}$$

From the E_{+} respectively the E_{-} part of the first equation of motion $D_{-}\phi = 0$ we obtain

$$\frac{\partial \phi_1}{\partial z} + a\phi_1 = 0 \tag{31}$$

$$\frac{\partial \phi_2}{\partial z} - a\phi_2 = 0 \tag{32}$$

Using Eqs.(31) and its complex conjugate the left hand side of Eq.(30) becomes

$$\frac{\partial a}{\partial x_+}+\frac{\partial a^*}{\partial x_-}=-2\frac{\partial^2}{\partial z\partial z^*}\ln\left(|\phi_1|^2\right)=-\frac{1}{2}\Delta\ln\left(|\phi_1|^2\right)$$

The equation (30) becomes

$$-\frac{1}{2}\Delta\ln\rho_1=\frac{1}{\kappa}\left(\rho_1-\rho_2\right) \tag{33}$$

The Eq.(32) allows to express a and a^* in terms of ϕ_2. The left hand side of Eq.(30) becomes

$$\frac{\partial a}{\partial x_+}+\frac{\partial a^*}{\partial x_-}=2\frac{\partial^2}{\partial z\partial z^*}\ln\left(|\phi_2|^2\right)=\frac{1}{2}\Delta\ln\left(|\phi_2|^2\right)$$

The other form of Eq.(30) is

$$\frac{1}{2}\Delta\ln\rho_2=\frac{1}{\kappa}\left(\rho_1-\rho_2\right) \tag{34}$$

The right hand side in Eqs.(33) and (34) is the same and if we substract the equations we obtain

$$\Delta\ln\rho_1+\Delta\ln\rho_2=0 \tag{35}$$

This means $\rho_1\rho_2=\exp(\sigma)$ where σ is a *harmonic function*, $\Delta\sigma=0$. We take $\sigma\equiv 0$, leading to $\rho_1=\rho_2^{-1}\equiv\rho$ and introduce a scalar function ψ, defined by $\rho=\exp(\psi)$. Then the Eqs.(33) and (34) take the unique form

$$\Delta\ln\rho=-\frac{2}{\kappa}\left(\rho-\frac{1}{\rho}\right) \tag{36}$$

which is the *sinh*-Poisson equation (also known as the elliptic *sinh*-Gordon equation)

$$\Delta\psi+\frac{4}{\kappa}\sinh\psi=0 \tag{37}$$

The model describes correctly the Self-Duality states and identifies the asymptotic relaxation states of the fluid (known to be solutions of the *sinh*-Poisson equation [2]) with the *self - duality* states.

However we would like to examine the model when the system is not at self-duality. Then the energy is not zero and $\rho_1\rho_2\neq 1$. It is only at SD that we have the relationship $\rho_1=\rho_2^{-1}$ and we can use a single ψ. We can however define ω on the basis of the gauge field A^μ, as $\omega\sim F_{12}\sim F_{+-}$. Before the SD state is reached we see the gauge field as a velocity that carries the matter ϕ.

4 Parallel between the field theoretical and the statistical approaches

One cannot establish a simple mapping from the notions and operations in the fluid model (ψ,$\mathbf{v}$,ω), the point-like vortex model (x_i, y_i) and the field-theoretical model (ϕ, A_μ). In the following we note few suggestive connections.

4.1 The condition of consistency

For an arbitrary position (x, y) in plane, the sum of the contributions of all point-like vortices, propagated through $G(x, y; x', y')$ the Green functions of the Laplacian (*i.e.* the right hand side of the Eq.(9)) gives the velocity that would have a point-like vortex if it were placed in that point (x, y). Knowing the local space variation of this velocity one can calculate the vorticity in that particular point. On the other hand the density of point-like vortices in that particular point (positive and negative) also determines the vorticity. We then dispose of the vorticity in (x, y) calculated in two ways: from the rotational of the velocity derived from the contributions of all point-like vortices (excepting the current point (x, y) to avoid singularity), and, on the other hand, from the density of positive/negative point-like accumulations in a differential area around (x, y). The consistency imposes that these two values of vorticity are identical. For the discrete model this remains an imaginary exercise but in FT this compatibility is ensured by the Gauss law (or constraint) which is the second of the equations of motion of the FT model, obtained after functional variation to the time-like component of the gauge field $A_0(x, y)$. It expresses the fact that F_{12} , which is the magnetic field B or the rotational of the velocity, is equal to the zero-component, (the "charge" density) of the current, which is the difference $\rho_1 - \rho_2$ or the vorticity, at SD. A similar conclusion is arrived at by Montgomery 1993: self-consistency means that the "most probable" state generates the velocity field in which the vortices are convected.

The condition is $\frac{i}{2\kappa}J^0 = F_{12}$ which must be read in this order: the vorticity (the density of point-like positive/negative vortices, more generally J^0) is equal with the rotational of the velocity, *i.e.* the curvature of the connection A_μ.

4.2 None of the two kinds of point-like vortices in a point can be zero

In the discrete model the value of the vorticity in every cell is obtained as an unbalance between the positive and negative vortices

$$\omega_i = -\left(N_i^+ - N_i^-\right) \tag{38}$$

Joyce and Montgomery [15] find the relation

$$N_i^+ N_i^- = const \tag{39}$$

which means that in the same state i the number of positve vortices is the inverse of the number of negative vortices. The state i is actually the space position (x, y). This excludes the situation that one of $N_i^\pm$ can be zero. The same relationship is derived in the FT model [Eq.(35)]. This becomes at SD a property of invariance of the FT model to the inversion: $\rho \to 1/\rho$.

4.3 The energy

The energy of the fluid is

$$E = \frac{1}{2}\int d^2r \left|\nabla\psi\right|^2 = -\frac{1}{2}\int d^2r\ \omega\psi \tag{40}$$

If we simply translate this expression in terms of FT variables at SD it results

$$\begin{aligned} E^{FT} &= \frac{1}{\kappa}\int d^2r\ \left(\rho - \frac{1}{\rho}\right)\ln\rho \\ &= \frac{1}{\kappa}\int d^2r \left(\rho\ln\rho + \frac{1}{\rho}\ln\frac{1}{\rho}\right) \end{aligned} \tag{41}$$

which is connected with the entropy $S = 2\beta E$ of the discrete system but expressed in terms of the variable ρ,

$$S = \ln W = \sum_i \left(N_i^+ \ln N_i^+ + N_i^- \ln N_i^-\right) \tag{42}$$

and suggests the identifications $N_i^+ \to \rho$ and $N_i^- \to 1/\rho$ at SD.

4.4 The helicity in the FT description

The conventional helicity density is zero in $2D$: $\mathbf{v} \cdot \omega = 0$. However the Chern - Simons term in the Lagrangian carries a similar significance (one easily recognizes that the CS term generalizes the product $\mathbf{A} \cdot \mathbf{B}$, *i.e.* the helicity of a magnetic field configuration). At stationarity, as is SD, the Chern - Simons term becomes

$$-\kappa\varepsilon^{\mu\nu 0}\mathrm{tr}\left(\left(\partial_\mu A_\nu\right) A_0 + \frac{2}{3}A_\mu A_\nu A_0\right) = -\kappa\varepsilon^{ij}\mathrm{tr}\left(A_i \dot{A}_j\right) - \kappa\mathrm{tr}\left(A_0 F_{12}\right) \quad (43)$$

$$= -\kappa\mathrm{tr}\left(A_0 F_{12}\right) \quad (44)$$

and from the Gauss constraint (H is the Cartan generator)

$$A_0 = -\frac{i}{4m\kappa}\left[\phi, \phi^\dagger\right] = -\frac{i}{4m\kappa}\left(\rho_1 - \rho_2\right) H = \left(\frac{i}{8m}\omega\right) H \quad (45)$$

and $F_{12} \equiv F_{xy} = B = \left(-i\omega/4\right) H$. From (45) we note that A_0 is purely imaginary. The field B depends on the matter functions $\rho_{1,2}$ via the Gauss constraint

$$B = F_{12} = \frac{i}{2\kappa}\left[\phi, \phi^\dagger\right] = \left(-\frac{i}{4}\omega\right) H \quad (46)$$

with the last equality valid at SD. At stationarity

$$\mathcal{L}_{CS} = -\kappa\mathrm{tr}\left(A_0 F_{12}\right) = -\omega^2 \frac{\kappa}{16m} \quad (47)$$

This part of the action functional is related to the helicity of the field. We note however that it has the same nature as the matter field self-interaction (last term in the Lagrangian) which means that at SD the physical vorticity is actually represented by two distinct functions: using the matter field $\sim \left[\phi, \phi^\dagger\right]$ and respectively using the gauge field F_{12}.

4.5 The Entropy

The statistical approach (SA) to the discretized model uses the entropy of the gas of point-like vortices and looks for its extremum under the constraints of constant number of positive and negative vortices (separately) and of constant energy. To draw a parallel between the statistical approach and the FT model we write the *partition function* for the FT Lagrangian. Since the field theory is purely classical, a partition function has only a meaning if we have a statistical ensemble of realizations of the fields, due to either a random initialization or to an external random factor [20], [21]. Without an in-depth investigation, we just indicate below the possible mapping between the specific quantities in the two approaches

$$\begin{aligned} Z &= \int D[\phi] D\left[\phi^\dagger\right] D[A^\mu] D\left[A^{\mu\dagger}\right] \exp\left(i \int d^2x dt\ \mathcal{L}\right) \\ &= \int D[\phi] D\left[\phi^\dagger\right] D[A_+] D[A_-] \delta(\Phi) \\ &\quad \times \exp\left\{ i \int d^2x \left[4\rho_1 \left| \frac{\partial}{\partial z} \ln \phi_1 + a \right|^2 + 4\rho_2 \left| \frac{\partial}{\partial z} \ln \phi_2 - a \right|^2 \right] \right\} \end{aligned} \tag{48}$$

with Jacobian 1 for the change of variables $\left(A^\mu, A^{\mu\dagger}\right) \rightarrow (A_+, A_-) \rightarrow (a, a^*)$ and $\delta(\Phi)$ is the Dirac functional expressing the Gauss constraint, denoted for simplicity $\Phi(\phi, A_\mu) = 0$. The following associations are suggested

$$\frac{N!}{\prod_i N_i^+!} \rightarrow \int^{(1)} D[\phi_1] D[\phi_1^*] D[a] D[a^*] \exp\left\{ i \int d^2x\ 4\rho_1 \left| \frac{\partial}{\partial z} \ln \phi_1 + a \right|^2 \right\} \tag{49}$$

and

$$\frac{N!}{\prod_i N_i^-!} \rightarrow \int^{(2)} D[\phi_2] D[\phi_2^*] D[a] D[a^*] \exp\left\{ i \int d^2x\ 4\rho_2 \left| \frac{\partial}{\partial z} \ln \phi_2 - a \right|^2 \right\} \tag{50}$$

The upperscripts (1) and (2) have the meaning that the integrations extends over function sub-space restricted by the Gauss law, which means that the two integrals are not independent factors in the product leading to (48). The same is the case in Eq.(11) where the two factors are connected by the the constraints $\sum_i N_i^+ = N^+ = N$ and $\sum_j N_j^- = N^- = N$ and by fixed total energy E. The self-duality necessarily calls for the equality of total positive and total negative vorticities (see Appendix A).

The Gauss constraint is

$$\delta(\Phi) \equiv \delta\left[(\partial_+ a + \partial_- a^*) - \frac{1}{\kappa} (\rho_1 - \rho_2) \right] \tag{51}$$

The partition function is calculated taking the saddle point solution, which is equivalent with Eqs.(31) and (32) leading to the *sinh*-Poisson equation: the argument in (51) of the δ function vanishes.

In the Eq.(49) the left hand side is the number of the possible configurations that the system of N^+ indiscernable point-like objects can take in i states, *i.e.* with occupation numbers N_i^+. In the right hand side we have, at SD when the exponent is zero, the volume of the functional subspace formed by states that fulfill the first equation that leads to SD. The same is valid for the second equation, for N^-. The vacuum is the state with the energy of the discrete system as

$$N_i^+ = N_i^- \tag{52}$$

which corresponds to the *vacuum* in FT at $\rho_i = 1$. This is equivalent with *pairing* of opposite vortices.

5 The equations of the field theoretical model close to the self-dual states

5.1 The equations for the matter field components

The Euler - Lagrange equations resulting from the Lagrangian (15) are

$$iD_0\phi = -\frac{1}{2m}D_+D_-\phi - \frac{1}{4m\kappa}\left[\left[\phi,\phi^\dagger\right],\phi\right] \tag{53}$$

and (the Gauss constraint)

$$\kappa\varepsilon^{\mu\nu\rho}F_{\nu\rho} = iJ^\mu \tag{54}$$

The calculations are detailed in Appendix B. These equations are valid in general, not only at self - duality. In contrast to the latter they are difficult to study since an explicit solution is not available. We will try to investigate the equations in a regime that is close to the SD state. We retain the time dependence (which necessarly is slow close to stationarity $\partial_0 \to 0$) maintain ρ_1 and ρ_2 unrelated ($\rho_1\rho_2 = 1$ exists only at SD) and assume the same algebraic structure as for SD states (see Appendices C and D).

We start by examining what can be obtained from the Gauss constraint since it is always valid

$$F_{12} = \frac{i}{2\kappa}\left[\phi,\phi^\dagger\right] \tag{55}$$

It provides a formal expression for the gauge potential components $A_{x,y}$. Inserting the algebraic ansatz the left hand side is

$$\begin{aligned} F_{12} &= \partial_x A_y - \partial_y A_x + [A_x, A_y] && (56)\\ \overline{F}_{12} &= \partial_x \overline{A}_y - \partial_y \overline{A}_x && (57)\end{aligned}$$

where we denote by *bar* the amplitudes along the gauge group generator H, $A_\pm = \overline{A}_\pm H$ and their combinations. The Gauss constraint becomes an equation for the field of vectors $\overline{\mathbf{A}} \equiv \left(\overline{A}_x, \overline{A}_y\right)$

$$curl\overline{\mathbf{A}} = \frac{i}{2\kappa}\left(\rho_1 - \rho_2\right) \tag{58}$$

The general solution contains the rotational of a vector field, which we take $\frac{i}{4}g\widehat{\mathbf{e}}_z$ with g a scalar function, plus the gradient of another scalar function, $\frac{i}{2}h$.

$$\overline{A}_x = \frac{i}{4}\frac{\partial g}{\partial y} + \frac{i}{2}\frac{\partial}{\partial x}h \ , \ \overline{A}_y = -\frac{i}{4}\frac{\partial g}{\partial x} + \frac{i}{2}\frac{\partial}{\partial y}h \tag{59}$$

If the scalar function g is found such that

$$-\frac{1}{4}i\frac{\partial^2 g}{\partial x^2} - \frac{1}{4}i\frac{\partial^2 g}{\partial y^2} = \frac{i}{2\kappa}\left(\rho_1 - \rho_2\right) \tag{60}$$

or

$$\Delta g = -\frac{2}{\kappa}\left(\rho_1 - \rho_2\right) \tag{61}$$

then the Gauss law is verified and we dispose of formal expressions for $\overline{A}_{x,y}$ in terms of $\rho_1 - \rho_2$. What we have done is just to eliminate the gauge field components in view of reducing the problem to only the matter field equation, Eq.(53).

The equation of motion (53) is expanded and, matching the coefficients of each generator $E_\pm$ we obtain two equations for the scalar function $\phi_{1,2}$. This is shown in detail in Appendix C. The equation resulting from E_+.

$$\begin{aligned}
& i\frac{\partial \phi_1}{\partial t} - 2ib\phi_1 \\
= \quad & -\frac{1}{2}\frac{\partial^2 \phi_1}{\partial x^2} - \frac{1}{2}\left[\frac{\partial\left(a - a^*\right)}{\partial x}\phi_1 + \left(a - a^*\right)\frac{\partial \phi_1}{\partial x}\right] \\
& -\frac{1}{2}\frac{\partial \phi_1}{\partial x}\left(a - a^*\right) - \frac{1}{2}\left(a - a^*\right)^2\phi_1 \\
& -\frac{1}{2}\frac{\partial^2 \phi_1}{\partial y^2} - \frac{i}{2}\left[\frac{\partial\left(a + a^*\right)}{\partial y}\phi_1 + \left(a + a^*\right)\frac{\partial \phi_1}{\partial y}\right] \\
& -\frac{i}{2}\frac{\partial \phi_1}{\partial y}\left(a + a^*\right) + \frac{1}{2}\left(a + a^*\right)^2\phi_1 \\
& -\frac{1}{m\kappa}\left(\rho_1 - \rho_2\right)\phi_1
\end{aligned} \tag{62}$$

The equation resulting from E_-.

$$\begin{aligned} & i\frac{\partial \phi_2}{\partial t}+2ib\phi_2 \qquad (63)\\ = & -\frac{1}{2}\frac{\partial^2\phi_2}{\partial x^2}+\frac{1}{2}\left[\frac{\partial\left(a-a^*\right)}{\partial x}\phi_2+\left(a-a^*\right)\frac{\partial \phi_2}{\partial x}\right] \\ & +\frac{1}{2}\frac{\partial \phi_2}{\partial x}\left(a-a^*\right)-\frac{1}{2}\left(a-a^*\right)^2\phi_2 \\ & -\frac{1}{2}\frac{\partial^2\phi_2}{\partial y^2}+\frac{i}{2}\left[\frac{\partial\left(a+a^*\right)}{\partial y}\phi_2+\left(a+a^*\right)\frac{\partial \phi_2}{\partial y}\right] \\ & +\frac{i}{2}\frac{\partial \phi_2}{\partial y}\left(a+a^*\right)+\frac{1}{2}\left(a+a^*\right)^2\phi_2 \\ & +\frac{1}{m\kappa}\left(\rho_1-\rho_2\right)\phi_2 \end{aligned}$$

With them we will derive equations for the two amplitudes $\rho_{1,2}$ and also for their combinations $\rho_1 \pm \rho_2$. For this we first introduce explicit expressions for the two functions ϕ_1 and ϕ_2,

$$\phi_1 = \sqrt{\rho_1}\exp\left(i\chi\right)=\exp\left(\frac{\psi_1}{2}+i\chi\right) \qquad (64)$$

$$\phi_2 = \sqrt{\rho_2}\exp\left(i\eta\right)=\exp\left(\frac{\psi_2}{2}+i\eta\right) \qquad (65)$$

It is now useful to look for the SD case, such as to get an orientation of what will be the structure of the equations amenable to the SD state. At SD we have a unique ψ, $\rho_1=\exp\left(\psi\right)=\rho_2^{-1}$ and

$$a = -\frac{\partial}{\partial z}\ln\phi_1=-\frac{\partial}{\partial z}\left(\frac{\psi}{2}+i\chi\right) \qquad (66)$$

$$a = \frac{\partial}{\partial z}\ln\phi_2=\frac{\partial}{\partial z}\left(\frac{\psi}{2}+i\eta\right) \qquad (67)$$

From Eq.(29) the expressions of the gauge potentials at SD are

$$A_x = \frac{1}{2}\left(a-a^*\right)H=\frac{i}{2}\left(\frac{1}{2}\frac{\partial\psi}{\partial y}-\frac{\partial\chi}{\partial x}\right)H=\frac{i}{2}\left(-\frac{1}{2}\frac{\partial\psi}{\partial y}+\frac{\partial\eta}{\partial x}\right)H \qquad (68)$$

$$A_y = \frac{i}{2}\left(a+a^*\right)H=-\frac{i}{2}\left(\frac{1}{2}\frac{\partial\psi}{\partial x}+\frac{\partial\chi}{\partial y}\right)H=\frac{i}{2}\left(-\frac{1}{2}\frac{\partial\psi}{\partial x}+\frac{\partial\eta}{\partial y}\right)H \qquad (69)$$

$$A_0 = -\frac{i}{4m\kappa}\left[\phi,\phi^\dagger\right]=-\frac{i}{4m\kappa}\left(\rho_1-\rho_2\right)H=\left(\frac{i}{8m}\omega\right)H\equiv bH \qquad (70)$$

We get the indication that at SD the (x,y) gauge components are purely imaginary and the first contribution in each of them is the *curl* of $\psi\widehat{\mathbf{e}}_z$. This part is the physical velocity, $-\nabla\psi \times \widehat{\mathbf{e}}_z$, if ψ is the streamfunction. Since all components of the gauge potential are laying along the Cartan generator H in the space of the gauge algebra the convection $[A_{\pm},]$ part of the covariant derivative operator does not affect the algebraic content of the matter field, ϕ, assumed to be a combination of the other two generators.

Returning to Eqs.(62) and (63) we introduce the definitions

$$v_x^{(1)} = \frac{2\overline{A}_x}{i} + \frac{\partial \chi}{\partial x} \ , \ v_y^{(1)} = \frac{2\overline{A}_y}{i} + \frac{\partial \chi}{\partial y} \tag{71}$$

$$v_x^{(2)} = -\frac{2\overline{A}_x}{i} + \frac{\partial \eta}{\partial x} \ , \ v_y^{(2)} = -\frac{2\overline{A}_y}{i} + \frac{\partial \eta}{\partial y} \tag{72}$$

and taking into account that $b + b^* = 0$ we derive the equations for the difference and for the sum $\rho_1 \pm \rho_2$.

$$\frac{\partial}{\partial t}(\rho_1 - \rho_2) + \frac{\partial}{\partial x}\left[v_x^{(1)}\rho_1 - v_x^{(2)}\rho_2\right] + \frac{\partial}{\partial y}\left[v_y^{(1)}\rho_1 - v_y^{(2)}\rho_2\right] = 0 \tag{73}$$

and similarly

$$\frac{\partial}{\partial t}(\rho_1 + \rho_2) + \frac{\partial}{\partial x}\left[v_x^{(1)}\rho_1 + v_x^{(2)}\rho_2\right] + \frac{\partial}{\partial y}\left[v_y^{(1)}\rho_1 + v_y^{(2)}\rho_2\right] = 0 \tag{74}$$

(The calculations are presented in detail in Appendix D). These equations generalize those of the Abelian model of Ref.[6].

We also derive equations for the two functions $\rho_{1,2}$.

$$\frac{\partial}{\partial t}\rho_1 + div\left(\mathbf{v}^{(1)}\rho_1\right) = 0 \tag{75}$$

$$\frac{\partial}{\partial t}\rho_2 + div\left(\mathbf{v}^{(2)}\rho_2\right) = 0 \tag{76}$$

5.2 The velocity fields

The first velocity field

$$\mathbf{v}^{(1)} = \frac{1}{2}\nabla g \times \widehat{\mathbf{e}}_z + \nabla(h+\chi) \tag{77}$$

and the second velocity field

$$\mathbf{v}^{(2)} = -\frac{1}{2}\nabla g \times \widehat{\mathbf{e}}_z + \nabla(-h+\eta) \tag{78}$$

differ by the phases of the functions ϕ_1 and ϕ_2, *i.e.* by χ and η. We try to learn more about the velocity fields $\mathbf{v}^{(1,2)}$ by taking the limit to SD.

The formal solutions of the equation

$$curl\overline{\mathbf{A}} = \frac{i}{2\kappa}\left(\rho_1 - \rho_2\right)$$

is expressed as

$$\begin{aligned}\overline{A}_x &= -\frac{\partial}{\partial y}\int d\mathbf{r}' G\left(\mathbf{r}-\mathbf{r}'\right)\left\{\frac{i}{2\kappa}\left[\rho_1\left(\mathbf{r}',t\right) - \rho_2\left(\mathbf{r}',t\right)\right]\right\} \\ &\quad + gauge\ term\end{aligned} \tag{79}$$

$$\begin{aligned}\overline{A}_y &= \frac{\partial}{\partial x}\int d\mathbf{r}' G\left(\mathbf{r}-\mathbf{r}'\right)\left\{\frac{i}{2\kappa}\left[\rho_1\left(\mathbf{r}',t\right) - \rho_2\left(\mathbf{r}',t\right)\right]\right\} \\ &\quad + gauge\ term\end{aligned} \tag{80}$$

and at SD, where we have a unique ψ,

$$\rho_1\left(\mathbf{r}',t\right) - \rho_2\left(\mathbf{r}',t\right) \rightarrow \frac{-\kappa}{2}\omega\left(x,y\right) = -\frac{\kappa}{2}\Delta\psi\left(x,y\right)$$

We can choose the gauge terms such as to cancel the gradients in Eqs.(59). Alternatively we can use Eq.(68)

$$v_x^{(1)} = \frac{2}{i}\overline{A}_x + \frac{\partial\chi}{\partial x} \rightarrow \frac{1}{2}\frac{\partial\psi}{\partial y}\ ,\ v_y^{(1)} = \frac{2}{i}\overline{A}_y + \frac{\partial\chi}{\partial x} \rightarrow -\frac{1}{2}\frac{\partial\psi}{\partial x} \tag{81}$$

Similarly for the second velocity field

$$v_x^{(2)} = -\frac{2}{i}\overline{A}_x + \frac{\partial\eta}{\partial x} \rightarrow -\frac{1}{2}\frac{\partial\psi}{\partial y}\ ,\ v_y^{(2)} = -\frac{2}{i}\overline{A}_y + \frac{\partial\eta}{\partial y} \rightarrow \frac{1}{2}\frac{\partial\psi}{\partial x} \tag{82}$$

At SD both velocity fields become divergenceless $\nabla\cdot\mathbf{v}^{(1)} = 0$, $\nabla\cdot\mathbf{v}^{(2)} = 0$ and they are opposite

$$\mathbf{v}^{(2)} = -\mathbf{v}^{(1)} \tag{83}$$

If we assume that these properties are approximately fulfilled in the states close (but not at) SD, we get

$$\frac{\partial}{\partial t}\left(\rho_1 - \rho_2\right) + \frac{\partial}{\partial x}\left[v_x^{(1)}\left(\rho_1 + \rho_2\right)\right] + \frac{\partial}{\partial y}\left[v_y^{(1)}\left(\rho_1 + \rho_2\right)\right] \approx 0 \tag{84}$$

and respectively

$$\frac{\partial}{\partial t}\left(\rho_1 + \rho_2\right) + \frac{\partial}{\partial x}\left[v_x^{(1)}\left(\rho_1 - \rho_2\right)\right] + \frac{\partial}{\partial y}\left[v_y^{(1)}\left(\rho_1 - \rho_2\right)\right] \approx 0 \tag{85}$$

After replacing the SD expression of $\mathbf{v}^{(1)}$ and taking into account that at SD $\rho_1 = \exp\left(\psi\right)$, $\rho_2 = \exp\left(-\psi\right)$, we see that both equations become a simple statement of the stationarity $\partial\left(\rho \pm 1/\rho\right)/\partial t = 0$.

5.3 The current of the matter field

The expressions of the matter current will help us to prove that the FT reproduces in the continuum limit the equations of the point-like vortices Eqs.(9). In field theory J^{μ} is calculated according to standard procedures

$$J^0 = \left[\phi, \phi^{\dagger}\right] \tag{86}$$

$$J^i = -\frac{i}{2m}\left(\left[\phi^{\dagger}, D_i\phi\right] - \left[(D_i\phi)^{\dagger}, \phi\right]\right) \tag{87}$$

Using the *algebraic ansatz for* ϕ and A_{μ} we obtain the following expressions

$$\begin{aligned} m\overline{J}^x &= -\rho_1 \frac{\partial \chi}{\partial x} + \rho_2 \frac{\partial \eta}{\partial x} + i(a - a^*)(\rho_1 + \rho_2) \\ &= -\rho_1 \frac{\partial \chi}{\partial x} + \rho_2 \frac{\partial \eta}{\partial x} - \frac{2\overline{A}_x}{i}(\rho_1 + \rho_2) \end{aligned} \tag{88}$$

$$\begin{aligned} m\overline{J}^y &= -\rho_1 \frac{\partial \chi}{\partial y} + \rho_2 \frac{\partial \eta}{\partial y} - (a + a^*)(\rho_1 + \rho_2) \\ &= -\rho_1 \frac{\partial \chi}{\partial y} + \rho_2 \frac{\partial \eta}{\partial y} - \frac{2\overline{A}_y}{i}(\rho_1 + \rho_2) \end{aligned} \tag{89}$$

$$\overline{J}^0 = \rho_1 - \rho_2 \tag{90}$$

in which the gauge potentials $\overline{A}_{x,y}$ appear. The detailed calculations are in the Appendices E and F.

We now examine these expressions close to SD. From the first equation of self-duality, $D_-\phi = 0$ we obtain the combinations of a and a^* as

$$a + a^* = -\frac{1}{2}\frac{\partial \psi}{\partial x} - \frac{\partial \chi}{\partial y} \tag{91}$$

$$a - a^* = i\left(\frac{1}{2}\frac{\partial \psi}{\partial y} - \frac{\partial \chi}{\partial x}\right) \tag{92}$$

Further, we take $\rho_1 \to \exp(\psi)$ and $\rho_2 \to \exp(-\psi)$. At SD the phases of ϕ_1 and ϕ_2 are opposite $\chi = -\eta$. Then it is obtained, close to SD

$$m\overline{J}^x \approx -(\rho_1 + \rho_2)\, v_x^{(1)} = -\frac{\partial}{\partial y}\frac{1}{2}(\rho_1 - \rho_2) = \frac{\kappa}{4}\frac{\partial}{\partial y}\omega \tag{93}$$

and

$$m\overline{J}^y \approx -(\rho_1 + \rho_2)\, v_y^{(1)} = \frac{\partial}{\partial x}\frac{1}{2}(\rho_1 - \rho_2) = -\frac{\kappa}{4}\frac{\partial}{\partial x}\omega \tag{94}$$

To this we have to add

$$\overline{J}^0 \approx \rho_1 - \rho_2 = -\frac{\kappa}{2}\omega \tag{95}$$

The formulas can be written in the form

$$\overline{J}^x \rightarrow -\frac{1}{2m}\left(\rho_1 + \rho_2\right)\frac{\partial \psi}{\partial y} \tag{96}$$

$$\overline{J}^y \rightarrow \frac{1}{2m}\left(\rho_1 + \rho_2\right)\frac{\partial \psi}{\partial x} \tag{97}$$

We note that these expression for $\overline{J}^{x,y} / \left(\rho_1 + \rho_2\right)$ coincide at SD with Eqs.(9).

6 Discussion

Detailed calculations regarding the properties of the velocity fields and the currents can be found in Appendixes A to F. One may find that the field-theoretical formulation of the $2D$ Euler fluid has a consistent background that justifies applications and/or extension.

6.1 Few comments

The FT is based on a dual representation of the same physical object: the vorticity. It is the density of matter $J^0 = \left[\phi, \phi^\dagger\right]$ and is the magnetic field $F_{12} = B \sim \left[\phi, \phi^\dagger\right]$; the Gauss law constrains them to be equal. This representation unfolds the nonlinearity of Eq.(1) but expresses it in a different way: the gauge-field-induced repulsion between elements of vorticity (part of the kinetic energy) is balanced by the two-body δ-function attraction represented by the last term in the Lagrangian (it is true for vortices of each sign; in addition, we must have made the option Eq.(25)). This permits that at self-duality the differential degree in the equations of motion to be decreased: the first SD equation (26) is *first-order* differential in contrast with Eq.(17) which is second order.

The FT reveals that the essential nature of self-organization is topological. Less visible in the case of the (present) Euler model, it is explicit in the FT models for fluids of single-sign vorticity (leading to the Liouville equation), etc. where the asymptotic states are mappings between compact manifolds and the energy is bounded from below by an integer multiple of the magnetic flux of a single vortex. Since $B \sim \omega$ the suggestion is clear: only the vorticity can self-organize, the combinations like the potential vorticity do not have this property. Essentially B and ω are flux-like quantities, we must think to them as $Bdx \wedge dy$ and $\omega dx \wedge dy$, *i.e.* they are differential two-forms. The integral over the plane is the degree of the topological mappings mentioned above. We note however that for the fluids with short range interaction like $2D$ plasma and the $2D$ atmosphere the self-organization (inherited from ω) is approximative and the potential vorticity dominates the dynamics via Ertel's theorem.

6.2 The approach to SD through states where the parameters do not obey the constraint Eq.(25)

The CS term and the matter self-interaction term combine to give a contribution to the energy, the second term in Eq.(21). When the parameters (coefficients of the CS respectively matter self-interaction terms) are not chosen as in Eq.(25) the energy of the system is non-zero even if we take the SD condition $D_{-}\phi = 0$. Approaching the SD state means that these two parameters must progressively become equal. Compared with the preceding part of this work, this gives another meaning to "being close to self-duality" but a FT description still remains to be elaborated. Few qualitative aspects of such a FT description are however available and we draw a parallel with the evolution of the physical fluid in the late phases of approaching stationary and coherent flow solutions of Eq.(2).

As is well known (and reviewed in the Introduction) in the late phase of fluid relaxation (equivalently, vorticity self-organization) the process of separation of opposite-sign elements of vorticity and coalescence of like-sign has led to formation of mesoscopic vortices of both signs. Their motion in plane is much slower than the rate of rotation of the fluid on the closed streamlines. The FT equivalent is that the energy term

$$\delta E \equiv \left(-\frac{g}{2} + \frac{1}{4m\kappa} \right) \mathrm{tr} \left(\left[\phi^{\dagger}, \phi \right]^{2} \right) \tag{98}$$

is very small. The merging of mesoscopic vortices is possible due to dissipation-mediated reconnections of streamlines. In the physical fluid, in such an event part of the energy is lost by dissipation and part of the energy related to the motion of the centres of the mesoscopic vortices that merge, is transferred to motion on streamlines. In FT we must see the term (98) approaching zero.

When the two parameters are not equal there is interaction between vortices. This has been studied for similar FT systems ([22], [19], [23], [24]). When the system is very close to SD one assumes that the mesoscopic vortices are not too different of the exact SD vortices. Then one inserts exact solutions of Eq.(2) into the expression of the energy (21), without assuming SD (Eq.(26) and (25)). Taking as parameters the positions of the centers of these exact SD vortices, it is possible to determine the force of interaction from variation of the energy to these parameters. The result depends decisively on the sign of the term (98). It is also possible to derive the relative motion of the vortices from their geodesic flow on the manifold generated by the positions in plane [25]. This argument works for several FT systems but the application to the present case is not straightforward: we have both positive and negative vortices and the energy is bounded from below by $E = 0$. We anticipate a more careful analysis and just mention the argument for the present case. At SD (*i.e.* $g - 1/(2m\kappa) = 0$) the total energy is zero and the solution consists of a dipole. This exact solution approximates the one of the phase just before reaching SD, when the field consisted of two mesoscopic vortices of opposite signs, in slow relative motion. We note that when $\delta E < 0$ (in Eq.(98)) this supplementary energy being negative means that there is attraction between vortices. We say that there is a predominance of the CS term (κ is large) from which it arises the second term in the paranthesis. Qualitatively, we say that the evolution toward SD must involve a decay of this attraction energy, *i.e.* at every reconnection a certain amount of the absolute magnitude of the CS term ($\sim$ helicity) must be removed. Since we know that at SD the CS part in the Lagrangian is

$$\mathcal{L}_{CS}^{SD} = -\kappa \mathrm{tr}\left(A_0 F_{12}\right) = -\kappa \frac{1}{16m} \omega^2$$

we can reformulate, saying that at every reconnection event a certain amount of enstrophy is removed. This seems to be compatible with the numerical simulations, where the evolution toward order is associated with decrease of the enstrophy.

We understand that the approach to SD and suppression of (98) implies the decrease of the topological content that is due to the Chern-Simons term. This is mediated by dissipative mechanisms which are missing from the basic formulation (15). We can get a hint on the necessary extension of the model from the *baryogenesis*, which involves the change of Chern-Simons topological number by transitions between states with different topological content [26], [27]. A simple application is prevented by the absence of the Higgs vacua and implicitely of the *sphaleron* solutions. This study is underway.

6.3 The conformal transformation as mappings between solutions of the FT equations of motion

The FT model inherits the conformal invariance of the the $2D$ Euler fluid (1): there is no intrinsic length in the physical system and the length of the side of the box L is just an arbitrary parameter. The Lagrangian (15) is invariant to conformal transformations [18], [28], [19] and their generators verify the following relation (t is the time)

$$\mathcal{E}t^2 - 2\mathcal{D}t + \mathcal{K} > 0$$

where $\mathcal{E}$ is energy *i.e.* the integral of Eq.(24), $\mathcal{D}$ and $\mathcal{K} > 0$ are generators of the dilation and special conformal transformations, $\mathbf{x} \rightarrow \mathbf{x}/(1 + at)$, where $a =$ const., explained in Ref.[18]. The conformal transformations allow to find new, time-dependent solutions of the equations of motion (17), starting from the static solutions of the SD equation (2).These new solutions have energy $E > 0$ which means that they cannot spontaneously evolve from the static SD solutions without an external input of energy. Each conformal transformation is a map in the function space connecting solutions of (17). It is not a necessary dynamic change of the behavior of the system but, since each function obtained by the conformal transformation is an extremum of the action, the path in the function space connecting such solutions is the most economic way for the system to access a particular type of behavior.

As noted in [29] when $\mathcal{E} > 0$ and $\mathcal{D} > \sqrt{\mathcal{E}\mathcal{K}}$ there is a finite time t^* such that for $t \rightarrow t^*$ the amplitude of the solution ϕ becomes zero anywhere on the plane with the exception of $r = 0$ where diverges. In particular, when the system is initialized in this region of parameters ($\mathcal{E} = 0, \mathcal{D} > 0, \mathcal{K} > 0$) the two opposite-sign vortices evolve to cuasi-singular concentrated spikes. When there is no spontaneous evolution toward singularity, we note that, for a one-dimensional solution of (2), the profile of $\psi(x)$ can be mapped to another solution $\psi'(x,t)$ which, for fixed t and $a > 0$ is more narrow, closer to the symmetry axis $x = 0$. The velocity $v'_y(x,t) = -d\psi'/dx$ is higher so there is need of energy for the system to evolve from the static solution to the time-dependent one. The shear increases and, with just small external drive, the sheared layer can evolve to onset of the Kelvin-Helmholtz instability.

6.4 The dynamics of the $2D$ physical fluid and its FT model

The ideal incompressible fluid in two dimensions evolves from a turbulent initial state to a stationary, highly ordered flow pattern via mergings of vortices and concentration of the vorticities of both signs into separate large scale vortices. The evolution has two components:

(1) isotopological motion with preservation of all streamlines and exact conservation of the energy

(2) fast events consisting of breaking up and reconnection of streamlines leading to change in topology of the flow. In particular merging of vortices *i.e.* generation of larger scale flow from two smaller vortices at their encounter is only possible by reconnection. A dissipative mechanism is necessary like molecular viscosity or collisions. However the amount of energy that is lost (by heat) in this way is very small and the total energy is approximately conserved. The events of reconnections (equivalently: the dissipative events) take place in a set with very small measure [30]. The main importance of reconnections is obviously the topological re-arrangement they make possible. In this way the system get closer to the state of SD which has a simple topological structure [2].

If we exclude any dissipative process and initialise the state such that its energy is not minimal (zero at SD) the fluid will continue to move, never reaching stationarity. This happens because the processes that would allow the system to access states of lower energy, and finally the lowest of all, the SD state, are forbidden since reconnections are not allowed.

For very small positive energy the system has only few mesoscopic vortices moving slowly as this state only precedes the full organization into the stationary vortex dipole solution of Eq.(2). Then the motion can be seen as consisting of the fast rotation in the vortices and the slow displacement of their centres. In the energy-plateau states of isotopological motion (between two reconnection events) the system creates accumulation of streamlines in few narrow regions and these generate conditions favorable for reconnection. The narrow regions are characterised by high values of the gradients of vorticity and any dissipation, if exists, will be easier exploited to start a reconnection event. The asymptotic SD state has all motion in the vortical rotation with no displacement of the centres of vortices.

The action functional reduces at stationarity to the square of an expression of (ϕ, A_μ) and the states extremizing the action are identified by taking to zero this expression. They are characterised by equality of the total amount of positive and negative vorticity, although the Lagrangian does not include this explicitely. By comparison, the statistical approach based on the variational treatment of the entropy must impose these properties and include them via Lagrange multipliers supplementing the entropy functional extremization.

Regarding the *negative temperature* determined in Taylor [31], it has been shown by Joyce and Montgomery [15] and by Edwards and Taylor [14] that the threshold energy is $E = 0$ and for any *positive* energy the temperature is negative. The FT model finds indeed that the SD state has $E = 0$ which must be interpreted as follows: the energy corresponds to the situation where there is no motion of the centres of the remaining vortices (the dipole) and the only motion is rotation along the streamlines of the two vortices. Since the system of point-like vortices is purely kinematic, the energy of the displacement along the streamlines is zero. It means that the only change in the matter function ϕ is given by the phase modification which is due to the potential $A_{x,y}$. This corresponds to the rotation of the fluid on the streamlines of the dipole. Is just an indefinite increase of the angular phase and this is expressed by $D_{-}\phi = 0$.

7 Conclusions

The field theoretical formalism for the Euler fluid finds that the asymptotic, highly organized, states are due to the property of self-duality. It derives in a very transparent way the *sinh*-Poisson equation. It implies that all other states, either with $E \neq 0$ or non-doubly periodic or with $\int d^2r\omega \neq 0$ cannot be stationary.

The fact that the asymptotic states exist due to the self-duality (as shown by the field theoretical formulation) may help to better understand the universal character of the vorticity concentration [32], [33]. In fluids with similar properties ($2D$ atmosphere, plasma in magnetic field) highly organized flows are observed [34]. We must remember that the evolution of the $2D$ Euler fluid to the coherent flow pattern [solution of Eq.(2)] takes place in the absence of gradients of pressure, of gradients of temperature, of buoyancy, of centrifugal forces, etc. Nothing was needed for the vorticity separation and concentration, except for the nature of the nonlinearity which supports inverse cascade, *i.e.* the intrinsic tendency to self-organization of the flow toward large scales. This process is similar to the Widom - Rowlinson phase transition by its universality and by the fact that besides the equation itself the input is quasi-inexistent. When formation of structures is described, as for example the tropical cyclones and tornadoes in atmosphere or the convection cells in plasma, etc. the necessary use of the conservation laws as dynamical equations should not make us to forget that inside the final pattern of flow there is also a *universal* structure. This tendency to self-organization is revealed or made more visible at relaxation but it does not depend on any particular circumstance. Also, the drive and dissipation in real systems can alter substantially the structure and actually can dominate the system's behavior but there is no way to simply suppress the tendency to self-organization, which will always be present. We may neglect the self-organization, on quantitative basis, but we should not ignore it [35], [36], [37].

Although the field theoretical formulation of the $2D$ Euler fluid proposes an interesting perspective on the fluid dynamics, it also has limitations: it cannot (simply) accomodate dissipation therefore the evolution of the FT variables actually reproduces isotopological motions of the fluid. If the energy in the initial state is not zero the FT system does not reach self-duality and the *sinh*-Poisson solutions.

The interest for the FT formulation also comes from the developments that it suggests: the connection with the Constant Mean Curvature (CMC) surfaces (a flow in the SD state has an associated CMC surface); the representation of the fluid "contour dynamics" as sections in a Riemann surface which is the solution of a supersymmetric extension of the model; the role of the Anti-de Sitter metrics in associating to the ideal fluid the geometric-algebraic structure that underlies the self-duality; etc. All these are certainly attractive fields of investigation.

Acknowledgement This work has been partly supported by the grant ERC - Like 4/2012 of UEFISCDI Romania.

Appendices

A Appendix A. The condition of zero total vorticity

In the statistical approach (SA) it is adopted from the start the condition that the total number of positive vortices equals the total number of negative vortices

$$N^{+} \equiv \sum_{i} N_{i}^{+} = \text{const} \quad , \quad N^{-} \equiv \sum_{i} N_{i}^{-} = \text{const} \tag{A.1}$$

and the *balance*

$$N_{+} = N_{-} \tag{A.2}$$

This is equivalent to the assumption that in the surface of interest the total amount of vorticity is zero. In FT there is no such assumption from the beginning and we can inquire if the system identifies as extremum (the SD state) the same situation *i.e.* zero total vorticity

$$\int d^2r \ \omega = 0 \tag{A.3}$$

This would mean

$$\int d^2r \ \rho_1 = \int d^2r \ \rho_2 \tag{A.4}$$

at SD, where $\rho_1 = \rho = \exp(\psi)$ and $\rho_2 = \rho^{-1} = \exp(-\psi)$. We consider that the sign of κ is fixed and from the equation at SD

$$\omega + \frac{2}{\kappa}\left(\rho - \frac{1}{\rho}\right) = 0 \tag{A.5}$$

we obtain in the regions where $\kappa\omega = +|\kappa\omega|$

$$\rho^+ = \frac{1}{4}\left(-|\kappa\omega| + \sqrt{|\kappa\omega|^2 + 16}\right) \tag{A.6}$$

with only the *positive* root $\rho \equiv \exp(\psi)$ being retained. The upperscript means that the result is valid in the regions with positive vorticity. In the same regions $1/\rho^+ = \exp(-\psi)$ is

$$\frac{1}{\rho^+} = \frac{1}{4}\left(|\kappa\omega| + \sqrt{|\kappa\omega|^2 + 16}\right) \tag{A.7}$$

In the regions where the vorticity is negative $\kappa\omega = -|\kappa\omega|$ we have, taking the positive root

$$\rho^- = \frac{1}{4}\left(|\kappa\omega| + \sqrt{|\kappa\omega|^2 + 16}\right) \tag{A.8}$$

and the inverse

$$\frac{1}{\rho^-} = \frac{1}{4}\left(-|\kappa\omega| + \sqrt{|\kappa\omega|^2 + 16}\right) \tag{A.9}$$

We have to prove Eq.(A.4), *i.e.*

$$\int d^2r \ \rho - \int d^2r \ (1/\rho) = 0 \tag{A.10}$$

Writting such as to exhibit the domains $\kappa\omega \gtrless 0$,

$$\int^+ d^2r \ \rho^+ + \int^- d^2r \ \rho^- = \int^+ d^2r \ \frac{1}{\rho^+} + \int^- d^2r \ \frac{1}{\rho^-} \tag{A.11}$$

we have

$$\int^{+} d^2r \frac{1}{4}\left(-|\kappa\omega| + \sqrt{|\kappa\omega|^2+16}\right) + \int^{-} d^2r \frac{1}{4}\left(|\kappa\omega| + \sqrt{|\kappa\omega|^2+16}\right)$$
$$= \int^{+} d^2r \frac{1}{4}\left(|\kappa\omega| + \sqrt{|\kappa\omega|^2+16}\right) + \int^{-} d^2r \frac{1}{4}\left(-|\kappa\omega| + \sqrt{\omega^2+16}\right)$$

where the upper sign at the integrals labels the regions where $\kappa\omega$ is positive respectively negative. After cancellations

$$\int^{+} d^2r \left(-\frac{1}{2}|\omega|\right) + \int^{-} d^2r \left(\frac{1}{2}|\omega|\right) = 0 \tag{A.12}$$

and this indeed means that the integrals of the vorticity over the region where it is positive equals the integral of the vorticity over the region where it is negative

$$\int^{+} d^2r \left(\frac{1}{2}|\omega|\right) = \int^{-} d^2r \left(\frac{1}{2}|\omega|\right) \tag{A.13}$$

equivalent with

$$N^{+} = \sum_i N_i^{+} = N^{-} = \sum_i N_i^{-} \tag{A.14}$$

In other words the SD gives that the total vorticity in the field is *zero*. We note that in FT this is not an assumption but a result.

B Appendix B. Derivation of the equations of motion

The Lagrangian of the model is

$$\begin{aligned} L &= -\kappa\varepsilon^{\mu\nu\rho}\mathrm{tr}\left(\left(\partial_\mu A_\nu\right)A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho\right) \\ &\quad + i\mathrm{tr}\left(\phi^\dagger\left(D_0\phi\right)\right) - \frac{1}{2m}\mathrm{tr}\left(\left(D_k\phi\right)^\dagger\left(D^k\phi\right)\right) \\ &\quad + \frac{1}{4m\kappa}\mathrm{tr}\left(\left[\phi,\phi^\dagger\right]^2\right) \end{aligned} \tag{B.1}$$

where

$$D_\mu = \partial_\mu + [A_\mu, \,] \tag{B.2}$$

and the metric is

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{B.3}$$

B.1 Preparation for the derivation of the equation of motion equivalent to the Gauss constraint

B.1.1 The Chern-Simons term

This part is presented in detail in [38] and here we only mention the principal steps. The Chern - Simons part of the gauge Lagrangean is

$$\mathcal{L}_{CS} = -\frac{1}{2}\kappa\varepsilon^{\mu\nu\rho}\mathrm{tr}\left(A_\mu\left(\partial_\nu A_\rho - \partial_\rho A_\nu\right) + \frac{2}{3}A_\mu\left[A_\nu, A_\rho\right]\right) \tag{B.4}$$

and expanded

$$\begin{aligned} \mathcal{L}_{CS} &= -\kappa\mathrm{tr}\left\{A_0\left(\partial_1 A_2\right) - A_0\left(\partial_2 A_1\right) - A_1\left(\partial_0 A_2\right)\right. \\ &+A_1\left(\partial_2 A_0\right) - A_2\left(\partial_1 A_0\right) + A_2\left(\partial_0 A_1\right) \\ &+\frac{2}{3}A_0A_1A_2 - \frac{2}{3}A_0A_2A_1 - \frac{2}{3}A_1A_0A_2 \\ &\left.+\frac{2}{3}A_1A_2A_0 - \frac{2}{3}A_2A_1A_0 + \frac{2}{3}A_2A_0A_1\right\} \end{aligned} \tag{B.5}$$

Using the properties of the Trace operator we obtain

$$\begin{aligned} \mathcal{L}_{CS} &= -\kappa\mathrm{tr}\left\{A_0\left(\partial_1 A_2\right) - A_0\left(\partial_2 A_1\right) - A_1\left(\partial_0 A_2\right)\right. \\ &+A_1\left(\partial_2 A_0\right) - A_2\left(\partial_1 A_0\right) + A_2\left(\partial_0 A_1\right) \\ &\left.+2A_0A_1A_2 - 2A_0A_2A_1\right\} \end{aligned} \tag{B.6}$$

or

$$\begin{aligned} \mathcal{L}_{CS} &= -\kappa\mathrm{tr}\left\{-A_1\partial_0 A_2 + A_2\partial_0 A_1 + 2A_0\partial_1 A_2 - 2A_0\partial_2 A_1\right. \\ &\left.+2A_0A_1A_2 - 2A_0A_2A_1\right\} \end{aligned} \tag{B.7}$$

This will be used for functional derivatives of the Lagrangian density.

B.1.2 The matter Lagrangean

This part is

$$\mathcal{L}_m = i\mathrm{tr}\left(\phi^\dagger\left(D_0\phi\right)\right) - \frac{1}{2m}\mathrm{tr}\left(D_\kappa\phi\right)^\dagger\left(D^k\phi\right) \tag{B.8}$$

$$\equiv \mathcal{L}_m^{(1)} + \mathcal{L}_m^{(2)} \tag{B.9}$$

The first term is

$$\begin{aligned} \mathcal{L}_m^{(1)} &= i\mathrm{tr}\left(\phi^\dagger\left(D_0\phi\right)\right) \\ &= i\mathrm{tr}\left\{\phi^\dagger\left(\frac{\partial\phi}{\partial t} + \left[A_0, \phi\right]\right)\right\} = i\mathrm{tr}\left(\phi^\dagger\frac{\partial\phi}{\partial t} + \phi^\dagger A_0\phi - \phi^\dagger\phi A_0\right) \end{aligned}$$

and this is the form that we will use for functional variation to A_0.

Now the other term

$$\begin{aligned}\mathcal{L}_m^{(2)} &\equiv -\frac{1}{2m}\mathrm{tr}\left[\left(D^k\phi\right)^\dagger\left(D_k\phi\right)\right] \\ = & -\frac{1}{2m}\mathrm{tr}\left[\left(\frac{\partial\phi^\dagger}{\partial x}+\phi^\dagger A^{1\dagger}-A^{1\dagger}\phi^\dagger\right)\left(\frac{\partial\phi}{\partial x}+A_1\phi-\phi A_1\right)\right. \\ & \left.+\left(\frac{\partial\phi^\dagger}{\partial y}+\phi^\dagger A^{2\dagger}-A^{2\dagger}\phi^\dagger\right)\left(\frac{\partial\phi}{\partial y}+A_2\phi-\phi A_2\right)\right]\end{aligned} \tag{B.10}$$

We expand the products

$$\begin{aligned}\mathcal{L}_m^{(2)} = & -\frac{1}{2m}\mathrm{tr}\left\{\frac{\partial\phi^\dagger}{\partial x}\frac{\partial\phi}{\partial x}+\frac{\partial\phi^\dagger}{\partial x}A_1\phi-\frac{\partial\phi^\dagger}{\partial x}\phi A_1\right. \\ & +\phi^\dagger A^{1\dagger}\frac{\partial\phi}{\partial x}+\phi^\dagger A^{1\dagger}A_1\phi-\phi^\dagger A^{1\dagger}\phi A_1 \\ & -A^{1\dagger}\phi^\dagger\frac{\partial\phi}{\partial x}-A^{1\dagger}\phi^\dagger A_1\phi+A^{1\dagger}\phi^\dagger\phi A_1 \\ & +\frac{\partial\phi^\dagger}{\partial y}\frac{\partial\phi}{\partial y}+\frac{\partial\phi^\dagger}{\partial y}A_2\phi-\frac{\partial\phi^\dagger}{\partial y}\phi A_2 \\ & +\phi^\dagger A^{2\dagger}\frac{\partial\phi}{\partial y}+\phi^\dagger A^{2\dagger}A_2\phi-\phi^\dagger A^{2\dagger}\phi A_2 \\ & \left.-A^{2\dagger}\phi^\dagger\frac{\partial\phi}{\partial y}-A^{2\dagger}\phi^\dagger A_2\phi+A^{2\dagger}\phi^\dagger\phi A_2\right\}\end{aligned} \tag{B.11}$$

and this form will be used in the functional derivations.

B.2 The Euler-Lagrange equations for the gauge field

The Euler-Lagrange equations

$$\frac{\partial}{\partial x^\mu}\frac{\delta\mathcal{L}}{\delta\left(\frac{\partial A_\alpha}{\partial x^\mu}\right)}-\frac{\delta\mathcal{L}}{\delta A_\alpha}=0 \tag{B.12}$$

We use distinct notations for the three components of the Lagrangean density, $\mathcal{L}=\mathcal{L}_{CS}+\mathcal{L}_m+V$ where $\mathcal{L}_{CS}$ is the gauge field (Chern - Simons) part, $\mathcal{L}_m$ is the "matter" part and V is the nonlinear self-interaction potential for the "matter" field. We use the detailed expressions for $\mathcal{L}_{CS}$ from Eq.(B.6) and $\mathcal{L}_m$ is given by the Eq.(B.11). The functional derivations are done separately on these two parts.

B.2.1 The variation to A_0

The equation of motion resulting from the variation to A_0 is

$$\frac{\partial}{\partial x^\mu}\frac{\delta\mathcal{L}}{\delta\left(\frac{\partial A_0}{\partial x^\mu}\right)}-\frac{\delta\mathcal{L}}{\delta A_0}=0 \tag{B.13}$$

or

$$\frac{\partial}{\partial x^0}\frac{\delta\mathcal{L}}{\delta\left(\partial_0 A_0\right)}+\frac{\partial}{\partial x^1}\frac{\delta\mathcal{L}}{\delta\left(\partial_1 A_0\right)}+\frac{\partial}{\partial x^2}\frac{\delta\mathcal{L}}{\delta\left(\partial_2 A_0\right)}-\frac{\delta\mathcal{L}}{\delta A_0}=0 \tag{B.14}$$

Functional derivations to A_0 of the gauge field (Chern-Simons) Lagrangean The gauge field Lagrangean is Eq.(B.6)

$$\begin{aligned}\mathcal{L}_{CS} \;=\;& (-\kappa)\,\mathrm{tr}\left\{A_0\left(\partial_1 A_2\right)-A_0\left(\partial_2 A_1\right)-A_1\left(\partial_0 A_2\right)\right.\\ &+A_1\left(\partial_2 A_0\right)-A_2\left(\partial_1 A_0\right)+A_2\left(\partial_0 A_1\right)\\ &\left.+2A_0A_1A_2-2A_0A_2A_1\right\}\end{aligned} \tag{B.15}$$

and we have to calculate

$$\frac{\partial}{\partial x^0}\frac{\delta\mathcal{L}_{CS}}{\delta\left(\partial_0 A_0\right)}+\frac{\partial}{\partial x^1}\frac{\delta\mathcal{L}_{CS}}{\delta\left(\partial_1 A_0\right)}+\frac{\partial}{\partial x^2}\frac{\delta\mathcal{L}_{CS}}{\delta\left(\partial_2 A_0\right)}-\frac{\delta\mathcal{L}_{CS}}{\delta A_0}$$

The calculations have been presented in detail in Ref.[38]. The result is

$$\kappa\varepsilon^{0\nu\rho}F_{\nu\rho}=iJ^0 \tag{B.16}$$

and the general form

$$\kappa\varepsilon^{\mu\nu\rho}F_{\nu\rho}=iJ^\mu \tag{B.17}$$

B.3 Euler-Lagrange equations for the *matter* field

We start from the Euler-Lagrange equation resulting from variation of the functional variable $\phi^\dagger$.

$$\frac{\partial}{\partial x^0}\frac{\delta\mathcal{L}}{\delta\left(\partial_0\phi^\dagger\right)}+\frac{\partial}{\partial x^1}\frac{\delta\mathcal{L}}{\delta\left(\partial_1\phi^\dagger\right)}+\frac{\partial}{\partial x^2}\frac{\delta\mathcal{L}}{\delta\left(\partial_2\phi^\dagger\right)}-\frac{\delta\mathcal{L}}{\delta\phi^\dagger}=0 \tag{B.18}$$

where $\mathcal{L} = \mathcal{L}_{CS} + \mathcal{L}_m + \mathcal{V}$. The Chern-Simons term is in Eq.(B.7) and the other two are

$$\begin{aligned}
\mathcal{L}_m &= i\mathrm{tr}\left(\phi^\dagger \frac{\partial \phi}{\partial t} + \phi^\dagger A_0 \phi - \phi^\dagger \phi A_0\right) \qquad \text{(B.19)} \\
&-\frac{1}{2m}\mathrm{tr}\left\{\frac{\partial \phi^\dagger}{\partial x}\frac{\partial \phi}{\partial x} + \frac{\partial \phi^\dagger}{\partial x} A_1 \phi - \frac{\partial \phi^\dagger}{\partial x}\phi A_1\right. \\
&+\phi^\dagger A^{1\dagger}\frac{\partial \phi}{\partial x} + \phi^\dagger A^{1\dagger} A_1 \phi - \phi^\dagger A^{1\dagger}\phi A_1 \\
&-A^{1\dagger}\phi^\dagger \frac{\partial \phi}{\partial x} - A^{1\dagger}\phi^\dagger A_1 \phi + A^{1\dagger}\phi^\dagger \phi A_1 \\
&+\frac{\partial \phi^\dagger}{\partial y}\frac{\partial \phi}{\partial y} + \frac{\partial \phi^\dagger}{\partial y} A_2 \phi - \frac{\partial \phi^\dagger}{\partial y}\phi A_2 \\
&+\phi^\dagger A^{2\dagger}\frac{\partial \phi}{\partial y} + \phi^\dagger A^{2\dagger} A_2 \phi - \phi^\dagger A^{2\dagger}\phi A_2 \\
&\left.-A^{2\dagger}\phi^\dagger \frac{\partial \phi}{\partial y} - A^{2\dagger}\phi^\dagger A_2 \phi + A^{2\dagger}\phi^\dagger \phi A_2\right\}
\end{aligned}$$

$$\mathcal{V} = \frac{1}{4m\kappa}\mathrm{tr}\left(\left[\phi^\dagger, \phi\right]^2\right) \qquad \text{(B.20)}$$

The contribution of $\mathcal{L}_{CS}$ (Chern-Simons) to the Euler Lagrange equation for the functional variable $\phi^\dagger$ This means

$$\frac{\partial}{\partial x^0}\frac{\delta \mathcal{L}_{CS}}{\delta\left(\partial_0 \phi^\dagger\right)} + \frac{\partial}{\partial x^1}\frac{\delta \mathcal{L}_{CS}}{\delta\left(\partial_1 \phi^\dagger\right)} + \frac{\partial}{\partial x^2}\frac{\delta \mathcal{L}_{CS}}{\delta\left(\partial_2 \phi^\dagger\right)} - \frac{\delta \mathcal{L}_{CS}}{\delta \phi^\dagger} = 0 \qquad \text{(B.21)}$$

The Lagrangian $\mathcal{L}_{CS}$ is the Chern-Simons Lagrangian and does not contain matter fields, ϕ and/or $\phi^\dagger$. It results that there is no contribution from it.

The contribution of $\mathcal{L}_m$ to the Euler Lagrange equation for the functional variable $\phi^\dagger$ The contribution from the "matter" Lagrangian is

$$\frac{\partial}{\partial x^0}\frac{\delta \mathcal{L}_m}{\delta\left(\partial_0 \phi^\dagger\right)} + \frac{\partial}{\partial x^1}\frac{\delta \mathcal{L}_m}{\delta\left(\partial_1 \phi^\dagger\right)} + \frac{\partial}{\partial x^2}\frac{\delta \mathcal{L}_m}{\delta\left(\partial_2 \phi^\dagger\right)} - \frac{\delta \mathcal{L}_m}{\delta \phi^\dagger} \qquad \text{(B.22)}$$

The first term

$$\frac{\partial}{\partial x^0}\frac{\delta \mathcal{L}_m}{\delta\left(\partial_0 \phi^\dagger\right)} \qquad \text{(B.23)}$$

Before calculating it we have to symetrise the roles of ϕ and $\phi^\dagger$ by integrating by parts the first term

$$i\mathrm{tr}\left(\phi^\dagger \frac{\partial \phi}{\partial t}\right) \qquad \text{(B.24)}$$

using

$$\frac{\partial}{\partial t}\left(\phi^{\dagger}\phi\right) = \frac{\partial \phi^{\dagger}}{\partial t}\phi + \phi^{\dagger}\frac{\partial \phi}{\partial t} \tag{B.25}$$

Then

$$\begin{aligned} \phi^{\dagger}\frac{\partial \phi}{\partial t} &= \frac{\partial}{\partial t}\left(\phi^{\dagger}\phi\right) - \frac{\partial \phi^{\dagger}}{\partial t}\phi \\ &\rightarrow -\frac{\partial \phi^{\dagger}}{\partial t}\phi \end{aligned} \tag{B.26}$$

and the first part of the matter Lagrangian now looks

$$i\mathrm{tr}\left(-\frac{\partial \phi^{\dagger}}{\partial t}\phi + \phi^{\dagger}A_0\phi - \phi^{\dagger}\phi A_0\right) \tag{B.27}$$

and

$$\begin{aligned} \frac{\partial}{\partial x^0}\frac{\delta \mathcal{L}_m}{\delta\left(\partial_0\phi^{\dagger}\right)} &= \frac{\partial}{\partial x^0} i\mathrm{tr}\frac{\delta}{\delta\left(\partial_0\phi^{\dagger}\right)}\left[-\left(\partial_0\phi^{\dagger}\right)\phi\right] \\ &= -i\frac{\partial}{\partial x^0}\left(\phi\right)^T \end{aligned} \tag{B.28}$$

There is no other contribution from $\mathcal{L}_m$ to this functional variation to $\left(\partial_0\phi^{\dagger}\right)$.

The next term is calculating after retaining from the full expression of $\mathcal{L}_m$ the part that has a nonvanishing contribution

$$\begin{aligned} & \frac{\partial}{\partial x^1}\frac{\delta \mathcal{L}_m}{\delta\left(\partial_1\phi^{\dagger}\right)} \\ = \; & \frac{\partial}{\partial x^1}\frac{\delta}{\delta\left(\partial_1\phi^{\dagger}\right)}\left(-\frac{1}{2m}\right)\mathrm{tr}\left\{\frac{\partial \phi^{\dagger}}{\partial x^1}\frac{\partial \phi}{\partial x^1} + \frac{\partial \phi^{\dagger}}{\partial x^1}A_1\phi - \frac{\partial \phi^{\dagger}}{\partial x^1}\phi A_1\right\} \end{aligned} \tag{B.29}$$

We have

$$\left(-\frac{1}{2m}\right)\frac{\partial}{\partial x^1}\frac{\delta}{\delta\left(\partial_1\phi^{\dagger}\right)}\mathrm{tr}\left\{\frac{\partial \phi^{\dagger}}{\partial x^1}\frac{\partial \phi}{\partial x^1}\right\} = \left(-\frac{1}{2m}\right)\frac{\partial}{\partial x^1}\left(\frac{\partial \phi}{\partial x^1}\right)^T \tag{B.30}$$

$$\begin{aligned} \frac{\partial}{\partial x^1}\frac{\delta}{\delta\left(\partial_1\phi^{\dagger}\right)}\left(-\frac{1}{2m}\right)\mathrm{tr}\left\{\frac{\partial \phi^{\dagger}}{\partial x}A_1\phi\right\} &= \left(-\frac{1}{2m}\right)\frac{\partial}{\partial x^1}\left(A_1\phi\right)^T \\ &= \left(-\frac{1}{2m}\right)\left(\frac{\partial \phi^T}{\partial x^1}A_1^T + \phi^T\frac{\partial A_1^T}{\partial x^1}\right) \end{aligned} \tag{B.31}$$

$$\frac{\partial}{\partial x^1}\frac{\delta}{\delta\left(\partial_1\phi^\dagger\right)}\left(-\frac{1}{2m}\right)\mathrm{tr}\left\{-\frac{\partial\phi^\dagger}{\partial x^1}\phi A_1\right\} = \left(\frac{1}{2m}\right)\frac{\partial}{\partial x^1}\left(\phi A_1\right)^T \tag{B.32}$$
$$= \left(\frac{1}{2m}\right)\left(\frac{\partial A_1^T}{\partial x^1}\phi^T + A_1^T\frac{\partial\phi^T}{\partial x^1}\right)$$

The result from this term is

$$\frac{\partial}{\partial x^1}\frac{\delta\mathcal{L}_m}{\delta\left(\partial_1\phi^\dagger\right)} \tag{B.33}$$
$$= \left(-\frac{1}{2m}\right)\left(\frac{\partial^2\phi}{\partial\left(x^1\right)^2}\right)^T + \left(-\frac{1}{2m}\right)\left(\frac{\partial\phi^T}{\partial x^1}A_1^T + \phi^T\frac{\partial A_1^T}{\partial x^1}\right) + \left(\frac{1}{2m}\right)\left(\frac{\partial A_1^T}{\partial x^1}\phi^T + A_1^T\frac{\partial\phi^T}{\partial x^1}\right)$$

We still can transform this expression

$$\frac{\partial}{\partial x^1}\frac{\delta\mathcal{L}_m}{\delta\left(\partial_1\phi^\dagger\right)} \tag{B.34}$$
$$= \left(-\frac{1}{2m}\right)\left(\left(\frac{\partial^2\phi}{\partial\left(x^1\right)^2}\right)^T + \frac{\partial\phi^T}{\partial x^1}A_1^T - \frac{\partial A_1^T}{\partial x^1}\phi^T + \phi^T\frac{\partial A_1^T}{\partial x^1} - A_1^T\frac{\partial\phi^T}{\partial x^1}\right)$$
$$= \left(-\frac{1}{2m}\right)\left\{\left(\frac{\partial^2\phi}{\partial\left(x^1\right)^2}\right)^T + \left[A_1,\frac{\partial\phi}{\partial x^1}\right]^T - \left[\phi,\frac{\partial A_1}{\partial x^1}\right]^T\right\}$$

We repeat the calculation for $x^2\ (\equiv y)$.

$$\frac{\partial}{\partial x^2}\frac{\delta\mathcal{L}_m}{\delta\left(\partial_2\phi^\dagger\right)} \tag{B.35}$$
$$= \frac{\partial}{\partial x^2}\frac{\delta}{\delta\left(\partial_2\phi^\dagger\right)}\left(-\frac{1}{2m}\right)\mathrm{tr}\left\{\frac{\partial\phi^\dagger}{\partial x^2}\frac{\partial\phi}{\partial x^2} + \frac{\partial\phi^\dagger}{\partial x^2}A_2\phi - \frac{\partial\phi^\dagger}{\partial x^2}\phi A_2\right\}$$

We take separately the terms

$$\frac{\partial}{\partial x^2}\frac{\delta}{\delta\left(\partial_2\phi^\dagger\right)}\left(-\frac{1}{2m}\right)\mathrm{tr}\left\{\frac{\partial\phi^\dagger}{\partial x^2}\frac{\partial\phi}{\partial x^2}\right\} = \left(-\frac{1}{2m}\right)\frac{\partial}{\partial x^2}\left(\frac{\partial\phi}{\partial x^2}\right)^T \tag{B.36}$$
$$= \left(-\frac{1}{2m}\right)\left(\frac{\partial^2\phi}{\partial\left(x^2\right)^2}\right)^T$$

$$\frac{\partial}{\partial x^2}\frac{\delta}{\delta\left(\partial_2\phi^\dagger\right)}\left(-\frac{1}{2m}\right)\mathrm{tr}\left\{\frac{\partial\phi^\dagger}{\partial x^2}A_2\phi\right\} = \left(-\frac{1}{2m}\right)\frac{\partial}{\partial x^2}\left(A_2\phi\right)^T \tag{B.37}$$
$$= \left(-\frac{1}{2m}\right)\left(\frac{\partial\phi^T}{\partial x^2}A_2^T+\phi^T\frac{\partial A_2^T}{\partial x^2}\right)$$

$$\frac{\partial}{\partial x^2}\frac{\delta}{\delta\left(\partial_2\phi^\dagger\right)}\left(-\frac{1}{2m}\right)\mathrm{tr}\left\{-\frac{\partial\phi^\dagger}{\partial x^2}\phi A_2\right\} = \left(\frac{1}{2m}\right)\frac{\partial}{\partial x^2}\left(\phi A_2\right)^T \tag{B.38}$$
$$= \left(\frac{1}{2m}\right)\left(\frac{\partial A_2^T}{\partial x^2}\phi^T+A_2^T\frac{\partial\phi^T}{\partial x^2}\right)$$

Adding the three parts

$$\frac{\partial}{\partial x^2}\frac{\delta}{\delta\left(\partial_2\phi^\dagger\right)}\mathcal{L}_m \tag{B.39}$$
$$= \left(-\frac{1}{2m}\right)\left(\frac{\partial^2\phi}{\partial\left(x^2\right)^2}\right)^T$$
$$+\left(-\frac{1}{2m}\right)\left(\frac{\partial\phi^T}{\partial x^2}A_2^T+\phi^T\frac{\partial A_2^T}{\partial x^2}\right)$$
$$+\left(\frac{1}{2m}\right)\left(\frac{\partial A_2^T}{\partial x^2}\phi^T+A_2^T\frac{\partial\phi^T}{\partial x^2}\right)$$

This expression can be transformed as

$$\frac{\partial}{\partial x^2}\frac{\delta}{\delta\left(\partial_2\phi^\dagger\right)}\mathcal{L}_m \tag{B.40}$$
$$= \left(-\frac{1}{2m}\right)\left(\frac{\partial^2\phi}{\partial\left(x^2\right)^2}\right)^T$$
$$+\left(-\frac{1}{2m}\right)\left\{\left(A_2\frac{\partial\phi}{\partial x^2}\right)^T+\left(\frac{\partial A_2}{\partial x^2}\phi\right)^T-\left(\phi\frac{\partial A_2}{\partial x^2}\right)^T-\left(\frac{\partial\phi}{\partial x^2}A_2\right)^T\right\}$$
$$= \left(-\frac{1}{2m}\right)\left\{\left(\frac{\partial^2\phi}{\partial\left(x^2\right)^2}\right)^T+\left[A_2,\frac{\partial\phi}{\partial x^2}\right]^T-\left[\phi,\frac{\partial A_2}{\partial x^2}\right]^T\right\}$$

Now the last term, retaining in the lagrangian $\mathcal{L}_m$ only the terms that can contribute to the functional derivative

$$
\begin{aligned}
& -\frac{\delta \mathcal{L}_m}{\delta \phi^\dagger} \qquad \text{(B.41)} \\
= & -\frac{\delta}{\delta \phi^\dagger} i \mathrm{tr} \left\{ \phi^\dagger A_0 \phi - \phi^\dagger \phi A_0 \right\} \\
& -\frac{\delta}{\delta \phi^\dagger} \left(-\frac{1}{2m} \right) \mathrm{tr} \left\{ \phi^\dagger A^{1\dagger} \frac{\partial \phi}{\partial x^1} + \phi^\dagger A^{1\dagger} A_1 \phi - \phi^\dagger A^{1\dagger} \phi A_1 \right. \\
& \qquad -A^{1\dagger} \phi^\dagger \frac{\partial \phi}{\partial x^1} - A^{1\dagger} \phi^\dagger A_1 \phi + A^{1\dagger} \phi^\dagger \phi A_1 \\
& \qquad +\phi^\dagger A^{2\dagger} \frac{\partial \phi}{\partial x^2} + \phi^\dagger A^{2\dagger} A_2 \phi - \phi^\dagger A^{2\dagger} \phi A_2 \\
& \qquad \left. -A^{2\dagger} \phi^\dagger \frac{\partial \phi}{\partial x^2} - A^{2\dagger} \phi^\dagger A_2 \phi + A^{2\dagger} \phi^\dagger \phi A_2 \right\}
\end{aligned}
$$

The first two terms are

$$
-\frac{\delta}{\delta \phi^\dagger} i \mathrm{tr} \left\{ \phi^\dagger A_0 \phi - \phi^\dagger \phi A_0 \right\} = -\frac{\delta}{\delta \phi^\dagger} i \mathrm{tr} \left\{ \phi^\dagger \left[A_0, \phi \right] \right\} = -i \left(\left[A_0, \phi \right] \right)^T \qquad \text{(B.42)}
$$

Derivation of the first line of the part $\left(D^k \phi \right)^\dagger (D_k \phi)$.

$$
\begin{aligned}
& -\frac{\delta}{\delta \phi^\dagger} \left(-\frac{1}{2m} \right) \mathrm{tr} \left\{ \phi^\dagger A^{1\dagger} \frac{\partial \phi}{\partial x^1} + \phi^\dagger A^{1\dagger} A_1 \phi - \phi^\dagger A^{1\dagger} \phi A_1 \right\} \qquad \text{(B.43)} \\
= & \left(\frac{1}{2m} \right) \frac{\delta}{\delta \phi^\dagger} \mathrm{tr} \left\{ \phi^\dagger A^{1\dagger} \frac{\partial \phi}{\partial x^1} + \phi^\dagger A^{1\dagger} A_1 \phi - \phi^\dagger A^{1\dagger} \phi A_1 \right\} \\
= & \left(\frac{1}{2m} \right) \frac{\delta}{\delta \phi^\dagger} \mathrm{tr} \left\{ \phi^\dagger A^{1\dagger} \left(\frac{\partial \phi}{\partial x^1} + \left[A_1, \phi \right] \right) \right\} = \left(\frac{1}{2m} \right) \frac{\delta}{\delta \phi^\dagger} \mathrm{tr} \left\{ \phi^\dagger A^{1\dagger} \left(D_1 \phi \right) \right\} \\
= & \left(\frac{1}{2m} \right) \left[A^{1\dagger} \left(D_1 \phi \right) \right]^T = \left(\frac{1}{2m} \right) \left(D_1 \phi \right)^T \left(A^{1\dagger} \right)^T
\end{aligned}
$$

Derivation of the second line of the part $\left(D^k \phi \right)^\dagger (D_k \phi)$.

$$
\begin{aligned}
& -\frac{\delta}{\delta \phi^\dagger} \left(-\frac{1}{2m} \right) \mathrm{tr} \left\{ -A^{1\dagger} \phi^\dagger \frac{\partial \phi}{\partial x^1} - A^{1\dagger} \phi^\dagger A_1 \phi + A^{1\dagger} \phi^\dagger \phi A_1 \right\} \qquad \text{(B.44)} \\
= & \left(\frac{1}{2m} \right) \left\{ - \left(A^{1\dagger} \right)^T \left(\frac{\partial \phi}{\partial x^1} \right)^T - \left(A^{1\dagger} \right)^T \left(A_1 \phi \right)^T + \left(A^{1\dagger} \right)^T \left(\phi A_1 \right)^T \right\} \\
= & \left(\frac{1}{2m} \right) (-) \left(A^{1\dagger} \right)^T \left\{ \left(\frac{\partial \phi}{\partial x^1} \right)^T + \left(A_1 \phi \right)^T - \left(\phi A_1 \right)^T \right\} \\
= & \left(-\frac{1}{2m} \right) \left(A^{1\dagger} \right)^T \left(D_1 \phi \right)^T
\end{aligned}
$$

Derivation of the third line of the part $\left(D^k\phi\right)^\dagger (D_k\phi)$.

$$
\begin{aligned}
& -\frac{\delta}{\delta\phi^\dagger}\left(-\frac{1}{2m}\right)\mathrm{tr}\left\{\phi^\dagger A^{2\dagger}\frac{\partial\phi}{\partial x^2}+\phi^\dagger A^{2\dagger}A_2\phi-\phi^\dagger A^{2\dagger}\phi A_2\right\} \quad \text{(B.45)}\\
= & \left(\frac{1}{2m}\right)\left\{\left(A^{2\dagger}\frac{\partial\phi}{\partial x^2}\right)^T+\left(A^{2\dagger}A_2\phi\right)^T-\left(A^{2\dagger}\phi A_2\right)^T\right\}\\
= & \left(\frac{1}{2m}\right)\left\{\left(\frac{\partial\phi}{\partial x^2}\right)^T+(A_2\phi)^T-(\phi A_2)^T\right\}\left(A^{2\dagger}\right)^T\\
= & \left(\frac{1}{2m}\right)(D_2\phi)^T\left(A^{2\dagger}\right)^T
\end{aligned}
$$

Derivation of the fourth (last) line of $\left(D^k\phi\right)^\dagger (D_k\phi)$.

$$
\begin{aligned}
& -\frac{\delta}{\delta\phi^\dagger}\left(-\frac{1}{2m}\right)\mathrm{tr}\left\{-A^{2\dagger}\phi^\dagger\frac{\partial\phi}{\partial y}-A^{2\dagger}\phi^\dagger A_2\phi+A^{2\dagger}\phi^\dagger\phi A_2\right\} \quad \text{(B.46)}\\
= & \left(\frac{1}{2m}\right)\left\{-\left(A^{2\dagger}\right)^T\left(\frac{\partial\phi}{\partial y}\right)^T-\left(A^{2\dagger}\right)^T(A_2\phi)^T+\left(A^{2\dagger}\right)^T(\phi A_2)^T\right\}\\
= & \left(\frac{1}{2m}\right)(-)\left(A^{2\dagger}\right)^T\left\{\left(\frac{\partial\phi}{\partial y}\right)^T+(A_2\phi)^T-(\phi A_2)^T\right\}\\
= & \left(\frac{1}{2m}\right)(-)\left(A^{2\dagger}\right)^T(D_2\phi)^T
\end{aligned}
$$

Putting together the four results on the five lines above:

$$
\begin{aligned}
& -\frac{\delta\mathcal{L}_m}{\delta\phi^\dagger} \quad \text{(B.47)}\\
= & -i\left([A_0,\phi]\right)^T\\
& +\left(\frac{1}{2m}\right)(D_1\phi)^T\left(A^{1\dagger}\right)^T\\
& -\left(\frac{1}{2m}\right)\left(A^{1\dagger}\right)^T(D_1\phi)^T\\
& +\left(\frac{1}{2m}\right)(D_2\phi)^T\left(A^{2\dagger}\right)^T\\
& -\left(\frac{1}{2m}\right)\left(A^{2\dagger}\right)^T(D_2\phi)^T
\end{aligned}
$$

or

$$
-\frac{\delta\mathcal{L}_m}{\delta\phi^\dagger}=-i\left([A_0,\phi]\right)^T+\left(\frac{1}{2m}\right)\left\{\left[D_1\phi,A^{1\dagger}\right]^T+\left[D_2\phi,A^{2\dagger}\right]^T\right\} \quad \text{(B.48)}
$$

Now we can write all the terms of the equation Euler-Lagrange resulting from the variation to the function $\phi^{\dagger}$.

$$\begin{aligned} & \text{contribution from the "matter" Lagrangian } \mathcal{L}_m \\ = & \frac{\partial}{\partial x^0}\frac{\delta \mathcal{L}_m}{\delta\left(\partial_0\phi^{\dagger}\right)} + \frac{\partial}{\partial x^1}\frac{\delta \mathcal{L}_m}{\delta\left(\partial_1\phi^{\dagger}\right)} + \frac{\partial}{\partial x^2}\frac{\delta \mathcal{L}_m}{\delta\left(\partial_2\phi^{\dagger}\right)} - \frac{\delta \mathcal{L}_m}{\delta\phi^{\dagger}} \end{aligned} \tag{B.49}$$

is

$$\begin{aligned} & -i\frac{\partial}{\partial x^0}\left(\phi\right)^T \\ & \left(-\frac{1}{2m}\right)\left\{\left(\frac{\partial^2\phi}{\partial\left(x^1\right)^2}\right)^T + \left[A_1, \frac{\partial\phi}{\partial x^1}\right]^T - \left[\phi, \frac{\partial A_1}{\partial x^1}\right]^T\right\} \\ & +\left(-\frac{1}{2m}\right)\left\{\left(\frac{\partial^2\phi}{\partial\left(x^2\right)^2}\right)^T + \left[A_2, \frac{\partial\phi}{\partial x^2}\right]^T - \left[\phi, \frac{\partial A_2}{\partial x^2}\right]^T\right\} \\ & -i\left(\left[A_0, \phi\right]\right)^T + \left(\frac{1}{2m}\right)\left\{\left[D_1\phi, A^{1\dagger}\right]^T + \left[D_2\phi, A^{2\dagger}\right]^T\right\} \end{aligned} \tag{B.50}$$

We take off the transpose operator T and try to recollect the expressions in a simpler form

$$\begin{aligned} & -i\frac{\partial}{\partial x^0}\phi - i\left[A_0, \phi\right] \\ & +\left(-\frac{1}{2m}\right)\left\{\frac{\partial^2\phi}{\partial x^2} + \left[A_1, \frac{\partial\phi}{\partial x^1}\right] - \left[\phi, \frac{\partial A_1}{\partial x^1}\right] - \left[D_1\phi, A^{1\dagger}\right]\right. \\ & \qquad \left.\frac{\partial^2\phi}{\partial y^2} + \left[A_2, \frac{\partial\phi}{\partial x^2}\right] - \left[\phi, \frac{\partial A_2}{\partial x^2}\right] - \left[D_2\phi, A^{2\dagger}\right]\right\} \end{aligned} \tag{B.51}$$

The terms that contain the *time* are

$$-iD_0\phi \tag{B.52}$$

The first group of terms (those that refers to the variable x^1).

$$\begin{aligned} & \frac{\partial^2\phi}{\partial\left(x^1\right)^2} + \left[A_1, \frac{\partial\phi}{\partial x^1}\right] - \left[\phi, \frac{\partial A_1}{\partial x^1}\right] - \left[D_1\phi, A^{1\dagger}\right] \\ = & \frac{\partial^2\phi}{\partial\left(x^1\right)^2} + \frac{\partial}{\partial x^1}\left[A_1, \phi\right] - \left[D_1\phi, A^{1\dagger}\right] \\ = & \frac{\partial}{\partial x^1}\left(\frac{\partial\phi}{\partial x^1} + \left[A_1, \phi\right]\right) - \left[D_1\phi, A^{1\dagger}\right] \\ = & \frac{\partial}{\partial x^1}D_1\phi + \left[A^{1\dagger}, D_1\phi\right] \end{aligned} \tag{B.53}$$

(to be multiplied by $\left(-\frac{1}{2m}\right)$. The second group of terms (those that refers to the variable x^2).

$$\begin{aligned} & \frac{\partial^2\phi}{\partial\left(x^2\right)^2}+\left[A_2,\frac{\partial\phi}{\partial x^2}\right]-\left[\phi,\frac{\partial A_2}{\partial x^2}\right]-\left[D_2\phi,A^{2\dagger}\right] \\ = & \frac{\partial^2\phi}{\partial\left(x^2\right)^2}+\frac{\partial}{\partial x^2}\left[A_2,\phi\right]-\left[D_2\phi,A^{2\dagger}\right] \\ = & \frac{\partial}{\partial x^2}\left(\frac{\partial\phi}{\partial x^2}+\left[A_2,\phi\right]\right)-\left[D_2\phi,A^{2\dagger}\right] \\ = & \frac{\partial}{\partial x^2}D_2\phi+\left[A^{2\dagger},D_2\phi\right] \end{aligned} \tag{B.54}$$

(to be multiplied by $\left(-\frac{1}{2m}\right)$.

The contribution of $\mathcal{V}$ to the Euler Lagrange equation for the functional variable $\phi^\dagger$ We recall that the full Lagrangian was

$$\mathcal{L}=\mathcal{L}_{CS}+\mathcal{L}_m+\mathcal{V} \tag{B.55}$$

where the potential is

$$\mathcal{V}=\frac{1}{4m\kappa}\mathrm{tr}\left(\left[\phi^\dagger,\phi\right]^2\right) \tag{B.56}$$

we have to calculate

$$\begin{aligned} & \text{contribution from the potential } \mathcal{V} \\ = & \frac{\partial}{\partial x^0}\frac{\delta\mathcal{V}}{\delta\left(\partial_0\phi^\dagger\right)}+\frac{\partial}{\partial x^1}\frac{\delta\mathcal{V}}{\delta\left(\partial_1\phi^\dagger\right)}+\frac{\partial}{\partial x^2}\frac{\delta\mathcal{V}}{\delta\left(\partial_2\phi^\dagger\right)}-\frac{\delta\mathcal{V}}{\delta\phi^\dagger} \end{aligned} \tag{B.57}$$

We find

$$\frac{\partial}{\partial x^0}\frac{\delta\mathcal{V}}{\delta\left(\partial_0\phi^\dagger\right)}=0 \tag{B.58}$$

$$\frac{\partial}{\partial x^1}\frac{\delta\mathcal{V}}{\delta\left(\partial_1\phi^\dagger\right)}=0 \tag{B.59}$$

$$\frac{\partial}{\partial x^2}\frac{\delta\mathcal{V}}{\delta\left(\partial_2\phi^\dagger\right)}=0 \tag{B.60}$$

$$
\begin{aligned}
& -\frac{\delta \mathcal{V}}{\delta \phi^\dagger} \qquad \text{(B.61)}\\
= & \frac{1}{4m\kappa}\left(-\frac{\delta}{\delta \phi^\dagger}\right) \operatorname{tr}\left(\left[\phi^\dagger, \phi\right]^2\right)\\
= & \frac{1}{4m\kappa}\left(-\frac{\delta}{\delta \phi^\dagger}\right) \operatorname{tr}\left[\left(\phi^\dagger\phi - \phi\phi^\dagger\right)^2\right]\\
= & \frac{1}{4m\kappa}\left(-\frac{\delta}{\delta \phi^\dagger}\right) \operatorname{tr}\left(\phi^\dagger\phi\phi^\dagger\phi - \phi^\dagger\phi\phi\phi^\dagger - \phi\phi^\dagger\phi^\dagger\phi + \phi\phi^\dagger\phi\phi^\dagger\right)
\end{aligned}
$$

The derivations use

$$
\frac{d}{d\mathbf{X}}\left(\mathbf{AXBX}\right) = \mathbf{A}^T\mathbf{X}^T\mathbf{B}^T + \mathbf{B}^T\mathbf{X}^T\mathbf{A}^T \qquad \text{(B.62)}
$$

The first term

$$
\begin{aligned}
& \left(-\frac{1}{4m\kappa}\right)\left(\frac{\delta}{\delta \phi^\dagger}\right) \operatorname{tr}\left(\phi^\dagger\phi\phi^\dagger\phi\right) \qquad \text{(B.63)}\\
= & \left(-\frac{1}{4m\kappa}\right)\left(\frac{\delta}{\delta \phi^\dagger}\right) \operatorname{tr}\left(\phi\phi^\dagger\phi\phi^\dagger\right) \quad \text{(applying cyclic permutation under tr)}\\
= & \left(-\frac{1}{4m\kappa}\right)\left(\phi^T\phi^{\dagger T}\phi^T + \phi^T\phi^{\dagger T}\phi^T\right)\\
= & \left(-\frac{1}{2m\kappa}\right)\left(\phi^T\phi^{\dagger T}\phi^T\right)
\end{aligned}
$$

The second term

$$
\begin{aligned}
& \frac{1}{4m\kappa}\left(-\frac{\delta}{\delta \phi^\dagger}\right) \operatorname{tr}\left(-\phi^\dagger\phi\phi\phi^\dagger\right) \qquad \text{(B.64)}\\
= & \left(\frac{1}{4m\kappa}\right)\left(\frac{\delta}{\delta \phi^\dagger}\right) \operatorname{tr}\left(\phi^\dagger\phi\phi\phi^\dagger\right)
\end{aligned}
$$

The type of this term is

$$
\frac{\delta}{\delta\mathbf{X}}\left(\mathbf{XAX}\right) = \left(\mathbf{AX}\right)^T + \left(\mathbf{XA}\right)^T \qquad \text{(B.65)}
$$

where

$$
\begin{aligned}
\mathbf{X} & \equiv \phi^\dagger \qquad \text{(B.66)}\\
\mathbf{A} & \equiv \phi\phi
\end{aligned}
$$

then it results

$$
\left(\frac{1}{4m\kappa}\right)\left(\frac{\delta}{\delta \phi^\dagger}\right) \operatorname{tr}\left(\phi^\dagger\phi\phi\phi^\dagger\right) = \left(\frac{1}{4m\kappa}\right)\left[\left(\phi\phi\phi^\dagger\right)^T + \left(\phi^\dagger\phi\phi\right)^T\right] \qquad \text{(B.67)}
$$

The third term

$$\begin{aligned}
&\frac{1}{4m\kappa}\left(-\frac{\delta}{\delta\phi^\dagger}\right)\mathrm{tr}\left(-\phi\phi^\dagger\phi^\dagger\phi\right) \\
= &\left(\frac{1}{4m\kappa}\right)\left(\frac{\delta}{\delta\phi^\dagger}\right)\mathrm{tr}\left(\phi\phi^\dagger\phi^\dagger\phi\right) = \left(\frac{1}{4m\kappa}\right)\left(\frac{\delta}{\delta\phi^\dagger}\right)\mathrm{tr}\left(\phi^\dagger\phi^\dagger\phi\phi\right)
\end{aligned} \tag{B.68}$$

This derivation has the type

$$\begin{aligned}
\frac{\delta}{\delta\mathbf{X}}(\mathbf{XXA}) &= T_1 + T_2 \\
T_1 &= \frac{\delta}{\delta\mathbf{X}}\left[\mathbf{X}(\mathbf{XA})\right] = (\mathbf{XA})^T \\
T_2 &= \frac{\delta}{\delta\mathbf{X}}\left[\mathbf{XAX}\right] = \frac{\delta}{\delta\mathbf{X}}\left[\mathbf{X}(\mathbf{AX})\right] = (\mathbf{AX})^T
\end{aligned} \tag{B.69}$$

where

$$\begin{aligned}
\mathbf{X} &\equiv \phi^\dagger \\
\mathbf{A} &\equiv \phi\phi
\end{aligned} \tag{B.70}$$

and we write

$$\left(\frac{1}{4m\kappa}\right)\left(\frac{\delta}{\delta\phi^\dagger}\right)\mathrm{tr}\left(\phi^\dagger\phi^\dagger\phi\phi\right) = \left(\frac{1}{4m\kappa}\right)\left[\left(\phi^\dagger\phi\phi\right)^T + \left(\phi\phi\phi^\dagger\right)^T\right] \tag{B.71}$$

The fourth term

$$\begin{aligned}
&\frac{1}{4m\kappa}\left(-\frac{\delta}{\delta\phi^\dagger}\right)\mathrm{tr}\left(\phi\phi^\dagger\phi\phi^\dagger\right) \\
= &\left(-\frac{1}{4m\kappa}\right)\left(\frac{\delta}{\delta\phi^\dagger}\right)\mathrm{tr}\left(\phi^\dagger\phi\phi^\dagger\phi\right) \\
= &\left(-\frac{1}{4m\kappa}\right)\left(\phi^T\phi^{\dagger T}\phi^T + \phi^T\phi^{\dagger T}\phi^T\right) \\
= &\left(-\frac{1}{2m\kappa}\right)\left(\phi^T\phi^{\dagger T}\phi^T\right)
\end{aligned} \tag{B.72}$$

Now let us collect all terms

$$
\begin{aligned}
-\frac{\delta\mathcal{V}}{\delta\phi^\dagger} &= \frac{1}{4m\kappa}\left(-\frac{\delta}{\delta\phi^\dagger}\right)\mathrm{tr}\left(\phi^\dagger\phi\phi^\dagger\phi - \phi^\dagger\phi\phi\phi^\dagger - \phi\phi^\dagger\phi^\dagger\phi + \phi\phi^\dagger\phi\phi^\dagger\right) \\
&= \left(-\frac{1}{2m\kappa}\right)\left(\phi^T\phi^{\dagger T}\phi^T\right) \\
&\quad + \left(\frac{1}{4m\kappa}\right)\left[\left(\phi\phi\phi^\dagger\right)^T + \left(\phi^\dagger\phi\phi\right)^T\right] \\
&\quad + \left(\frac{1}{4m\kappa}\right)\left[\left(\phi^\dagger\phi\phi\right)^T + \left(\phi\phi\phi^\dagger\right)^T\right] \\
&\quad \left(-\frac{1}{2m\kappa}\right)\left(\phi^T\phi^{\dagger T}\phi^T\right)
\end{aligned} \tag{B.73}
$$

This can be written

$$
\begin{aligned}
-\frac{\delta\mathcal{V}}{\delta\phi^\dagger} &= \left(-\frac{1}{2m\kappa}\right)\left\{\phi^T\phi^{\dagger T}\phi^T - \left(\phi\phi\phi^\dagger\right)^T - \left(\phi^\dagger\phi\phi\right)^T + \phi^T\phi^{\dagger T}\phi^T\right\} \\
&= \left(-\frac{1}{2m\kappa}\right)\left\{\phi\phi^\dagger\phi - \phi\phi\phi^\dagger - \phi^\dagger\phi\phi + \phi\phi^\dagger\phi\right\}^T \\
&= \left(-\frac{1}{2m\kappa}\right)\left\{\phi\left(\phi^\dagger\phi - \phi\phi^\dagger\right) - \left(\phi^\dagger\phi - \phi\phi^\dagger\right)\phi\right\}^T \\
&= \left(-\frac{1}{2m\kappa}\right)\left\{\phi\left[\phi^\dagger,\phi\right] - \left[\phi^\dagger,\phi\right]\phi\right\}^T \\
&= \left(-\frac{1}{2m\kappa}\right)\left[\phi,\left[\phi^\dagger,\phi\right]\right]^T \\
&= \left(\frac{1}{2m\kappa}\right)\left[\left[\phi^\dagger,\phi\right],\phi\right]^T
\end{aligned} \tag{B.74}
$$

And finally

$$
\begin{aligned}
&\text{contribution from the potential } \mathcal{V} \\
&= \frac{\partial}{\partial x^0}\frac{\delta\mathcal{V}}{\delta\left(\partial_0\phi^\dagger\right)} + \frac{\partial}{\partial x^1}\frac{\delta\mathcal{V}}{\delta\left(\partial_1\phi^\dagger\right)} + \frac{\partial}{\partial x^2}\frac{\delta\mathcal{V}}{\delta\left(\partial_2\phi^\dagger\right)} - \frac{\delta\mathcal{V}}{\delta\phi^\dagger} \\
&= \left(\frac{1}{2m\kappa}\right)\left[\left[\phi^\dagger,\phi\right],\phi\right]^T
\end{aligned} \tag{B.75}
$$

All contributions We collect all results

$$\begin{aligned} & 0 \\ & \left[-iD_0\phi + \left(-\frac{1}{2m}\right)\left(D^{k\dagger}D_k\phi\right)\right]^T \\ & + \left(\frac{1}{2m\kappa}\right)\left[\left[\phi^\dagger,\phi\right],\phi\right]^T \\ = & \ 0 \end{aligned} \tag{B.76}$$

or

$$iD_0\phi = -\frac{1}{2m}\left(D^{k\dagger}D_k\phi\right) + \frac{1}{2m\kappa}\left[\left[\phi^\dagger,\phi\right],\phi\right] \tag{B.77}$$

Final form of the equations of motion as derived from Euler-Lagrange eqs. The equations of motion that represent the Euler-Lagrange equations for the Lagrangian are

$$\begin{aligned} iD_0\phi &= -\frac{1}{2m}\left(D^k D_\kappa\right)\phi \\ & \quad -\frac{1}{2m\kappa}\left[\left[\phi,\phi^\dagger\right],\phi\right] \end{aligned} \tag{B.78}$$

$$\kappa\varepsilon^{\mu\nu\rho}F_{\nu\rho} = iJ^\mu \tag{B.79}$$

C Appendix C. Detailed form of the equation of motion for the *matter* field

The first equation of motion is

$$iD_0\phi = -\frac{1}{2m}\left(D^k D_\kappa\right)\phi - \frac{1}{2m\kappa}\left[\left[\phi,\phi^\dagger\right],\phi\right] \tag{C.1}$$

We have to calculate

$$D_0\phi = \frac{\partial\phi}{\partial t} + A_0\phi - \phi A_0 \tag{C.2}$$

$$\mathbf{D}^2\phi = D_k D^k\phi \tag{C.3}$$

We write explicitely the covariant derivative operators

$$\begin{aligned} & i\left(\frac{\partial \phi}{\partial t}+A_0\phi-\phi A_0\right) \\ = & -\frac{1}{2m}\left\{\left(\frac{\partial}{\partial x}+[A_1,]\right)\left(\frac{\partial}{\partial x}+\left[A^1,\right]\right)+\left(\frac{\partial}{\partial y}+[A_2,]\right)\left(\frac{\partial}{\partial y}+\left[A^2,\right]\right)\right\}\phi \\ & -\frac{1}{2m\kappa}\left[\left[\phi,\phi^\dagger\right],\phi\right] \end{aligned} \tag{C.4}$$

C.1 Calculation of the term $D_k D^k \phi$

We calculate separately the terms in the RHS.

First the x term, $D_x D^x \phi$ is expanded

$$\begin{aligned} & \left(\frac{\partial}{\partial x}+[A_1,]\right)\left(\frac{\partial}{\partial x}+\left[A^1,\right]\right)\phi \\ = & \left(\frac{\partial}{\partial x}+[A_1,]\right)\left(\frac{\partial \phi}{\partial x}+A^1\phi-\phi A^1\right) \\ = & \frac{\partial^2\phi}{\partial x^2}+\frac{\partial}{\partial x}\left(A^1\phi\right)-\frac{\partial}{\partial x}\left(\phi A^1\right) \\ & +A_1\left(\frac{\partial \phi}{\partial x}+A^1\phi-\phi A^1\right)-\left(\frac{\partial \phi}{\partial x}+A^1\phi-\phi A^1\right)A_1 \\ = & \frac{\partial^2\phi}{\partial x^2}+\frac{\partial A^1}{\partial x}\phi+A^1\frac{\partial \phi}{\partial x}-\frac{\partial \phi}{\partial x}A^1-\phi\frac{\partial A^1}{\partial x} \\ & +A_1\frac{\partial \phi}{\partial x}+\left(A_1\right)^2\phi-A_1\phi A^1 \\ & -\frac{\partial \phi}{\partial x}A_1-A^1\phi A_1+\phi A^1 A_1 \end{aligned} \tag{C.5}$$

Since we have

$$A^1 = A_1 \equiv A_x \tag{C.6}$$

we can simplify the expression

$$\begin{aligned} & \left(\frac{\partial}{\partial x}+[A_1,]\right)\left(\frac{\partial}{\partial x}+\left[A^1,\right]\right)\phi \\ = & \frac{\partial^2\phi}{\partial x^2}+\frac{\partial A_x}{\partial x}\phi-\phi\frac{\partial A_x}{\partial x} \\ & +2A_x\frac{\partial \phi}{\partial x}-2\frac{\partial \phi}{\partial x}A_x \\ & +A_x^2\phi+\phi A_x^2-2A_x\phi A_x \end{aligned} \tag{C.7}$$

The same calculation is made for the y term

$$\begin{aligned}
& \left(\frac{\partial}{\partial y}+[A_2,]\right)\left(\frac{\partial}{\partial y}+\left[A^2,\right]\right)\phi \qquad \text{(C.8)}\\
= & \left(\frac{\partial}{\partial y}+[A_2,]\right)\left(\frac{\partial\phi}{\partial y}+A^2\phi-\phi A^2\right)\\
= & \frac{\partial^2\phi}{\partial y^2}+\frac{\partial}{\partial y}\left(A^2\phi\right)-\frac{\partial}{\partial y}\left(\phi A^2\right)\\
& +A_2\left(\frac{\partial\phi}{\partial y}+A^2\phi-\phi A^2\right)-\left(\frac{\partial\phi}{\partial y}+A^2\phi-\phi A^2\right)A_2\\
= & \frac{\partial^2\phi}{\partial y^2}+\frac{\partial A^2}{\partial y}\phi+A^2\frac{\partial\phi}{\partial y}-\frac{\partial\phi}{\partial y}A^2-\phi\frac{\partial A^2}{\partial y}\\
& +A_2\frac{\partial\phi}{\partial y}+\left(A_2\right)^2\phi-A_2\phi A^2\\
& -\frac{\partial\phi}{\partial y}A_2-A^2\phi A_2+\phi A^2A_2
\end{aligned}$$

Since we have

$$A^2=A_2\equiv A_y \qquad \text{(C.9)}$$

we can simplify the expression

$$\begin{aligned}
& \left(\frac{\partial}{\partial y}+[A_2,]\right)\left(\frac{\partial}{\partial y}+\left[A^2,\right]\right)\phi \qquad \text{(C.10)}\\
= & \frac{\partial^2\phi}{\partial y^2}+\frac{\partial A_y}{\partial y}\phi-\phi\frac{\partial A_y}{\partial y}\\
& +2A_y\frac{\partial\phi}{\partial y}-2\frac{\partial\phi}{\partial y}A_y\\
& +A_y^2\phi+\phi A_y^2-2A_y\phi A_y
\end{aligned}$$

Now we can sum the two terms

$$D_kD^k\phi=\left(D_xD^x+D_yD^y\right)\phi$$

$$\begin{aligned}
& \left[\left(\frac{\partial}{\partial x}+[A_1,]\right)\left(\frac{\partial}{\partial x}+\left[A^1,\right]\right)+\left(\frac{\partial}{\partial y}+[A_2,]\right)\left(\frac{\partial}{\partial y}+\left[A^2,\right]\right)\right]\phi\\
= & \frac{\partial^2\phi}{\partial x^2}+\frac{\partial A_x}{\partial x}\phi-\phi\frac{\partial A_x}{\partial x}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial A_y}{\partial y}\phi-\phi\frac{\partial A_y}{\partial y} \qquad \text{(C.11)}\\
& +2A_x\frac{\partial\phi}{\partial x}-2\frac{\partial\phi}{\partial x}A_x+2A_y\frac{\partial\phi}{\partial y}-2\frac{\partial\phi}{\partial y}A_y\\
& +A_x^2\phi+\phi A_x^2-2A_x\phi A_x+A_y^2\phi+\phi A_y^2-2A_y\phi A_y
\end{aligned}$$

This expression must be multiplied by the numerical factor $-\frac{1}{2m}$.

This expresion of $D_k D^k \phi$ will be compared later with $D_+ D_- \phi$.

C.1.1 Calculation of $D_+ D_- \phi$ in terms of $A_{x,y}$

We will find the detailed expression of the term

$$\begin{aligned} & D_+ D_- \phi \\ = & \left(\partial_+ + [A_+,]\right)\left(\partial_- + [A_-,]\right)\phi \end{aligned} \tag{C.12}$$

where

$$\begin{aligned} A_+ &= A_x + iA_y \\ A_- &= A_x - iA_y \end{aligned} \tag{C.13}$$

$$\begin{aligned} \partial_+ &= \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \\ \partial_- &= \frac{\partial}{\partial x} - i\frac{\partial}{\partial y} \end{aligned} \tag{C.14}$$

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} + [A_x + iA_y,]\right)\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y} + [A_x - iA_y,]\right)\phi \tag{C.15}$$

Separately, the second operator in the product (second paranthesis) is

$$\begin{aligned} & \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y} + [A_x - iA_y,]\right)\phi \\ = & \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y} + A_x\phi - \phi A_x - iA_y\phi + i\phi A_y \end{aligned} \tag{C.16}$$

Now we apply the first operator (first paranthesis, $\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}+[A_x+iA_y,]\right)$) on this expression

$$
\begin{aligned}
&\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}+[A_x+iA_y,]\right)\left(\frac{\partial\phi}{\partial x}-i\frac{\partial\phi}{\partial y}+A_x\phi-\phi A_x-iA_y\phi+i\phi A_y\right) \qquad \text{(C.17)}\\
= \;&\frac{\partial^2\phi}{\partial x^2}-i\underbrace{\frac{\partial^2\phi}{\partial x\partial y}}+\frac{\partial A_x}{\partial x}\phi+A_x\frac{\partial\phi}{\partial x}-\frac{\partial\phi}{\partial x}A_x-\phi\frac{\partial A_x}{\partial x}-i\frac{\partial A_y}{\partial x}\phi-\underline{iA_y\frac{\partial\phi}{\partial x}}+\underleftarrow{i\frac{\partial\phi}{\partial x}A_y}+i\phi\frac{\partial A_y}{\partial x}\\
&+i\underbrace{\frac{\partial^2\phi}{\partial y\partial x}}+\frac{\partial^2\phi}{\partial y^2}+i\frac{\partial A_x}{\partial y}\phi+\underline{\underline{iA_x\frac{\partial\phi}{\partial y}}}-\underleftrightarrow{i\frac{\partial\phi}{\partial y}A_x}-i\phi\frac{\partial A_x}{\partial y}+\frac{\partial A_y}{\partial y}\phi+A_y\frac{\partial\phi}{\partial y}-\frac{\partial\phi}{\partial y}A_y-\phi\frac{\partial A_y}{\partial y}\\
&+A_x\frac{\partial\phi}{\partial x}-\underline{\underline{iA_x\frac{\partial\phi}{\partial y}}}+A_x^2\phi-A_x\phi A_x-iA_xA_y\phi+\underbrace{iA_x\phi A_y}\\
&-\frac{\partial\phi}{\partial x}A_x+\underleftrightarrow{i\frac{\partial\phi}{\partial y}A_x}-A_x\phi A_x+\phi A_x^2+\underleftrightarrow{iA_y\phi A_x}-i\phi A_yA_x\\
&+\underline{iA_y\frac{\partial\phi}{\partial x}}+A_y\frac{\partial\phi}{\partial y}+iA_yA_x\phi-\underleftrightarrow{iA_y\phi A_x}+A_y^2\phi-A_y\phi A_y\\
&-\underleftarrow{i\frac{\partial\phi}{\partial x}A_y}-\frac{\partial\phi}{\partial y}A_y-\underbrace{iA_x\phi A_y}+i\phi A_xA_y-A_y\phi A_y+\phi A_y^2
\end{aligned}
$$

There are 44 terms. Few terms, 14, cancel and others 30 are grouped.

The result is

$$
\begin{aligned}
&D_+D_-\phi= \qquad \text{(C.18)}\\
= \;&\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\\
&+2A_x\frac{\partial\phi}{\partial x}-2\frac{\partial\phi}{\partial x}A_x+2A_y\frac{\partial\phi}{\partial y}-2\frac{\partial\phi}{\partial y}A_y\\
&+\frac{\partial A_x}{\partial x}\phi-\phi\frac{\partial A_x}{\partial x}-i\frac{\partial A_y}{\partial x}\phi+i\phi\frac{\partial A_y}{\partial x}+i\frac{\partial A_x}{\partial y}\phi-i\phi\frac{\partial A_x}{\partial y}-\phi\frac{\partial A_y}{\partial y}+\frac{\partial A_y}{\partial y}\phi\\
&+A_x^2\phi-2A_x\phi A_x-iA_xA_y\phi+\phi A_x^2-i\phi A_yA_x+iA_yA_x\phi+A_y^2\phi-2A_y\phi A_y\\
&+i\phi A_xA_y+\phi A_y^2
\end{aligned}
$$

This is $D_+D_-\phi$.

Now we compare this with $D_kD^k\phi$ from Eq.(C.11) (we do not multiply yet by $-\frac{1}{2m}$) and substract

$$\begin{aligned} & D_k D^k \phi - D_+ D_- \phi \\ = & \left[-i \frac{\partial A_y}{\partial x} \phi + i\phi \frac{\partial A_y}{\partial x} + i \frac{\partial A_x}{\partial y} \phi - i\phi \frac{\partial A_x}{\partial y} \right. \\ & \left. -i A_x A_y \phi - i\phi A_y A_x + i A_y A_x \phi + i\phi A_x A_y \right] \end{aligned} \tag{C.19}$$

We have

$$\begin{aligned} & D_+ D_- - D_k D^k \\ = & -i \frac{\partial A_y}{\partial x} \phi + i\phi \frac{\partial A_y}{\partial x} + i \frac{\partial A_x}{\partial y} \phi - i\phi \frac{\partial A_x}{\partial y} \\ & -i A_x A_y \phi - i\phi A_y A_x + i A_y A_x \phi + i\phi A_x A_y \\ = & \left(-i \frac{\partial A_y}{\partial x} + i \frac{\partial A_x}{\partial y} - i A_x A_y + i A_y A_x \right) \phi \\ & + \phi \left(i \frac{\partial A_y}{\partial x} - i \frac{\partial A_x}{\partial y} - i A_y A_x + i A_x A_y \right) \\ = & -i \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} + A_x A_y - A_y A_x \right) \phi \\ & + i\phi \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} + A_x A_y - A_y A_x \right) \end{aligned} \tag{C.20}$$

This can be written

$$\begin{aligned} & D_+ D_- - D_k D^k \\ = & -i F_{xy} \phi + i\phi F_{xy} \\ = & -i \left[F_{xy}, \phi \right] \end{aligned} \tag{C.21}$$

or

$$D_k D^k = D_+ D_- + i \left[F_{xy}, \phi \right] \tag{C.22}$$

Now we replace with the formula derived by us for F_{12},

$$F_{xy} = F_{12} = \frac{i}{2\kappa} \left[\phi, \phi^\dagger \right] \tag{C.23}$$

and obtain

$$\begin{aligned} D_k D^k \phi &= D_+ D_- \phi + i \left[\frac{i}{2\kappa} \left[\phi, \phi^\dagger \right], \phi \right] \\ &= D_+ D_- \phi - \frac{1}{2\kappa} \left[\left[\phi, \phi^\dagger \right], \phi \right] \end{aligned} \tag{C.24}$$

At this moment the first equation of motion can be written

$$
\begin{aligned}
iD_0\phi &= -\frac{1}{2m}\mathbf{D}^2\phi - \frac{1}{2m\kappa}\left[\left[\phi,\phi^\dagger\right],\phi\right] &\text{(C.25)}\\
iD_0\phi &= -\frac{1}{2m}\left\{D_+D_-\phi - \frac{1}{2\kappa}\left[\left[\phi,\phi^\dagger\right],\phi\right]\right\} - \frac{1}{2m\kappa}\left[\left[\phi,\phi^\dagger\right],\phi\right]\\
&= -\frac{1}{2m}D_+D_-\phi + \frac{1}{4m\kappa}\left[\left[\phi,\phi^\dagger\right],\phi\right] - \frac{1}{2m\kappa}\left[\left[\phi,\phi^\dagger\right],\phi\right]\\
&= -\frac{1}{2m}D_+D_-\phi - \frac{1}{4m\kappa}\left[\left[\phi,\phi^\dagger\right],\phi\right]
\end{aligned}
$$

The last term in the right hand side of the expression of D_kD^k is

$$
-\frac{1}{2\kappa}\left[\left[\phi,\phi^\dagger\right],\phi\right] \quad \text{(C.26)}
$$

The full expression of the **first equation of motion** in detailed form is obtained from the Eqs.(C.2), (C.11) and (C.26).

We have

$$
\begin{aligned}
iD_0\phi &= -\frac{1}{2m}\mathbf{D}^2\phi - \frac{1}{2m\kappa}\left[\left[\phi,\phi^\dagger\right],\phi\right] &\text{(C.27)}\\
&= -\frac{1}{2m}\left\{D_+D_-\phi - \frac{1}{2\kappa}\left[\left[\phi,\phi^\dagger\right],\phi\right]\right\} - \frac{1}{2m\kappa}\left[\left[\phi,\phi^\dagger\right],\phi\right]\\
&= -\frac{1}{2m}D_+D_-\phi - \frac{1}{4m\kappa}\left[\left[\phi,\phi^\dagger\right],\phi\right]
\end{aligned}
$$

$$
iD_0\phi = -\frac{1}{2m}D_+D_-\phi - \frac{1}{4m\kappa}\left[\left[\phi,\phi^\dagger\right],\phi\right] \quad \text{(C.28)}
$$

We **NOTE** that this equation is valid in general not only at self-duality.

C.2 An expression for the time-component of the gauge potential A_0 at SD

We note that in the derivation of the Bogomolnyi form of the energy it was not necessary to impose the static states. Then at this moment the states may still have a time evolution, although they verify the lowest energy condition

$$
D_-\phi = 0 \quad \text{(C.29)}
$$

In this case we can combine the spatial components of the current density

$$\begin{aligned}
J^{+} &= J^{x} + iJ^{y} \qquad \text{(C.30)}\\
&= -\frac{i}{2m}\left(\left[\phi^{\dagger},(D^{x}\phi)\right]-\left[(D^{x}\phi)^{\dagger},\phi\right]\right)+i\left\{-\frac{i}{2m}\left(\left[\phi^{\dagger},(D^{y}\phi)\right]-\left[(D^{y}\phi)^{\dagger},\phi\right]\right)\right\}\\
&= -\frac{i}{2m}\left\{\left[\phi^{\dagger},(D^{x}\phi)\right]+i\left[\phi^{\dagger},(D^{y}\phi)\right]\right.\\
&\qquad\left.-\left[(D^{x}\phi)^{\dagger},\phi\right]-i\left[(D^{y}\phi)^{\dagger},\phi\right]\right\}\\
&= -\frac{i}{2m}\left(\left[\phi^{\dagger},\left(D^{+}\phi\right)\right]-\left[\left(D^{-}\phi\right)^{\dagger},\phi\right]\right)
\end{aligned}$$

and inserting in the equation written above the equation at Self-Duality $D_{-}\phi = 0$ we get

$$J^{+} = -\frac{i}{2m}\left(\left[\phi^{\dagger},\left(D^{+}\phi\right)\right]\right) \quad \text{at Self-Duality} \qquad \text{(C.31)}$$

We return to the expression of the current in the second (gauge-field) equation of motion, which is the Gauss law

$$\kappa\varepsilon^{\mu\nu\rho}F_{\nu\rho} = iJ^{\mu} \qquad \text{(C.32)}$$

and take the x and y components

$$\begin{aligned}
\kappa\varepsilon^{x\mu\nu}F_{\mu\nu} &= iJ^{x} \qquad \text{(C.33)}\\
\kappa\varepsilon^{y\mu\nu}F_{\mu\nu} &= iJ^{y}
\end{aligned}$$

$$\begin{aligned}
\kappa\left(\varepsilon^{xy0}F_{y0}+\varepsilon^{x0y}F_{0y}\right) &= iJ^{x} \qquad \text{(C.34)}\\
2\kappa F_{y0} &= iJ^{x}\\
2\kappa\left(\partial_{y}A_{0}-\partial_{0}A_{y}+[A_{y},A_{0}]\right) &= iJ^{x}
\end{aligned}$$

and analogous

$$\begin{aligned}
\kappa\left(\varepsilon^{yx0}F_{x0}+\varepsilon^{y0x}F_{0x}\right) &= iJ^{y} \qquad \text{(C.35)}\\
-2\kappa F_{x0} &= iJ^{y}\\
-2\kappa\left(\partial_{x}A_{0}-\partial_{0}A_{x}+[A_{x},A_{0}]\right) &= iJ^{y}
\end{aligned}$$

Now we combine them

$$
\begin{aligned}
i\left(J^x + iJ^y\right) &= 2\kappa\left(\partial_y A_0 - \partial_0 A_y + \left[A_y, A_0\right]\right. \\
&\quad \left. -i\left(\partial_x A_0 - \partial_0 A_x + \left[A_x, A_0\right]\right)\right) \\
&= 2\kappa\left(\left(\partial_y - i\partial_x\right) A_0\right. \\
&\quad -\partial_0\left(A_y - iA_x\right) \\
&\quad \left. +\left[A_y - iA_x, A_0\right]\right) \\
&= \frac{2\kappa}{i}\left(\left(\partial_x + i\partial_y\right) A_0\right. \\
&\quad -\partial_0\left(A_x + iA_y\right) \\
&\quad \left. +\left[A_x + iA_y, A_0\right]\right)
\end{aligned}
\tag{C.36}
$$

$$
\begin{aligned}
-J^+ &= 2\kappa\left(\partial_+ A_0 - \partial_0 A_+ + \left[A_+, A_0\right]\right) \\
&= 2\kappa\left(D_+ A_0 - \partial_0 A_+\right)
\end{aligned}
\tag{C.37}
$$

where we have introduced the notation

$$
D_+ \equiv \partial_+ + \left[A_+, \right]
\tag{C.38}
$$

Now we have two expressions for the current density J^+ at Self-Duality

$$
\begin{aligned}
J^+ &= -\frac{i}{2m}\left(\left[\phi^\dagger, \left(D^+\phi\right)\right]\right) \\
J^+ &= -2\kappa\left(D^+ A_0 - \partial_0 A_+\right)
\end{aligned}
\tag{C.39}
$$

At stationarity

$$
\partial_0 A_+ = 0
\tag{C.40}
$$

and from the two expressions of the current we have

$$
J^+ = -\frac{i}{2m}\left(\left[\phi^\dagger, \left(D^+\phi\right)\right]\right) = -2\kappa\left(D^+ A_0\right) \quad \text{at SD}
\tag{C.41}
$$

This allows us to identify the expression of the time-component of the potential, A_0

$$
A_0 = \frac{i}{4m\kappa}\left[\phi, \phi^\dagger\right] \quad \text{at SD}
\tag{C.42}
$$

We can replace

$$
\left[\phi, \phi^\dagger\right] = \left(\rho_1 - \rho_2\right) H
\tag{C.43}
$$

and further, since at SD we have introduced $\omega = \Delta \ln \rho_1 = \Delta \ln\left(1/\rho_2\right) = \Delta\psi$,

$$
\rho_1 - \rho_2 = -\frac{\kappa}{2}\omega \quad \text{at SD}
\tag{C.44}
$$

This shows that the zero component of the potential of interaction has algebraic content reduced to the Cartan generator

$$A_0 \sim \frac{i}{4m\kappa} (\rho_1 - \rho_2) H \tag{C.45}$$

and that it is purely imaginary. The magnitude of A_0 at SD is given by the *vorticity*.

Using this suggestion and following the text [38] we take the temporal component of the potential in the form

$$\begin{aligned} A^0 &\equiv bH \\ A^{0\dagger} &= A_0^{*T} \equiv -b^* H \end{aligned} \tag{C.46}$$

Taking into account the metric we have

$$A_0 = -A^0 = -bH \tag{C.47}$$

and we can identify, at *self-duality*:

$$\begin{aligned} A_0 &= \frac{i}{4m\kappa} \left[\phi, \phi^\dagger\right] = \frac{i}{4m\kappa} (\rho_1 - \rho_2) H \\ &= -bH \end{aligned} \tag{C.48}$$

or

$$\begin{aligned} b &= -\frac{i}{4m\kappa} (\rho_1 - \rho_2) \\ &= \text{imaginary} \quad (b^* + b = 0) \end{aligned} \tag{C.49}$$

and, after identifications at *SD*,

$$b = \frac{i}{8m\kappa} \omega \quad \text{at SD} \tag{C.50}$$

In detail, the operator of covariant derivative to time

$$\begin{aligned} & i \left(\frac{\partial \phi}{\partial t} + A_0 \phi - \phi A_0 \right) \\ = \;& i \left\{ \frac{\partial}{\partial t} (\phi_1 E_+ + \phi_2 E_-) + [A_0, \phi_1 E_+ + \phi_2 E_-] \right\} \\ = \;& i \frac{\partial \phi_1}{\partial t} E_+ + i \frac{\partial \phi_2}{\partial t} E_- + i (-b) \phi_1 [H, E_+] + i (-b) \phi_2 [H, E_-] \\ = \;& i \frac{\partial \phi_1}{\partial t} E_+ + i \frac{\partial \phi_2}{\partial t} E_- - 2ib\phi_1 E_+ + 2ib\phi_2 E_- \end{aligned} \tag{C.51}$$

Collecting the factors of the ladder generators

$$\begin{aligned} & i\left(\frac{\partial\phi}{\partial t}+A_0\phi-\phi A_0\right) && \text{(C.52)} \\ = & \left(i\frac{\partial\phi_1}{\partial t}-2ib\phi_1\right)E_+ +\left(i\frac{\partial\phi_2}{\partial t}+2ib\phi_2\right)E_- \end{aligned}$$

C.2.1 The first part of the first term in the RHS of the FIRST equation of motion, adopting the algebraic *ansatz*

We can try to replace the algebraic ansatz in the first term (for the x component) of the Eq.(C.4)

$$\left(\frac{\partial}{\partial x}+[A_1,]\right)\left(\frac{\partial\phi}{\partial x}+A^1\phi-\phi A^1\right) \tag{C.53}$$

taking

$$\phi=\phi_1E_+ +\phi_2E_- \tag{C.54}$$

and

$$\begin{aligned} A_x &= \frac{1}{2}\left(a-a^*\right)H && \text{(C.55)} \\ A_y &= \frac{i}{2}\left(a+a^*\right)H \end{aligned}$$

Then the second paranthesis is

$$\begin{aligned} & \frac{\partial\phi}{\partial x}+A^1\phi-\phi A^1 && \text{(C.56)} \\ = & \frac{\partial}{\partial x}\left(\phi_1E_+ +\phi_2E_-\right)+\left[\frac{1}{2}\left(a-a^*\right)H,\phi_1E_+ +\phi_2E_-\right] \\ = & \frac{\partial\phi_1}{\partial x}E_+ +\frac{\partial\phi_2}{\partial x}E_- +\frac{1}{2}\left(a-a^*\right)\phi_1\left[H,E_+\right]+\frac{1}{2}\left(a-a^*\right)\phi_2\left[H,E_-\right] \end{aligned}$$

Here we must use the commutators of the generators and obtain

$$\begin{aligned} & \frac{\partial\phi}{\partial x}+A^1\phi-\phi A^1 && \text{(C.57)} \\ = & \frac{\partial\phi_1}{\partial x}E_+ +\frac{\partial\phi_2}{\partial x}E_- +\frac{1}{2}\left(a-a^*\right)\phi_1 2E_+ -\frac{1}{2}\left(a-a^*\right)\phi_2 2E_- \\ = & \left[\frac{\partial\phi_1}{\partial x}+\left(a-a^*\right)\phi_1\right]E_+ +\left[\frac{\partial\phi_2}{\partial x}-\left(a-a^*\right)\phi_2\right]E_- \end{aligned}$$

The first paranthesis

$$\frac{\partial}{\partial x}+[A_1,] \tag{C.58}$$

is an operator which is applied on the second paranthesis

$$\begin{aligned}
& \left(\frac{\partial}{\partial x}+[A_1,]\right)\left\{\left[\frac{\partial\phi_1}{\partial x}+(a-a^*)\,\phi_1\right]E_+ + \left[\frac{\partial\phi_2}{\partial x}-(a-a^*)\,\phi_2\right]E_-\right\} \\
= & \left(\frac{\partial}{\partial x}+[A_1,]\right)\left[\frac{\partial\phi_1}{\partial x}+(a-a^*)\,\phi_1\right]E_+ + \left(\frac{\partial}{\partial x}+[A_1,]\right)\left[\frac{\partial\phi_2}{\partial x}-(a-a^*)\,\phi_2\right]E_- \\
\equiv & \; I^1 + II^1 \qquad \text{(C.59)}
\end{aligned}$$

The first part is

$$I^1 \equiv \left(\frac{\partial}{\partial x}+[A_1,]\right)\left\{\left[\frac{\partial\phi_1}{\partial x}+(a-a^*)\,\phi_1\right]E_+\right\}$$

and is written in detail

$$\begin{aligned}
I^1 &= \left(\frac{\partial}{\partial x}+[A_1,]\right)\left[\frac{\partial\phi_1}{\partial x}+(a-a^*)\,\phi_1\right]E_+ \qquad \text{(C.60)} \\
&= \frac{\partial^2\phi_1}{\partial x^2}E_+ + \left[\frac{\partial\,(a-a^*)}{\partial x}\phi_1+(a-a^*)\,\frac{\partial\phi_1}{\partial x}\right]E_+ \\
&\quad +\frac{\partial\phi_1}{\partial x}\,[A_1,E_+] \\
&\quad +(a-a^*)\,\phi_1\,[A_1,E_+]
\end{aligned}$$

and we have

$$\begin{aligned}
[A_1,E_+] &= \frac{1}{2}\,(a-a^*)\,[H,E_+] \qquad \text{(C.61)} \\
&= \frac{1}{2}\,(a-a^*)\,2E_+ \\
&= (a-a^*)\,E_+
\end{aligned}$$

Then the first part I becomes

$$\begin{aligned}
I^1 &= \left(\frac{\partial}{\partial x}+[A_1,]\right)\left[\frac{\partial\phi_1}{\partial x}+(a-a^*)\,\phi_1\right]E_+ \qquad \text{(C.62)} \\
&= \frac{\partial^2\phi_1}{\partial x^2}E_+ + \left[\frac{\partial\,(a-a^*)}{\partial x}\phi_1+(a-a^*)\,\frac{\partial\phi_1}{\partial x}\right]E_+ \\
&\quad +\frac{\partial\phi_1}{\partial x}\,(a-a^*)\,E_+ + (a-a^*)^2\,\phi_1 E_+
\end{aligned}$$

The second part is

$$II^1 \equiv \left(\frac{\partial}{\partial x}+[A_1,]\right)\left\{\left[\frac{\partial\phi_2}{\partial x}-(a-a^*)\,\phi_2\right]E_-\right\} \qquad \text{(C.63)}$$

Now we expand the second part II^1

$$\begin{aligned} II^1 &= \left(\frac{\partial}{\partial x} + [A_1,]\right)\left[\frac{\partial \phi_2}{\partial x} - (a - a^*)\,\phi_2\right] E_- \\ &= \frac{\partial^2 \phi_2}{\partial x^2} E_- - \left[\frac{\partial (a - a^*)}{\partial x}\phi_2 + (a - a^*)\,\frac{\partial \phi_2}{\partial x}\right] E_- \\ &\quad + \frac{\partial \phi_2}{\partial x}[A_1, E_-] - (a - a^*)\,\phi_2 [A_1, E_-] \end{aligned} \tag{C.64}$$

The commutator is

$$\begin{aligned} [A_1, E_-] &= \frac{1}{2}(a - a^*)[H, E_-] \\ &= \frac{1}{2}(a - a^*)(-2E_-) \\ &= -(a - a^*)\,E_- \end{aligned} \tag{C.65}$$

and the second term becomes

$$\begin{aligned} II^1 &= \left(\frac{\partial}{\partial x} + [A_1,]\right)\left[\frac{\partial \phi_2}{\partial x} - (a - a^*)\,\phi_2\right] E_- \\ &= \frac{\partial^2 \phi_2}{\partial x^2} E_- - \left[\frac{\partial (a - a^*)}{\partial x}\phi_2 + (a - a^*)\,\frac{\partial \phi_2}{\partial x}\right] E_- \\ &\quad - \frac{\partial \phi_2}{\partial x}(a - a^*)\,E_- + (a - a^*)^2\,\phi_2 E_- \end{aligned} \tag{C.66}$$

Now we collect the two terms I^1 from Eq.(C.62) and II^1 from Eq.(C.66)

$$\begin{aligned} & D_x D^x \phi \\ &= \left(\frac{\partial}{\partial x} + [A_1,]\right)\left(\frac{\partial \phi}{\partial x} + A^1\phi - \phi A^1\right) \\ &= I^1 + II^1 \end{aligned} \tag{C.67}$$

$$
\begin{aligned}
& \left(\frac{\partial}{\partial x}+[A_1,]\right)\left(\frac{\partial \phi}{\partial x}+A^1\phi-\phi A^1\right) && \text{(C.68)}\\
= & \left(\frac{\partial}{\partial x}+[A_1,]\right)\left\{\left[\frac{\partial \phi_1}{\partial x}+(a-a^*)\,\phi_1\right]E_+ + \left[\frac{\partial \phi_2}{\partial x}-(a-a^*)\,\phi_2\right]E_-\right\}\\
= & \frac{\partial^2\phi_1}{\partial x^2}E_+ + \left[\frac{\partial (a-a^*)}{\partial x}\phi_1+(a-a^*)\frac{\partial \phi_1}{\partial x}\right]E_+\\
& +\frac{\partial \phi_1}{\partial x}(a-a^*)\,E_+ + (a-a^*)^2\,\phi_1 E_+\\
& +\frac{\partial^2\phi_2}{\partial x^2}E_- - \left[\frac{\partial (a-a^*)}{\partial x}\phi_2+(a-a^*)\frac{\partial \phi_2}{\partial x}\right]E_-\\
& -\frac{\partial \phi_2}{\partial x}(a-a^*)\,E_- + (a-a^*)^2\,\phi_2 E_-
\end{aligned}
$$

This is the first part of the first term in the RHS of the FIRST equation of motion.

C.2.2 The second part of the first term in the RHS of the FIRST equation of motion, with the algebraic *ansatz*

This part is very similar to the previous one, with x replaced by y and A_1 replaced by A_2.

$$
\begin{aligned}
& D_y D^y \phi && \text{(C.69)}\\
= & \left(\frac{\partial}{\partial y}+[A_2,]\right)\left(\frac{\partial}{\partial y}+\left[A^2,\right]\right)\phi && \text{(C.70)}\\
= & \left(\frac{\partial}{\partial y}+[A_2,]\right)\left(\frac{\partial \phi}{\partial y}+\left[A^2,\phi\right]\right)
\end{aligned}
$$

The second paranthesis can be written in more detail, using the *algebraic ansatz*:

$$
\begin{aligned}
A_- &= A_x - iA_y = aH && \text{(C.71)}\\
A_+ &= A_x + iA_y = -a^* H
\end{aligned}
$$

$$
\begin{aligned}
\phi &= \phi_1 E_+ + \phi_2 E_- && \text{(C.72)}\\
A_y &\equiv A_2 = \frac{i}{2}(a+a^*)\,H
\end{aligned}
$$

It is

$$\begin{aligned}
&\frac{\partial \phi}{\partial y}+\left[A^2,\phi\right] \qquad (C.73)\\
=\;& \frac{\partial}{\partial y}\left(\phi_1 E_+ + \phi_2 E_-\right)+\left[\frac{i}{2}\left(a+a^*\right)H,\phi_1 E_+ + \phi_2 E_-\right]\\
=\;& \frac{\partial \phi_1}{\partial y}E_+ + \frac{\partial \phi_2}{\partial y}E_- + \frac{i}{2}\left(a+a^*\right)\phi_1\left[H,E_+\right]+\frac{i}{2}\left(a+a^*\right)\phi_2\left[H,E_-\right]\\
=\;& \frac{\partial \phi_1}{\partial y}E_+ + \frac{\partial \phi_2}{\partial y}E_- + i\left(a+a^*\right)\phi_1 E_+ - i\left(a+a^*\right)\phi_2 E_-\\
=\;& \left[\frac{\partial \phi_1}{\partial y}+i\left(a+a^*\right)\phi_1\right]E_+ + \left[\frac{\partial \phi_2}{\partial y}-i\left(a+a^*\right)\phi_2\right]E_-
\end{aligned}$$

On this expression we have to apply the operator from the first paranthesis

$$\begin{aligned}
&\left(\frac{\partial}{\partial y}+\left[A_2,\right]\right)\left\{\left[\frac{\partial \phi_1}{\partial y}+i\left(a+a^*\right)\phi_1\right]E_+ + \left[\frac{\partial \phi_2}{\partial y}-i\left(a+a^*\right)\phi_2\right]E_-\right\}\\
\equiv\;& I^2+II^2 \qquad (C.74)
\end{aligned}$$

The first part

$$\begin{aligned}
I^2 &= \left(\frac{\partial}{\partial y}+\left[A_2,\right]\right)\left\{\left[\frac{\partial \phi_1}{\partial y}+i\left(a+a^*\right)\phi_1\right]E_+\right\} \qquad (C.75)\\
&= \frac{\partial^2 \phi_1}{\partial y^2}E_+ + i\left[\frac{\partial\left(a+a^*\right)}{\partial y}\phi_1+\left(a+a^*\right)\frac{\partial \phi_1}{\partial y}\right]E_+\\
&\quad +\frac{\partial \phi_1}{\partial y}\left[A_2,E_+\right]+i\left(a+a^*\right)\phi_1\left[A_2,E_+\right]
\end{aligned}$$

Here we replace

$$A_2=\frac{i}{2}\left(a+a^*\right)H \qquad (C.76)$$

and we have the commutator

$$\begin{aligned}
\left[A_2,E_+\right] &= \frac{i}{2}\left(a+a^*\right)\left[H,E_+\right] \qquad (C.77)\\
&= i\left(a+a^*\right)E_+
\end{aligned}$$

and obtain

$$\begin{aligned}
I^2 &= \frac{\partial^2 \phi_1}{\partial y^2}E_+ + i\left[\frac{\partial\left(a+a^*\right)}{\partial y}\phi_1+\left(a+a^*\right)\frac{\partial \phi_1}{\partial y}\right]E_+ \qquad (C.78)\\
&\quad +\frac{\partial \phi_1}{\partial y}i\left(a+a^*\right)E_+ - \left(a+a^*\right)^2\phi_1 E_+
\end{aligned}$$

Now we expand the expression of the second part

$$\begin{aligned} II^2 &= \left(\frac{\partial}{\partial y}+[A_2,]\right)\left\{\left[\frac{\partial\phi_2}{\partial y}-i\left(a+a^*\right)\phi_2\right]E_-\right\} \\ &= \frac{\partial^2\phi_2}{\partial y^2}E_- - i\left[\frac{\partial\left(a+a^*\right)}{\partial y}\phi_2+\left(a+a^*\right)\frac{\partial\phi_2}{\partial y}\right]E_- \\ &\quad +\frac{\partial\phi_2}{\partial y}\left[A_2,E_-\right]-i\left(a+a^*\right)\phi_2\left[A_2,E_-\right] \end{aligned} \tag{C.79}$$

As before we use

$$\begin{aligned} \left[A_2,E_-\right] &= \frac{i}{2}\left(a+a^*\right)\left[H,E_-\right] \\ &= -i\left(a+a^*\right)E_- \end{aligned} \tag{C.80}$$

to replace the commutators

$$\begin{aligned} II^2 &= \frac{\partial^2\phi_2}{\partial y^2}E_- - i\left[\frac{\partial\left(a+a^*\right)}{\partial y}\phi_2+\left(a+a^*\right)\frac{\partial\phi_2}{\partial y}\right]E_- \\ &\quad +\frac{\partial\phi_2}{\partial y}\left(-\right)i\left(a+a^*\right)E_- - \left(a+a^*\right)^2\phi_2E_- \end{aligned} \tag{C.81}$$

The final formula for this first part of the Right Hand Side is

$$I^2+II^2 \tag{C.82}$$

$$\begin{aligned} &\left(\frac{\partial}{\partial y}+[A_2,]\right)\left\{\left[\frac{\partial\phi_1}{\partial y}+i\left(a+a^*\right)\phi_1\right]E_+ + \left[\frac{\partial\phi_2}{\partial y}-i\left(a+a^*\right)\phi_2\right]E_-\right\} \\ =\ &\frac{\partial^2\phi_1}{\partial y^2}E_+ + i\left[\frac{\partial\left(a+a^*\right)}{\partial y}\phi_1+\left(a+a^*\right)\frac{\partial\phi_1}{\partial y}\right]E_+ \\ &+\frac{\partial^2\phi_2}{\partial y^2}E_- - i\left[\frac{\partial\left(a+a^*\right)}{\partial y}\phi_2+\left(a+a^*\right)\frac{\partial\phi_2}{\partial y}\right]E_- \\ &+\frac{\partial\phi_1}{\partial y}i\left(a+a^*\right)E_+ - \left(a+a^*\right)^2\phi_1E_+ \\ &+\frac{\partial\phi_2}{\partial y}\left(-\right)i\left(a+a^*\right)E_- - \left(a+a^*\right)^2\phi_2E_- \end{aligned} \tag{C.83}$$

C.2.3 The full first term in the RHS of the FIRST equation of motion with the algebraic *ansatz*

This term is

$$\begin{aligned} &-\frac{1}{2}\left\{\left(\frac{\partial}{\partial x}+[A_1,]\right)\left(\frac{\partial}{\partial x}+\left[A^1,\right]\right)+\left(\frac{\partial}{\partial y}+[A_2,]\right)\left(\frac{\partial}{\partial y}+\left[A^2,\right]\right)\right\}\phi \\ =\ &-\frac{1}{2}\left(I^1+II^1+I^2+II^2\right) \end{aligned} \tag{C.84}$$

and it is constructed on the basis of the Eqs.(C.68) and (C.83). We write separately the coefficients of E_+ and of E_-.

The coefficient of E_+ (not yet multiplied by $-1/2$) is

$$\begin{aligned} & \frac{\partial^2 \phi_1}{\partial x^2} + \left[\frac{\partial (a - a^*)}{\partial x} \phi_1 + (a - a^*) \frac{\partial \phi_1}{\partial x} \right] \\ & + \frac{\partial \phi_1}{\partial x} (a - a^*) + (a - a^*)^2 \phi_1 \\ & + \frac{\partial^2 \phi_1}{\partial y^2} + i \left[\frac{\partial (a + a^*)}{\partial y} \phi_1 + (a + a^*) \frac{\partial \phi_1}{\partial y} \right] \\ & + \frac{\partial \phi_1}{\partial y} i (a + a^*) - (a + a^*)^2 \phi_1 \end{aligned} \tag{C.85}$$

The coefficient of E_- (not yet multiplied by $-1/2$) is

$$\begin{aligned} & \frac{\partial^2 \phi_2}{\partial x^2} - \left[\frac{\partial (a - a^*)}{\partial x} \phi_2 + (a - a^*) \frac{\partial \phi_2}{\partial x} \right] \\ & - \frac{\partial \phi_2}{\partial x} (a - a^*) + (a - a^*)^2 \phi_2 \\ & + \frac{\partial^2 \phi_2}{\partial y^2} - i \left[\frac{\partial (a + a^*)}{\partial y} \phi_2 + (a + a^*) \frac{\partial \phi_2}{\partial y} \right] \\ & + \frac{\partial \phi_2}{\partial y} (-) i (a + a^*) - (a + a^*)^2 \phi_2 \end{aligned} \tag{C.86}$$

C.2.4 The last term in the RHS of the first equation of motion, with the algebraic *ansatz*

This term is

$$-\frac{1}{2m\kappa} \left[\left[\phi, \phi^\dagger \right], \phi \right] \tag{C.87}$$

This is calculated in **xxx_clean.tex**. The steps and the result are:

$$\begin{aligned} \left[\phi, \phi^\dagger \right] &= (\phi_1^* \phi_1 - \phi_2^* \phi_2) H \\ &= (\rho_1 - \rho_2) H \end{aligned} \tag{C.88}$$

where we have introduced the notations

$$\begin{aligned} \rho_1 &\equiv |\phi_1|^2 \\ \rho_2 &\equiv |\phi_2|^2 \end{aligned} \tag{C.89}$$

The next step is to calculate

$$\left[\left[\phi, \phi^\dagger \right], \phi \right] = \left[(\rho_1 - \rho_2) H, \phi_1 E_+ + \phi_2 E_- \right] \tag{C.90}$$

This is

$$\left[\left[\phi,\phi^{\dagger}\right],\phi\right] = (\rho_1-\rho_2)\,\phi_1\,[H,E_+] + (\rho_1-\rho_2)\,\phi_2\,[H,E_-] \tag{C.91}$$
$$= 2\,(\rho_1-\rho_2)\,(\phi_1 E_+ - \phi_2 E_-)$$

Finally

$$-\frac{1}{2m\kappa}\left[\left[\phi,\phi^{\dagger}\right],\phi\right] = -\frac{1}{2m\kappa}2\,(\rho_1-\rho_2)\,(\phi_1 E_+ - \phi_2 E_-) \tag{C.92}$$
$$= -\frac{1}{m\kappa}\,(\rho_1-\rho_2)\,(\phi_1 E_+ - \phi_2 E_-)$$

C.2.5 The full equations obtained from the FIRST (matter) equation of motion after adopting the algebraic *ansatz*

Here are the terms that results by equating the coefficients of the two ladder generators.

The equation resulting from E_+. We use Eqs.(C.52), (C.85) and (C.92)

$$i\frac{\partial\phi_1}{\partial t} - 2ib\phi_1 \tag{C.93}$$
$$= -\frac{1}{2}\frac{\partial^2\phi_1}{\partial x^2} - \frac{1}{2}\left[\frac{\partial\,(a-a^*)}{\partial x}\phi_1 + (a-a^*)\,\frac{\partial\phi_1}{\partial x}\right]$$
$$-\frac{1}{2}\frac{\partial\phi_1}{\partial x}\,(a-a^*) - \frac{1}{2}\,(a-a^*)^2\,\phi_1$$
$$-\frac{1}{2}\frac{\partial^2\phi_1}{\partial y^2} - \frac{i}{2}\left[\frac{\partial\,(a+a^*)}{\partial y}\phi_1 + (a+a^*)\,\frac{\partial\phi_1}{\partial y}\right]$$
$$-\frac{i}{2}\frac{\partial\phi_1}{\partial y}\,(a+a^*) + \frac{1}{2}\,(a+a^*)^2\,\phi_1$$
$$-\frac{1}{m\kappa}\,(\rho_1-\rho_2)\,\phi_1$$

The equation resulting from E_-. We use Eqs.(C.52), (C.86) and (C.92)

$$\begin{aligned}
& i\frac{\partial \phi_2}{\partial t} + 2ib\phi_2 \\
= \; & -\frac{1}{2}\frac{\partial^2 \phi_2}{\partial x^2} + \frac{1}{2}\left[\frac{\partial (a - a^*)}{\partial x}\phi_2 + (a - a^*)\frac{\partial \phi_2}{\partial x}\right] \\
& +\frac{1}{2}\frac{\partial \phi_2}{\partial x}(a - a^*) - \frac{1}{2}(a - a^*)^2 \phi_2 \\
& -\frac{1}{2}\frac{\partial^2 \phi_2}{\partial y^2} + \frac{i}{2}\left[\frac{\partial (a + a^*)}{\partial y}\phi_2 + (a + a^*)\frac{\partial \phi_2}{\partial y}\right] \\
& +\frac{i}{2}\frac{\partial \phi_2}{\partial y}(a + a^*) + \frac{1}{2}(a + a^*)^2 \phi_2 \\
& +\frac{1}{m\kappa}(\rho_1 - \rho_2)\phi_2
\end{aligned} \tag{C.94}$$

D Appendix D. Applications of the equations of motion

We examine how the equations of motion can be transformed into a form that gives the time evolution of the vorticity, defined as

$$\rho_1 - \rho_2 \tag{D.1}$$

D.1 Derivation of the equation for $\rho_1 = |\phi_1|^2$

The equation resulting from E_+.

This is the equation for ϕ_1.

$$\begin{aligned}
& i\frac{\partial \phi_1}{\partial t} - 2ib\phi_1 \\
= \; & -\frac{1}{2}\frac{\partial^2 \phi_1}{\partial x^2} - \frac{1}{2}\left[\frac{\partial (a - a^*)}{\partial x}\phi_1 + (a - a^*)\frac{\partial \phi_1}{\partial x}\right] \\
& -\frac{1}{2}\frac{\partial \phi_1}{\partial x}(a - a^*) - \frac{1}{2}(a - a^*)^2 \phi_1 \\
& -\frac{1}{2}\frac{\partial^2 \phi_1}{\partial y^2} - \frac{i}{2}\left[\frac{\partial (a + a^*)}{\partial y}\phi_1 + (a + a^*)\frac{\partial \phi_1}{\partial y}\right] \\
& -\frac{i}{2}\frac{\partial \phi_1}{\partial y}(a + a^*) + \frac{1}{2}(a + a^*)^2 \phi_1 \\
& -\frac{1}{m\kappa}(\rho_1 - \rho_2)\phi_1
\end{aligned} \tag{D.2}$$

Now we write this equation for the complex conjugate, ϕ_1^*.

$$\begin{aligned}
& -i\frac{\partial \phi_1^*}{\partial t} + 2ib^*\phi_1^* \\
= \quad & -\frac{1}{2}\frac{\partial^2 \phi_1^*}{\partial x^2} - \frac{1}{2}\left[\frac{\partial\left(a^* - a\right)}{\partial x}\phi_1^* + \left(a^* - a\right)\frac{\partial \phi_1^*}{\partial x}\right] \\
& -\frac{1}{2}\frac{\partial \phi_1^*}{\partial x}\left(a^* - a\right) - \frac{1}{2}\left(a^* - a\right)^2\phi_1^* \\
& -\frac{1}{2}\frac{\partial^2 \phi_1^*}{\partial y^2} + \frac{i}{2}\left[\frac{\partial\left(a^* + a\right)}{\partial y}\phi_1^* + \left(a^* + a\right)\frac{\partial \phi_1^*}{\partial y}\right] \\
& +\frac{i}{2}\frac{\partial \phi_1^*}{\partial y}\left(a^* + a\right) + \frac{1}{2}\left(a^* + a\right)^2\phi_1^* \\
& -\frac{1}{m\kappa}\left(\rho_1 - \rho_2\right)\phi_1^*
\end{aligned} \tag{D.3}$$

Now we multiply the first equation (for ϕ_1) with ϕ_1^*.

$$\begin{aligned}
& i\phi_1^*\frac{\partial \phi_1}{\partial t} - 2ib\phi_1^*\phi_1 \\
= \quad & -\frac{1}{2}\phi_1^*\frac{\partial^2 \phi_1}{\partial x^2} - \frac{1}{2}\left[\frac{\partial\left(a - a^*\right)}{\partial x}\phi_1^*\phi_1 + \left(a - a^*\right)\phi_1^*\frac{\partial \phi_1}{\partial x}\right] \\
& -\frac{1}{2}\phi_1^*\frac{\partial \phi_1}{\partial x}\left(a - a^*\right) - \frac{1}{2}\left(a - a^*\right)^2\phi_1^*\phi_1 \\
& -\frac{1}{2}\phi_1^*\frac{\partial^2 \phi_1}{\partial y^2} - \frac{i}{2}\left[\frac{\partial\left(a + a^*\right)}{\partial y}\phi_1^*\phi_1 + \left(a + a^*\right)\phi_1^*\frac{\partial \phi_1}{\partial y}\right] \\
& -\frac{i}{2}\phi_1^*\frac{\partial \phi_1}{\partial y}\left(a + a^*\right) + \frac{1}{2}\left(a + a^*\right)^2\phi_1^*\phi_1 \\
& -\frac{1}{\kappa}\left(\rho_1 - \rho_2\right)\phi_1^*\phi_1
\end{aligned} \tag{D.4}$$

Similarly, we multiply the second equation (for ϕ_1^*) by ϕ_1.

$$\begin{aligned}
& -i\phi_1 \frac{\partial \phi_1^*}{\partial t} + 2ib^* \phi_1 \phi_1^* \\
= \quad & -\frac{1}{2}\phi_1 \frac{\partial^2 \phi_1^*}{\partial x^2} - \frac{1}{2}\left[\frac{\partial (a^* - a)}{\partial x}\phi_1\phi_1^* + (a^* - a)\,\phi_1 \frac{\partial \phi_1^*}{\partial x}\right] \\
& -\frac{1}{2}\phi_1 \frac{\partial \phi_1^*}{\partial x}(a^* - a) - \frac{1}{2}(a^* - a)^2 \phi_1\phi_1^* \\
& -\frac{1}{2}\phi_1 \frac{\partial^2 \phi_1^*}{\partial y^2} + \frac{i}{2}\left[\frac{\partial (a^* + a)}{\partial y}\phi_1\phi_1^* + (a^* + a)\,\phi_1 \frac{\partial \phi_1^*}{\partial y}\right] \\
& +\frac{i}{2}\phi_1 \frac{\partial \phi_1^*}{\partial y}(a^* + a) + \frac{1}{2}(a^* + a)^2 \phi_1\phi_1^* \\
& -\frac{1}{m\kappa}(\rho_1 - \rho_2)\,\phi_1\phi_1^*
\end{aligned} \tag{D.5}$$

Here we begin the combination of the first two equations, one for ϕ_1 and the second for ϕ_1^*. We will work line by line. We *substract* the two equations, with the intention of getting a time derivative of the modulus $\phi_1\phi_1^*$.

$$\begin{aligned}
& \text{first line } \left(i\phi_1^* \frac{\partial \phi_1}{\partial t} - 2ib\phi_1^*\phi_1\right) - \left(-i\phi_1 \frac{\partial \phi_1^*}{\partial t} + 2ib^*\phi_1\phi_1^*\right) \\
= \quad & i\frac{\partial}{\partial t}(\phi_1\phi_1^*) - 2i\,(b + b^*)\,|\phi_1|^2
\end{aligned} \tag{D.6}$$

$$\text{first term of the second line } \left(-\frac{1}{2}\phi_1^* \frac{\partial^2 \phi_1}{\partial x^2}\right) - \left(-\frac{1}{2}\phi_1 \frac{\partial^2 \phi_1^*}{\partial x^2}\right) \tag{D.7}$$

$$\begin{aligned}
& \text{second term of the second line} \\
& \left(-\frac{1}{2}\left[\frac{\partial (a - a^*)}{\partial x}\phi_1^*\phi_1 + (a - a^*)\,\phi_1^* \frac{\partial \phi_1}{\partial x}\right]\right) - \\
& -\left(-\frac{1}{2}\left[\frac{\partial (a^* - a)}{\partial x}\phi_1\phi_1^* + (a^* - a)\,\phi_1 \frac{\partial \phi_1^*}{\partial x}\right]\right) \\
= \quad & -\frac{1}{2}\left[\frac{\partial (a - a^*)}{\partial x}(\phi_1^*\phi_1 + \phi_1\phi_1^*)\right. \\
& \left. + (a - a^*)\left(\phi_1^* \frac{\partial \phi_1}{\partial x} + \phi_1 \frac{\partial \phi_1^*}{\partial x}\right)\right] \\
= \quad & -\frac{\partial (a - a^*)}{\partial x}|\phi_1|^2 - \frac{1}{2}(a - a^*)\frac{\partial}{\partial x}\left(|\phi_1|^2\right)
\end{aligned} \tag{D.8}$$

$$\begin{aligned}
&\text{first term of the third line} \\
&\left(-\frac{1}{2}\phi_1^*\frac{\partial\phi_1}{\partial x}\left(a-a^*\right)\right)-\left(-\frac{1}{2}\phi_1\frac{\partial\phi_1^*}{\partial x}\left(a^*-a\right)\right) \\
=\;& -\frac{1}{2}\left(a-a^*\right)\left(\phi_1^*\frac{\partial\phi_1}{\partial x}+\phi_1\frac{\partial\phi_1^*}{\partial x}\right) \\
=\;& -\frac{1}{2}\left(a-a^*\right)\frac{\partial}{\partial x}\left|\phi_1\right|^2
\end{aligned} \tag{D.9}$$

$$\begin{aligned}
&\text{second term of the third line} \\
&\left(-\frac{1}{2}\left(a-a^*\right)^2\phi_1^*\phi_1\right)-\left(-\frac{1}{2}\left(a^*-a\right)^2\phi_1\phi_1^*\right) \\
=\;& 0
\end{aligned}$$

$$\begin{aligned}
&\text{first term of the fourth line} \\
&\left(-\frac{1}{2}\phi_1^*\frac{\partial^2\phi_1}{\partial y^2}\right)-\left(-\frac{1}{2}\phi_1\frac{\partial^2\phi_1^*}{\partial y^2}\right) \\
=\;& -\frac{1}{2}\left(\phi_1^*\frac{\partial^2\phi_1}{\partial y^2}-\phi_1\frac{\partial^2\phi_1^*}{\partial y^2}\right)
\end{aligned} \tag{D.10}$$

$$\begin{aligned}
&\text{second term of the fourth line} \\
&\left(-\frac{i}{2}\left[\frac{\partial\left(a+a^*\right)}{\partial y}\phi_1^*\phi_1+\left(a+a^*\right)\phi_1^*\frac{\partial\phi_1}{\partial y}\right]\right) \\
&-\left(\frac{i}{2}\left[\frac{\partial\left(a^*+a\right)}{\partial y}\phi_1\phi_1^*+\left(a^*+a\right)\phi_1\frac{\partial\phi_1^*}{\partial y}\right]\right) \\
=\;& -i\frac{\partial\left(a+a^*\right)}{\partial y}\left|\phi_1\right|^2-\frac{i}{2}\left(a+a^*\right)\frac{\partial}{\partial y}\left|\phi_1\right|^2
\end{aligned} \tag{D.11}$$

$$\begin{aligned}
&\text{first term of the fifth line} \\
&\left(-\frac{i}{2}\phi_1^*\frac{\partial\phi_1}{\partial y}\left(a+a^*\right)\right)-\left(\frac{i}{2}\phi_1\frac{\partial\phi_1^*}{\partial y}\left(a^*+a\right)\right) \\
=\;& -\frac{i}{2}\left(a+a^*\right)\frac{\partial}{\partial y}\left|\phi_1\right|^2
\end{aligned} \tag{D.12}$$

$$\begin{aligned}
&\text{second term of the fifth line} \\
&\left(\frac{1}{2}\left(a+a^*\right)^2\phi_1^*\phi_1\right)-\left(\frac{1}{2}\left(a^*+a\right)^2\phi_1\phi_1^*\right) \\
=\;& 0
\end{aligned} \tag{D.13}$$

$$
\begin{aligned}
& \text{term of the sixth line} \\
& \frac{1}{m\kappa}\left(-\left(\rho_1-\rho_2\right)\phi_1^*\phi_1\right)-\left(-\left(\rho_1-\rho_2\right)\phi_1\phi_1^*\right) \\
= \; & 0
\end{aligned} \tag{D.14}
$$

What results:

$$
\begin{aligned}
& i\frac{\partial}{\partial t}\left|\phi_1\right|^2-2i\left(b+b^*\right)\left|\phi_1\right|^2 \quad \text{first line} \\
= \; & -\frac{1}{2}\left(\phi_1^*\frac{\partial^2\phi_1}{\partial x^2}-\phi_1\frac{\partial^2\phi_1^*}{\partial x^2}\right) \quad \text{first term of the second line} \\
& -\frac{\partial\left(a-a^*\right)}{\partial x}\left|\phi_1\right|^2-\frac{1}{2}\left(a-a^*\right)\frac{\partial}{\partial x}\left(\left|\phi_1\right|^2\right) \text{ second term of the second line} \\
& -\frac{1}{2}\left(a-a^*\right)\frac{\partial}{\partial x}\left|\phi_1\right|^2 \quad \text{first term of the third line} \\
& -\frac{1}{2}\left(\phi_1^*\frac{\partial^2\phi_1}{\partial y^2}-\phi_1\frac{\partial^2\phi_1^*}{\partial y^2}\right) \quad \text{first term of the fourth line} \\
& -i\frac{\partial\left(a+a^*\right)}{\partial y}\left|\phi_1\right|^2-\frac{i}{2}\left(a+a^*\right)\frac{\partial}{\partial y}\left|\phi_1\right|^2 \quad \text{second term of the fourth line} \\
& -\frac{i}{2}\left(a+a^*\right)\frac{\partial}{\partial y}\left|\phi_1\right|^2 \quad \text{first term of the fifth line}
\end{aligned} \tag{D.15}
$$

D.1.1 Derivation of the equation for $\rho_2=\left|\phi_2\right|^2$

The equation resulting from E_-. We use Eqs.(C.52), (C.86) and (C.92)

$$
\begin{aligned}
& i\frac{\partial\phi_2}{\partial t}+2ib\phi_2 \\
= \; & -\frac{1}{2}\frac{\partial^2\phi_2}{\partial x^2}+\frac{1}{2}\left[\frac{\partial\left(a-a^*\right)}{\partial x}\phi_2+\left(a-a^*\right)\frac{\partial\phi_2}{\partial x}\right] \\
& +\frac{1}{2}\frac{\partial\phi_2}{\partial x}\left(a-a^*\right)-\frac{1}{2}\left(a-a^*\right)^2\phi_2 \\
& -\frac{1}{2}\frac{\partial^2\phi_2}{\partial y^2}+\frac{i}{2}\left[\frac{\partial\left(a+a^*\right)}{\partial y}\phi_2+\left(a+a^*\right)\frac{\partial\phi_2}{\partial y}\right] \\
& +\frac{i}{2}\frac{\partial\phi_2}{\partial y}\left(a+a^*\right)+\frac{1}{2}\left(a+a^*\right)^2\phi_2 \\
& +\frac{1}{m\kappa}\left(\rho_1-\rho_2\right)\phi_2
\end{aligned} \tag{D.16}
$$

Now we write this equation after taking the complex cojugate

$$
\begin{aligned}
& -i\frac{\partial \phi_2^*}{\partial t} - 2ib^*\phi_2^* \\
= \; & -\frac{1}{2}\frac{\partial^2 \phi_2^*}{\partial x^2} + \frac{1}{2}\left[\frac{\partial\left(a^* - a\right)}{\partial x}\phi_2^* + \left(a^* - a\right)\frac{\partial \phi_2^*}{\partial x}\right] \\
& +\frac{1}{2}\frac{\partial \phi_2^*}{\partial x}\left(a^* - a\right) - \frac{1}{2}\left(a^* - a\right)^2\phi_2^* \\
& -\frac{1}{2}\frac{\partial^2 \phi_2^*}{\partial y^2} - \frac{i}{2}\left[\frac{\partial\left(a^* + a\right)}{\partial y}\phi_2^* + \left(a^* + a\right)\frac{\partial \phi_2^*}{\partial y}\right] \\
& -\frac{i}{2}\frac{\partial \phi_2^*}{\partial y}\left(a^* + a\right) + \frac{1}{2}\left(a^* + a\right)^2\phi_2^* \\
& +\frac{1}{m\kappa}\left(\rho_1 - \rho_2\right)\phi_2^*
\end{aligned}
\tag{D.17}
$$

The first equation is multiplied with ϕ_2^* and the result is

$$
\begin{aligned}
& i\phi_2^*\frac{\partial \phi_2}{\partial t} + 2ib\phi_2^*\phi_2 \\
= \; & -\frac{1}{2}\phi_2^*\frac{\partial^2 \phi_2}{\partial x^2} + \frac{1}{2}\left[\frac{\partial\left(a - a^*\right)}{\partial x}\phi_2^*\phi_2 + \left(a - a^*\right)\phi_2^*\frac{\partial \phi_2}{\partial x}\right] \\
& +\frac{1}{2}\phi_2^*\frac{\partial \phi_2}{\partial x}\left(a - a^*\right) - \frac{1}{2}\left(a - a^*\right)^2\phi_2^*\phi_2 \\
& -\frac{1}{2}\phi_2^*\frac{\partial^2 \phi_2}{\partial y^2} + \frac{i}{2}\left[\frac{\partial\left(a + a^*\right)}{\partial y}\phi_2^*\phi_2 + \left(a + a^*\right)\phi_2^*\frac{\partial \phi_2}{\partial y}\right] \\
& +\frac{i}{2}\phi_2^*\frac{\partial \phi_2}{\partial y}\left(a + a^*\right) + \frac{1}{2}\left(a + a^*\right)^2\phi_2^*\phi_2 \\
& +\frac{1}{m\kappa}\left(\rho_1 - \rho_2\right)\phi_2^*\phi_2
\end{aligned}
\tag{D.18}
$$

and the equation for ϕ_2^* is multiplied by ϕ_2 with the result

$$\begin{aligned} & -i\phi_2\frac{\partial\phi_2^*}{\partial t}-2ib^*\phi_2\phi_2^* \\ = & -\frac{1}{2}\phi_2\frac{\partial^2\phi_2^*}{\partial x^2}+\frac{1}{2}\left[\frac{\partial\left(a^*-a\right)}{\partial x}\phi_2\phi_2^*+\left(a^*-a\right)\phi_2\frac{\partial\phi_2^*}{\partial x}\right] \\ & +\frac{1}{2}\phi_2\frac{\partial\phi_2^*}{\partial x}\left(a^*-a\right)-\frac{1}{2}\left(a^*-a\right)^2\phi_2\phi_2^* \\ & -\frac{1}{2}\phi_2\frac{\partial^2\phi_2^*}{\partial y^2}-\frac{i}{2}\left[\frac{\partial\left(a^*+a\right)}{\partial y}\phi_2\phi_2^*+\left(a^*+a\right)\phi_2\frac{\partial\phi_2^*}{\partial y}\right] \\ & -\frac{i}{2}\phi_2\frac{\partial\phi_2^*}{\partial y}\left(a^*+a\right)+\frac{1}{2}\left(a^*+a\right)^2\phi_2\phi_2^* \\ & +\frac{1}{m\kappa}\left(\rho_1-\rho_2\right)\phi_2\phi_2^* \end{aligned} \tag{D.19}$$

Now we will substract the two equations, in order to obtain the time derivative $\partial/\partial t$ of the product $\phi_2^*\phi_2$. The terms are written one by one

$$\begin{aligned} & \text{first term on the first line} \\ i\phi_2^*\frac{\partial\phi_2}{\partial t}+i\phi_2\frac{\partial\phi_2^*}{\partial t} &= i\frac{\partial}{\partial t}\left|\phi_2\right|^2 \end{aligned} \tag{D.20}$$

$$\begin{aligned} & \text{the second term of the first line} \\ & 2ib\phi_2^*\phi_2+2ib^*\phi_2\phi_2^* \\ = & \ 2i\left(b+b^*\right)\left|\phi_2\right|^2 \end{aligned} \tag{D.21}$$

$$\begin{aligned} & \text{the first term of the second line} \\ & -\frac{1}{2}\phi_2^*\frac{\partial^2\phi_2}{\partial x^2}+\frac{1}{2}\phi_2\frac{\partial^2\phi_2^*}{\partial x^2} \end{aligned} \tag{D.22}$$

$$\begin{aligned} & \text{the second term of the second line} \\ & \frac{1}{2}\left[\frac{\partial\left(a-a^*\right)}{\partial x}\phi_2^*\phi_2+\left(a-a^*\right)\phi_2^*\frac{\partial\phi_2}{\partial x}\right] \\ & -\frac{1}{2}\left[\frac{\partial\left(a^*-a\right)}{\partial x}\phi_2\phi_2^*+\left(a^*-a\right)\phi_2\frac{\partial\phi_2^*}{\partial x}\right] \\ = & \ \frac{\partial\left(a-a^*\right)}{\partial x}\left|\phi_2\right|^2+\frac{1}{2}\left(a-a^*\right)\frac{\partial}{\partial x}\left(\left|\phi_2\right|^2\right) \end{aligned} \tag{D.23}$$

$$\begin{aligned} & \text{the first term of the third line} \\ & \frac{1}{2}\phi_2^*\frac{\partial\phi_2}{\partial x}\left(a-a^*\right)-\frac{1}{2}\phi_2\frac{\partial\phi_2^*}{\partial x}\left(a^*-a\right) \\ = & \ \frac{1}{2}\left(a-a^*\right)\frac{\partial}{\partial x}\left(\left|\phi_2\right|^2\right) \end{aligned} \tag{D.24}$$

$$\begin{aligned}&\text{the second term of the third line}\\&-\frac{1}{2}\left(a-a^*\right)^2\phi_2^*\phi_2+\frac{1}{2}\left(a^*-a\right)^2\phi_2\phi_2^*\\=\;&0\end{aligned}\tag{D.25}$$

$$\begin{aligned}&\text{the first term of the fourth line}\\&-\frac{1}{2}\phi_2^*\frac{\partial^2\phi_2}{\partial y^2}+\frac{1}{2}\phi_2\frac{\partial^2\phi_2^*}{\partial y^2}\end{aligned}\tag{D.26}$$

$$\begin{aligned}&\text{the second term of the fourth line}\\&\frac{i}{2}\left[\frac{\partial\left(a+a^*\right)}{\partial y}\phi_2^*\phi_2+\left(a+a^*\right)\phi_2^*\frac{\partial\phi_2}{\partial y}\right]\\&+\frac{i}{2}\left[\frac{\partial\left(a^*+a\right)}{\partial y}\phi_2\phi_2^*+\left(a^*+a\right)\phi_2\frac{\partial\phi_2^*}{\partial y}\right]\\=\;&i\frac{\partial\left(a+a^*\right)}{\partial y}\left|\phi_2\right|^2+\frac{i}{2}\left(a+a^*\right)\frac{\partial}{\partial y}\left(\left|\phi_2\right|^2\right)\end{aligned}\tag{D.27}$$

$$\begin{aligned}&\text{the first term in the fifth line}\\&\frac{i}{2}\phi_2^*\frac{\partial\phi_2}{\partial y}\left(a+a^*\right)+\frac{i}{2}\phi_2\frac{\partial\phi_2^*}{\partial y}\left(a^*+a\right)\\=\;&\frac{i}{2}\left(a+a^*\right)\frac{\partial}{\partial y}\left(\left|\phi_2\right|^2\right)\end{aligned}\tag{D.28}$$

$$\begin{aligned}&\text{the second term in the fifth line}\\&\frac{1}{2}\left(a+a^*\right)^2\phi_2^*\phi_2-\frac{1}{2}\left(a^*+a\right)^2\phi_2\phi_2^*\\=\;&0\end{aligned}\tag{D.29}$$

$$\begin{aligned}&\text{the term of the sixth line}\\&\frac{1}{m\kappa}\left(\rho_1-\rho_2\right)\phi_2^*\phi_2-\frac{1}{m\kappa}\left(\rho_1-\rho_2\right)\phi_2\phi_2^*\\=\;&0\end{aligned}\tag{D.30}$$

What results

$$
\begin{aligned}
& i\frac{\partial}{\partial t}\left|\phi_2\right|^2 + 2i\left(b+b^*\right)\left|\phi_2\right|^2 \quad \text{first line} \\
= \; & -\frac{1}{2}\phi_2^*\frac{\partial^2\phi_2}{\partial x^2} + \frac{1}{2}\phi_2\frac{\partial^2\phi_2^*}{\partial x^2} \quad \text{first term of the second line} \\
& +\frac{\partial\left(a-a^*\right)}{\partial x}\left|\phi_2\right|^2 + \frac{1}{2}\left(a-a^*\right)\frac{\partial}{\partial x}\left(\left|\phi_2\right|^2\right) \quad \text{the second term of the second line} \\
& +\frac{1}{2}\left(a-a^*\right)\frac{\partial}{\partial x}\left(\left|\phi_2\right|^2\right) \quad \text{the first term of the third line} \\
& -\frac{1}{2}\phi_2^*\frac{\partial^2\phi_2}{\partial y^2} + \frac{1}{2}\phi_2\frac{\partial^2\phi_2^*}{\partial y^2} \quad \text{the first term of the fourth line} \\
& +i\frac{\partial\left(a+a^*\right)}{\partial y}\left|\phi_2\right|^2 + \frac{i}{2}\left(a+a^*\right)\frac{\partial}{\partial y}\left(\left|\phi_2\right|^2\right) \quad \text{the second term of the fourth line} \\
& +\frac{i}{2}\left(a+a^*\right)\frac{\partial}{\partial y}\left(\left|\phi_2\right|^2\right) \quad \text{the first term of the fifth line}
\end{aligned} \tag{D.31}
$$

D.1.2 Derivation of the equation for the difference $\Omega \equiv \left|\phi_1\right|^2 - \left|\phi_2\right|^2$

Now let us substract the two equations such as to obtain the combination

$$\Omega \equiv \left|\phi_1\right|^2 - \left|\phi_2\right|^2 \tag{D.32}$$

and

$$\Xi \equiv \left|\phi_1\right|^2 + \left|\phi_2\right|^2 \tag{D.33}$$

.

$$i\frac{\partial}{\partial t}\left|\phi_1\right|^2 - i\frac{\partial}{\partial t}\left|\phi_2\right|^2 = i\frac{\partial}{\partial t}\Omega \text{ first terms on the first lines} \tag{D.34}$$

$$-2i\left(b+b^*\right)\left|\phi_1\right|^2 - 2i\left(b+b^*\right)\left|\phi_2\right|^2 = -2i\left(b+b^*\right)\Xi \text{ second terms of the first lines} \tag{D.35}$$

$$
\begin{aligned}
& -\frac{1}{2}\phi_1^*\frac{\partial^2\phi_1}{\partial x^2} + \frac{1}{2}\phi_1\frac{\partial^2\phi_1^*}{\partial x^2} + \frac{1}{2}\phi_2^*\frac{\partial^2\phi_2}{\partial x^2} - \frac{1}{2}\phi_2\frac{\partial^2\phi_2^*}{\partial x^2} \\
& -\frac{1}{2}\phi_1^*\frac{\partial^2\phi_1}{\partial y^2} + \frac{1}{2}\phi_1\frac{\partial^2\phi_1^*}{\partial y^2} + \frac{1}{2}\phi_2^*\frac{\partial^2\phi_2}{\partial y^2} - \frac{1}{2}\phi_2\frac{\partial^2\phi_2^*}{\partial y^2} \text{ terms with second order derivations}
\end{aligned} \tag{D.36}
$$

$$
\begin{aligned}
& -\frac{\partial\left(a-a^*\right)}{\partial x}\left|\phi_1\right|^2 - \frac{1}{2}\left(a-a^*\right)\frac{\partial}{\partial x}\left(\left|\phi_1\right|^2\right) - \frac{\partial\left(a-a^*\right)}{\partial x}\left|\phi_2\right|^2 - \frac{1}{2}\left(a-a^*\right)\frac{\partial}{\partial x}\left(\left|\phi_2\right|^2\right) \\
= \; & -\frac{\partial\left(a-a^*\right)}{\partial x}\Xi - \frac{1}{2}\left(a-a^*\right)\frac{\partial}{\partial x}\Xi \quad \text{the second terms of the second lines}
\end{aligned} \tag{D.37}
$$

$$
\begin{aligned}
& -\frac{1}{2}\left(a-a^{*}\right)\frac{\partial}{\partial x}\left(|\phi_1|^2\right)-\frac{1}{2}\left(a-a^{*}\right)\frac{\partial}{\partial x}\left(|\phi_2|^2\right) \\
= & -\frac{1}{2}\left(a-a^{*}\right)\frac{\partial}{\partial x}\Xi \quad \text{the first terms of the third lines}
\end{aligned} \tag{D.38}
$$

$$
\begin{aligned}
& -i\frac{\partial\left(a+a^{*}\right)}{\partial y}|\phi_1|^2-\frac{i}{2}\left(a+a^{*}\right)\frac{\partial}{\partial y}\left(|\phi_1|^2\right)-i\frac{\partial\left(a+a^{*}\right)}{\partial y}|\phi_2|^2-\frac{i}{2}\left(a+a^{*}\right)\frac{\partial}{\partial y}\left(|\phi_2|^2\right) \\
= & -i\frac{\partial\left(a+a^{*}\right)}{\partial y}\Xi-\frac{i}{2}\left(a+a^{*}\right)\frac{\partial}{\partial y}\Xi \quad \text{the second terms of the fourth lines}
\end{aligned} \tag{D.39}
$$

$$
\begin{aligned}
& -\frac{i}{2}\left(a+a^{*}\right)\frac{\partial}{\partial y}|\phi_1|^2-\frac{i}{2}\left(a+a^{*}\right)\frac{\partial}{\partial y}\left(|\phi_2|^2\right) \\
= & -\frac{i}{2}\left(a+a^{*}\right)\frac{\partial}{\partial y}\Xi \quad \text{the first term of the fifth line}
\end{aligned} \tag{D.40}
$$

We now collect the results

$$
\begin{aligned}
& i\frac{\partial}{\partial t}\Omega-2i\left(b+b^{*}\right)\Xi \\
= & -\frac{1}{2}\phi_1^{*}\frac{\partial^2\phi_1}{\partial x^2}+\frac{1}{2}\phi_1\frac{\partial^2\phi_1^{*}}{\partial x^2}+\frac{1}{2}\phi_2^{*}\frac{\partial^2\phi_2}{\partial x^2}-\frac{1}{2}\phi_2\frac{\partial^2\phi_2^{*}}{\partial x^2} \\
& -\frac{1}{2}\phi_1^{*}\frac{\partial^2\phi_1}{\partial y^2}+\frac{1}{2}\phi_1\frac{\partial^2\phi_1^{*}}{\partial y^2}+\frac{1}{2}\phi_2^{*}\frac{\partial^2\phi_2}{\partial y^2}-\frac{1}{2}\phi_2\frac{\partial^2\phi_2^{*}}{\partial y^2} \\
& -\frac{\partial\left(a-a^{*}\right)}{\partial x}\Xi-\frac{1}{2}\left(a-a^{*}\right)\frac{\partial}{\partial x}\Xi \\
& \qquad -\frac{1}{2}\left(a-a^{*}\right)\frac{\partial}{\partial x}\Xi \\
& -i\frac{\partial\left(a+a^{*}\right)}{\partial y}\Xi-\frac{i}{2}\left(a+a^{*}\right)\frac{\partial}{\partial y}\Xi \\
& \qquad -\frac{i}{2}\left(a+a^{*}\right)\frac{\partial}{\partial y}\Xi
\end{aligned} \tag{D.41}
$$

The result can still be transformed

$$
\begin{aligned}
& i\frac{\partial}{\partial t}\Omega \\
= \; & 2i\left(b+b^{*}\right)\Xi \\
& -\frac{1}{2}\phi_1^{*}\frac{\partial^2\phi_1}{\partial x^2}+\frac{1}{2}\phi_1\frac{\partial^2\phi_1^{*}}{\partial x^2}+\frac{1}{2}\phi_2^{*}\frac{\partial^2\phi_2}{\partial x^2}-\frac{1}{2}\phi_2\frac{\partial^2\phi_2^{*}}{\partial x^2} \\
& -\frac{1}{2}\phi_1^{*}\frac{\partial^2\phi_1}{\partial y^2}+\frac{1}{2}\phi_1\frac{\partial^2\phi_1^{*}}{\partial y^2}+\frac{1}{2}\phi_2^{*}\frac{\partial^2\phi_2}{\partial y^2}-\frac{1}{2}\phi_2\frac{\partial^2\phi_2^{*}}{\partial y^2} \\
& -\frac{\partial}{\partial x}\left[\left(a-a^{*}\right)\Xi\right] \\
& -i\frac{\partial}{\partial y}\left[\left(a+a^{*}\right)\Xi\right]
\end{aligned}
\tag{D.42}
$$

Now, if we re-insert the components of the potential

$$
\begin{aligned}
a-a^{*} &= 2A_x/H \equiv 2\overline{A}_x \\
i\left(a+a^{*}\right) &= 2A_y/H \equiv 2\overline{A}_y
\end{aligned}
\tag{D.43}
$$

and keep the complex coefficients b of the zero-component potential A_0

$$
\begin{aligned}
& i\frac{\partial}{\partial t}\Omega-2i\left(b+b^{*}\right)\Xi \\
= \; & F\left(\Delta;\phi_1,\phi_2\right) \\
& -\frac{\partial}{\partial x}\left(2\overline{A}_x\Xi\right)-\frac{\partial}{\partial y}\left(2\overline{A}_y\Xi\right)
\end{aligned}
\tag{D.44}
$$

where we have introduced the notation $F\left(\Delta;\phi_1,\phi_2\right)$ for the terms that contain second order derivatives.

$$
\begin{aligned}
& i\frac{\partial}{\partial t}\left(\rho_1-\rho_2\right)-2i\left(b+b^{*}\right)\left(\rho_1+\rho_2\right)+\frac{\partial}{\partial x}\left[2\overline{A}_x\left(\rho_1+\rho_2\right)\right]+\frac{\partial}{\partial y}\left[2\overline{A}_y\left(\rho_1+\rho_2\right)\right] \\
= \; & F\left(\Delta;\phi_1,\phi_2\right)
\end{aligned}
\tag{D.45}
$$

We transform the first two terms of the second-order differential terms

$$
\begin{aligned}
&F\left(\Delta;\phi_1,\phi_2\right)\\
&\qquad -\frac{1}{2}\phi_1^*\frac{\partial^2\phi_1}{\partial x^2}+\frac{1}{2}\phi_1\frac{\partial^2\phi_1^*}{\partial x^2} \\
&\quad = -\frac{1}{2}\frac{\partial}{\partial x}\left[\phi_1^*\frac{\partial\phi_1}{\partial x}\right]+\frac{1}{2}\left[\left(\frac{\partial\phi_1^*}{\partial x}\right)\left(\frac{\partial\phi_1}{\partial x}\right)\right] \\
&\qquad +\frac{1}{2}\frac{\partial}{\partial x}\left[\phi_1\frac{\partial\phi_1^*}{\partial x}\right]-\frac{1}{2}\left[\left(\frac{\partial\phi_1}{\partial x}\right)\left(\frac{\partial\phi_1^*}{\partial x}\right)\right] \\
&\quad = -\frac{1}{2}\frac{\partial}{\partial x}\left[\phi_1^*\frac{\partial\phi_1}{\partial x}-\phi_1\frac{\partial\phi_1^*}{\partial x}\right] \\
&\quad = -\frac{1}{2}\frac{\partial}{\partial x}\left[\left(\phi_1^*\right)^2\frac{\partial}{\partial x}\left(\frac{\phi_1}{\phi_1^*}\right)\right]
\end{aligned}
\tag{D.46}
$$

and take also the other pairs

$$
\begin{aligned}
&-\frac{1}{2}\phi_1^*\frac{\partial^2\phi_1}{\partial y^2}+\frac{1}{2}\phi_1\frac{\partial^2\phi_1^*}{\partial y^2} \\
&= -\frac{1}{2}\frac{\partial}{\partial y}\left[\left(\phi_1^*\right)^2\frac{\partial}{\partial y}\left(\frac{\phi_1}{\phi_1^*}\right)\right]
\end{aligned}
\tag{D.47}
$$

$$
\begin{aligned}
&\frac{1}{2}\phi_2^*\frac{\partial^2\phi_2}{\partial x^2}-\frac{1}{2}\phi_2\frac{\partial^2\phi_2^*}{\partial x^2} \\
&= \frac{1}{2}\frac{\partial}{\partial x}\left[\left(\phi_2^*\right)^2\frac{\partial}{\partial x}\left(\frac{\phi_2}{\phi_2^*}\right)\right]
\end{aligned}
\tag{D.48}
$$

$$
\begin{aligned}
&\frac{1}{2}\phi_2^*\frac{\partial^2\phi_2}{\partial y^2}-\frac{1}{2}\phi_2\frac{\partial^2\phi_2^*}{\partial y^2} \\
&= \frac{1}{2}\frac{\partial}{\partial y}\left[\left(\phi_2^*\right)^2\frac{\partial}{\partial y}\left(\frac{\phi_2}{\phi_2^*}\right)\right]
\end{aligned}
\tag{D.49}
$$

Then

$$
\begin{aligned}
&F\left(\Delta;\phi_1,\phi_2\right) \\
&= -\frac{1}{2}\phi_1^*\frac{\partial^2\phi_1}{\partial x^2}+\frac{1}{2}\phi_1\frac{\partial^2\phi_1^*}{\partial x^2}+\frac{1}{2}\phi_2^*\frac{\partial^2\phi_2}{\partial x^2}-\frac{1}{2}\phi_2\frac{\partial^2\phi_2^*}{\partial x^2} \\
&\quad -\frac{1}{2}\phi_1^*\frac{\partial^2\phi_1}{\partial y^2}+\frac{1}{2}\phi_1\frac{\partial^2\phi_1^*}{\partial y^2}+\frac{1}{2}\phi_2^*\frac{\partial^2\phi_2}{\partial y^2}-\frac{1}{2}\phi_2\frac{\partial^2\phi_2^*}{\partial y^2} \\
&= -\frac{1}{2}\frac{\partial}{\partial x}\left[\left(\phi_1^*\right)^2\frac{\partial}{\partial x}\left(\frac{\phi_1}{\phi_1^*}\right)\right]-\frac{1}{2}\frac{\partial}{\partial y}\left[\left(\phi_1^*\right)^2\frac{\partial}{\partial y}\left(\frac{\phi_1}{\phi_1^*}\right)\right] \\
&\quad +\frac{1}{2}\frac{\partial}{\partial x}\left[\left(\phi_2^*\right)^2\frac{\partial}{\partial x}\left(\frac{\phi_2}{\phi_2^*}\right)\right]+\frac{1}{2}\frac{\partial}{\partial y}\left[\left(\phi_2^*\right)^2\frac{\partial}{\partial y}\left(\frac{\phi_2}{\phi_2^*}\right)\right]
\end{aligned}
\tag{D.50}
$$

We replace the functions ϕ_1 , ϕ_2 and their conjugates with

$$\begin{aligned} \phi_1 &= \rho_1^{1/2} \exp(i\chi) \\ \phi_2 &= \rho_2^{1/2} \exp(i\eta) \end{aligned} \tag{D.51}$$

Then we obtain

$$\begin{aligned} & -\frac{1}{2}\frac{\partial}{\partial x}\left[(\phi_1^*)^2 \frac{\partial}{\partial x}\left(\frac{\phi_1}{\phi_1^*}\right)\right] \\ = & -\frac{1}{2}\frac{\partial}{\partial x}\left[\rho_1 \exp(-2i\chi) \frac{\partial}{\partial x} \exp(2i\chi)\right] \\ = & -\frac{1}{2} 2i \frac{\partial}{\partial x}\left[\rho_1 \frac{\partial \chi}{\partial x}\right] = -i\frac{\partial}{\partial x}\left[\rho_1 \frac{\partial \chi}{\partial x}\right] \end{aligned} \tag{D.52}$$

$$\begin{aligned} & -\frac{1}{2}\frac{\partial}{\partial y}\left[(\phi_1^*)^2 \frac{\partial}{\partial y}\left(\frac{\phi_1}{\phi_1^*}\right)\right] \\ = & -i\frac{\partial}{\partial y}\left[\rho_1 \frac{\partial \chi}{\partial y}\right] \end{aligned} \tag{D.53}$$

$$\begin{aligned} & \frac{1}{2}\frac{\partial}{\partial x}\left[(\phi_2^*)^2 \frac{\partial}{\partial x}\left(\frac{\phi_2}{\phi_2^*}\right)\right] \\ = & \; i\frac{\partial}{\partial x}\left[\rho_2 \frac{\partial \eta}{\partial x}\right] \end{aligned} \tag{D.54}$$

$$\begin{aligned} & \frac{1}{2}\frac{\partial}{\partial y}\left[(\phi_2^*)^2 \frac{\partial}{\partial y}\left(\frac{\phi_2}{\phi_2^*}\right)\right] \\ = & \; i\frac{\partial}{\partial y}\left[\rho_2 \frac{\partial \eta}{\partial y}\right] \end{aligned} \tag{D.55}$$

Then

$$\begin{aligned} & F(\Delta; \phi_1, \phi_2) \\ = & -i\frac{\partial}{\partial x}\left[\rho_1 \frac{\partial \chi}{\partial x}\right] - i\frac{\partial}{\partial y}\left[\rho_1 \frac{\partial \chi}{\partial y}\right] \\ & + i\frac{\partial}{\partial x}\left[\rho_2 \frac{\partial \eta}{\partial x}\right] + i\frac{\partial}{\partial y}\left[\rho_2 \frac{\partial \eta}{\partial y}\right] \end{aligned} \tag{D.56}$$

We simply introduce this expression for F in the equation derived before for the difference $\rho_1 - \rho_2$ and write

$$\begin{aligned} & i\frac{\partial}{\partial t}(\rho_1 - \rho_2) - 2i(b + b^*)(\rho_1 + \rho_2) + \frac{\partial}{\partial x}\left[2\overline{A}_x(\rho_1 + \rho_2)\right] + \frac{\partial}{\partial y}\left[2\overline{A}_y(\rho_1 + \rho_2)\right] \\ = & -i\frac{\partial}{\partial x}\left[\rho_1 \frac{\partial \chi}{\partial x}\right] - i\frac{\partial}{\partial y}\left[\rho_1 \frac{\partial \chi}{\partial y}\right] + i\frac{\partial}{\partial x}\left[\rho_2 \frac{\partial \eta}{\partial x}\right] + i\frac{\partial}{\partial y}\left[\rho_2 \frac{\partial \eta}{\partial y}\right] \end{aligned} \tag{D.57}$$

or

$$
\begin{aligned}
& \frac{\partial}{\partial t}\left(\rho_1-\rho_2\right)-2\left(b+b^*\right)\left(\rho_1+\rho_2\right) \qquad \text{(D.58)} \\
& +\frac{\partial}{\partial x}\left[\left(\frac{2\overline{A}_x}{i}+\frac{\partial \chi}{\partial x}\right)\rho_1+\left(\frac{2\overline{A}_x}{i}-\frac{\partial \eta}{\partial x}\right)\rho_2\right] \\
& +\frac{\partial}{\partial y}\left[\left(\frac{2\overline{A}_y}{i}+\frac{\partial \chi}{\partial y}\right)\rho_1+\left(\frac{2\overline{A}_y}{i}-\frac{\partial \eta}{\partial y}\right)\rho_2\right] \\
= \; & 0
\end{aligned}
$$

This equation is derived from the equations of motion under the *algebraic ansatz*.

There is no other approximation.

Here we can introduce definitions

$$
v_x^{(1)} \equiv \frac{2\overline{A}_x}{i}+\frac{\partial \chi}{\partial x}\ ,\ v_y^{(1)}=\frac{2\overline{A}_y}{i}+\frac{\partial \chi}{\partial y} \qquad \text{(D.59)}
$$

$$
v_x^{(2)} \equiv -\frac{2\overline{A}_x}{i}+\frac{\partial \eta}{\partial x}\ ,\ v_y^{(2)}=-\frac{2\overline{A}_y}{i}+\frac{\partial \eta}{\partial y} \qquad \text{(D.60)}
$$

and we can write

$$
\begin{aligned}
& \frac{\partial}{\partial t}\left(\rho_1-\rho_2\right)-2\left(b+b^*\right)\left(\rho_1+\rho_2\right) \qquad \text{(D.61)} \\
& +\frac{\partial}{\partial x}\left[v_x^{(1)}\rho_1-v_x^{(2)}\rho_2\right]+\frac{\partial}{\partial y}\left[v_y^{(1)}\rho_1-v_y^{(2)}\rho_2\right] \\
= \; & 0
\end{aligned}
$$

The equations derived until now, for ρ_1, ρ_2 and $(\rho_1-\rho_2)$ have involved ONLY the second equation of motion

$$
iD_0\phi=-\frac{1}{2m}D_kD^k\phi-\frac{1}{2m\kappa}\left[\left[\phi,\phi^\dagger\right],\phi\right] \qquad \text{(D.62)}
$$

and the potentials $A_{x,y}$, which under algebraic ansatz, are given in terms of a and a^*. In addition we use the expression of A_0 and its algebraic ansatz, which is imaginary, $b \in Im\mathbf{R}$.

Nothing else, in particular the *second equation of motion*, or the Gauss constraint. This has not been yet invoked.

D.1.3 Approximate form of the equation for $\Omega = \rho_1 - \rho_2$ close to self-duality

When we are close to the SD state, we can approximate: A_0 is purely imaginar close to SD, and

$$b + b^* \approx 0 \tag{D.63}$$

$$\rho_1 = \rho_2^{-1} = \rho = \exp(\psi) \tag{D.64}$$

$$\chi \approx -\eta \tag{D.65}$$

and we will keep however the two functions ρ_1 and ρ_2. The approximation will only consists of taking the two phases as almost equal and opposed.

The terms in the expression of $F(\Delta; \phi_1, \phi_2)$ become

$$-i\frac{\partial}{\partial x}\left[\rho_1 \frac{\partial \chi}{\partial x}\right] - i\frac{\partial}{\partial x}\left[\rho_2 \frac{\partial \chi}{\partial x}\right] = -i\frac{\partial}{\partial x}\left[(\rho_1 + \rho_2)\frac{\partial \chi}{\partial x}\right] \tag{D.66}$$

and

$$-i\frac{\partial}{\partial y}\left[\rho_1 \frac{\partial \chi}{\partial y}\right] - i\frac{\partial}{\partial y}\left[\rho_2 \frac{\partial \chi}{\partial y}\right] = -i\frac{\partial}{\partial y}\left[(\rho_1 + \rho_2)\frac{\partial \chi}{\partial y}\right] \tag{D.67}$$

which gives

$$\begin{aligned} & F(\Delta; \phi_1, \phi_2) \\ = \quad & -i\frac{\partial}{\partial x}\left[(\rho_1 + \rho_2)\frac{\partial \chi}{\partial x}\right] - i\frac{\partial}{\partial y}\left[(\rho_1 + \rho_2)\frac{\partial \chi}{\partial y}\right] \end{aligned} \tag{D.68}$$

At this point, the *approximative* (due to the assumption $\chi \approx -\eta$) form of the equation for the time-variation of

$$\Omega \equiv \rho_1 - \rho_2 \tag{D.69}$$

is

$$\begin{aligned} & i\frac{\partial}{\partial t}(\rho_1 - \rho_2) + \frac{\partial}{\partial x}\left[2\overline{A}_x(\rho_1 + \rho_2)\right] + \frac{\partial}{\partial y}\left[2\overline{A}_y(\rho_1 + \rho_2)\right] \\ & \approx -i\frac{\partial}{\partial x}\left[(\rho_1 + \rho_2)\frac{\partial \chi}{\partial x}\right] - i\frac{\partial}{\partial y}\left[(\rho_1 + \rho_2)\frac{\partial \chi}{\partial y}\right] \end{aligned} \tag{D.70}$$

or

$$\begin{aligned} & \frac{\partial}{\partial t}(\rho_1 - \rho_2) \\ & + \frac{\partial}{\partial x}\left[(\rho_1 + \rho_2)\left(\frac{2\overline{A}_x}{i} + \frac{\partial \chi}{\partial x}\right)\right] \\ & + \frac{\partial}{\partial y}\left[(\rho_1 + \rho_2)\left(\frac{2\overline{A}_y}{i} + \frac{\partial \chi}{\partial y}\right)\right] \\ \approx \quad & 0 \quad \text{close to SD} \end{aligned} \tag{D.71}$$

NOTE. The expression for the potential in the simpler problem of the Liouville equation is

$$A_{\mu}=\partial_{\mu}\chi+\widehat{\mathbf{e}}_{z}\times\nabla\ln\rho \tag{D.72}$$

where we note that in our case the components of the potential are *imaginary*. Then $2\overline{A}_x$ and the term $i\partial_x\chi$ may lead to the *physical* part of the velocity

$$v^{phys}\equiv\widehat{\mathbf{e}}_{z}\times\nabla\ln\rho \;\text{ at SD} \tag{D.73}$$

And (still a problem with the factors 2) we have

$$\frac{\partial}{\partial t}\left(\rho_1-\rho_2\right)+\frac{\partial}{\partial x}\left[v_x^{phys}\left(\rho_1+\rho_2\right)\right]+\frac{\partial}{\partial y}\left[v_y^{phys}\left(\rho_1+\rho_2\right)\right]=0 \tag{D.74}$$

This is *NOT* the equation of continuity. **END.**

D.2 Derivation of the equation for the sum $\Xi=\rho_1+\rho_2$

Another operation that we can make with the two equations (for $|\phi_1|^2$ and respectively $|\phi_2|^2$) consists of adding them. This will obtain in the left hand side the time derivative of the *sum* of the two functions, *i.e.* Ξ.

The sum of the Eqs.(D.15) and (D.31) is made term by term

$$\begin{aligned}
& i\frac{\partial}{\partial t}\left|\phi_1\right|^2-2i\left(b+b^*\right)\left|\phi_1\right|^2 \\
& +i\frac{\partial}{\partial t}\left|\phi_2\right|^2+2i\left(b+b^*\right)\left|\phi_2\right|^2 \\
= & \; i\frac{\partial}{\partial t}\Xi-2i\left(b+b^*\right)\Omega \quad \text{first line}
\end{aligned} \tag{D.75}$$

The terms with second order derivatives

$$\begin{aligned}
& -\frac{1}{2}\phi_1^*\frac{\partial^2\phi_1}{\partial x^2}+\frac{1}{2}\phi_1\frac{\partial^2\phi_1^*}{\partial x^2}-\frac{1}{2}\phi_1^*\frac{\partial^2\phi_1}{\partial y^2}+\frac{1}{2}\phi_1\frac{\partial^2\phi_1^*}{\partial y^2} \\
& -\frac{1}{2}\phi_2^*\frac{\partial^2\phi_2}{\partial x^2}+\frac{1}{2}\phi_2\frac{\partial^2\phi_2^*}{\partial x^2}-\frac{1}{2}\phi_2^*\frac{\partial^2\phi_2}{\partial y^2}+\frac{1}{2}\phi_2\frac{\partial^2\phi_2^*}{\partial y^2}\text{terms with second order derivations}
\end{aligned} \tag{D.76}$$

$$\begin{aligned}
& -\frac{\partial\left(a-a^*\right)}{\partial x}\left|\phi_1\right|^2-\frac{1}{2}\left(a-a^*\right)\frac{\partial}{\partial x}\left(\left|\phi_1\right|^2\right)+\frac{\partial\left(a-a^*\right)}{\partial x}\left|\phi_2\right|^2+\frac{1}{2}\left(a-a^*\right)\frac{\partial}{\partial x}\left(\left|\phi_2\right|^2\right) \\
= & -\frac{\partial\left(a-a^*\right)}{\partial x}\Omega-\frac{1}{2}\left(a-a^*\right)\frac{\partial}{\partial x}\Omega \text{ terms of the second lines}
\end{aligned} \tag{D.77}$$

$$
\begin{aligned}
& -\frac{1}{2}\left(a-a^*\right)\frac{\partial}{\partial x}\left|\phi_1\right|^2+\frac{1}{2}\left(a-a^*\right)\frac{\partial}{\partial x}\left(\left|\phi_2\right|^2\right) \\
= & -\frac{1}{2}\left(a-a^*\right)\frac{\partial}{\partial x}\Omega \text{ terms of the third lines}
\end{aligned}
\tag{D.78}
$$

$$
\begin{aligned}
& -i\frac{\partial\left(a+a^*\right)}{\partial y}\left|\phi_1\right|^2-\frac{i}{2}\left(a+a^*\right)\frac{\partial}{\partial y}\left|\phi_1\right|^2+i\frac{\partial\left(a+a^*\right)}{\partial y}\left|\phi_2\right|^2+\frac{i}{2}\left(a+a^*\right)\frac{\partial}{\partial y}\left(\left|\phi_2\right|^2\right) \\
= & -i\frac{\partial\left(a+a^*\right)}{\partial y}\Omega-\frac{i}{2}\left(a+a^*\right)\frac{\partial}{\partial y}\Omega \text{ terms of the fourth lines}
\end{aligned}
\tag{D.79}
$$

$$
\begin{aligned}
& -\frac{i}{2}\left(a+a^*\right)\frac{\partial}{\partial y}\left|\phi_1\right|^2+\frac{i}{2}\left(a+a^*\right)\frac{\partial}{\partial y}\left(\left|\phi_2\right|^2\right) \\
= & -\frac{i}{2}\left(a+a^*\right)\frac{\partial}{\partial y}\Omega \text{ terms of the fifth lines}
\end{aligned}
\tag{D.80}
$$

Let consider what results

$$
\begin{aligned}
& i\frac{\partial}{\partial t}\Xi-2i\left(b+b^*\right)\Omega \\
= & -\frac{1}{2}\phi_1^*\frac{\partial^2\phi_1}{\partial x^2}+\frac{1}{2}\phi_1\frac{\partial^2\phi_1^*}{\partial x^2}-\frac{1}{2}\phi_1^*\frac{\partial^2\phi_1}{\partial y^2}+\frac{1}{2}\phi_1\frac{\partial^2\phi_1^*}{\partial y^2} \\
& -\frac{1}{2}\phi_2^*\frac{\partial^2\phi_2}{\partial x^2}+\frac{1}{2}\phi_2\frac{\partial^2\phi_2^*}{\partial x^2}-\frac{1}{2}\phi_2^*\frac{\partial^2\phi_2}{\partial y^2}+\frac{1}{2}\phi_2\frac{\partial^2\phi_2^*}{\partial y^2} \\
& -\frac{\partial\left(a-a^*\right)}{\partial x}\Omega-\frac{1}{2}\left(a-a^*\right)\frac{\partial}{\partial x}\Omega-\frac{1}{2}\left(a-a^*\right)\frac{\partial}{\partial x}\Omega \\
& -i\frac{\partial\left(a+a^*\right)}{\partial y}\Omega-\frac{i}{2}\left(a+a^*\right)\frac{\partial}{\partial y}\Omega-\frac{i}{2}\left(a+a^*\right)\frac{\partial}{\partial y}\Omega
\end{aligned}
\tag{D.81}
$$

$$
\begin{aligned}
& i\frac{\partial}{\partial t}\Xi-2i\left(b+b^*\right)\Omega \\
= & G\left(\Delta;\phi_1,\phi_2\right) \\
& -\frac{\partial}{\partial x}\left[\left(a-a^*\right)\Omega\right]-i\frac{\partial}{\partial y}\left[\left(a+a^*\right)\Omega\right]
\end{aligned}
\tag{D.82}
$$

We will have to work on the function G as for the previous case for F.

The treatment of the pairs of terms is identical

$$
\begin{aligned}
& -\frac{1}{2}\phi_1^* \frac{\partial^2 \phi_1}{\partial x^2} + \frac{1}{2}\phi_1 \frac{\partial^2 \phi_1^*}{\partial x^2} \\
= & -\frac{1}{2}\frac{\partial}{\partial x}\left[\phi_1^* \frac{\partial \phi_1}{\partial x}\right] + \frac{1}{2}\left[\left(\frac{\partial \phi_1^*}{\partial x}\right)\left(\frac{\partial \phi_1}{\partial x}\right)\right] \\
& +\frac{1}{2}\frac{\partial}{\partial x}\left[\phi_1 \frac{\partial \phi_1^*}{\partial x}\right] - \frac{1}{2}\left[\left(\frac{\partial \phi_1}{\partial x}\right)\left(\frac{\partial \phi_1^*}{\partial x}\right)\right] \\
= & -\frac{1}{2}\frac{\partial}{\partial x}\left[\phi_1^* \frac{\partial \phi_1}{\partial x} - \phi_1 \frac{\partial \phi_1^*}{\partial x}\right] \\
= & -\frac{1}{2}\frac{\partial}{\partial x}\left[\left(\phi_1^*\right)^2 \frac{\partial}{\partial x}\left(\frac{\phi_1}{\phi_1^*}\right)\right]
\end{aligned}
\tag{D.83}
$$

$$
\begin{aligned}
& -\frac{1}{2}\phi_1^* \frac{\partial^2 \phi_1}{\partial y^2} + \frac{1}{2}\phi_1 \frac{\partial^2 \phi_1^*}{\partial y^2} \\
= & -\frac{1}{2}\frac{\partial}{\partial y}\left[\left(\phi_1^*\right)^2 \frac{\partial}{\partial y}\left(\frac{\phi_1}{\phi_1^*}\right)\right]
\end{aligned}
\tag{D.84}
$$

$$
\begin{aligned}
& -\frac{1}{2}\phi_2^* \frac{\partial^2 \phi_2}{\partial x^2} + \frac{1}{2}\phi_2 \frac{\partial^2 \phi_2^*}{\partial x^2} \\
= & -\frac{1}{2}\frac{\partial}{\partial x}\left[\left(\phi_2^*\right)^2 \frac{\partial}{\partial x}\left(\frac{\phi_2}{\phi_2^*}\right)\right]
\end{aligned}
\tag{D.85}
$$

$$
\begin{aligned}
& -\frac{1}{2}\phi_2^* \frac{\partial^2 \phi_2}{\partial y^2} + \frac{1}{2}\phi_2 \frac{\partial^2 \phi_2^*}{\partial y^2} \\
= & -\frac{1}{2}\frac{\partial}{\partial y}\left[\left(\phi_2^*\right)^2 \frac{\partial}{\partial y}\left(\frac{\phi_2}{\phi_2^*}\right)\right]
\end{aligned}
\tag{D.86}
$$

The function G becomes

$$
\begin{aligned}
& G\left(\Delta;\phi_1,\phi_2\right) \\
= & -\frac{1}{2}\frac{\partial}{\partial x}\left[\left(\phi_1^*\right)^2 \frac{\partial}{\partial x}\left(\frac{\phi_1}{\phi_1^*}\right)\right] - \frac{1}{2}\frac{\partial}{\partial y}\left[\left(\phi_1^*\right)^2 \frac{\partial}{\partial y}\left(\frac{\phi_1}{\phi_1^*}\right)\right] \\
& -\frac{1}{2}\frac{\partial}{\partial x}\left[\left(\phi_2^*\right)^2 \frac{\partial}{\partial x}\left(\frac{\phi_2}{\phi_2^*}\right)\right] - \frac{1}{2}\frac{\partial}{\partial y}\left[\left(\phi_2^*\right)^2 \frac{\partial}{\partial y}\left(\frac{\phi_2}{\phi_2^*}\right)\right]
\end{aligned}
\tag{D.87}
$$

We note the difference relative to the expression of F, that the two terms involving ϕ_2 are now with the opposite sign.

$$\begin{aligned} & G\left(\Delta;\phi_1,\phi_2\right) \\ = & -i\frac{\partial}{\partial x}\left[\rho_1\frac{\partial\chi}{\partial x}\right]-i\frac{\partial}{\partial y}\left[\rho_1\frac{\partial\chi}{\partial y}\right] \\ & -i\frac{\partial}{\partial x}\left[\rho_2\frac{\partial\eta}{\partial x}\right]-i\frac{\partial}{\partial y}\left[\rho_2\frac{\partial\eta}{\partial y}\right] \end{aligned} \tag{D.88}$$

We insert this in the equation for the sum Ξ

$$\begin{aligned} & i\frac{\partial}{\partial t}\Xi-2i\left(b+b^*\right)\left(\rho_1-\rho_2\right)+\frac{\partial}{\partial x}\left[\left(a-a^*\right)\left(\rho_1-\rho_2\right)\right]+i\frac{\partial}{\partial y}\left[\left(a+a^*\right)\left(\rho_1-\rho_2\right)\right] \\ = & -i\frac{\partial}{\partial x}\left[\rho_1\frac{\partial\chi}{\partial x}\right]-i\frac{\partial}{\partial y}\left[\rho_1\frac{\partial\chi}{\partial y}\right]-i\frac{\partial}{\partial x}\left[\rho_2\frac{\partial\eta}{\partial x}\right]-i\frac{\partial}{\partial y}\left[\rho_2\frac{\partial\eta}{\partial y}\right] \end{aligned} \tag{D.89}$$

and replace the potentials

$$\begin{aligned} & i\frac{\partial}{\partial t}\left(\rho_1+\rho_2\right)-2i\left(b+b^*\right)\left(\rho_1-\rho_2\right)+\frac{\partial}{\partial x}\left[2\overline{A}_x\left(\rho_1-\rho_2\right)\right]+\frac{\partial}{\partial y}\left[2\overline{A}_y\left(\rho_1-\rho_2\right)\right] \\ = & -i\frac{\partial}{\partial x}\left[\rho_1\frac{\partial\chi}{\partial x}\right]-i\frac{\partial}{\partial y}\left[\rho_1\frac{\partial\chi}{\partial y}\right]-i\frac{\partial}{\partial x}\left[\rho_2\frac{\partial\eta}{\partial x}\right]-i\frac{\partial}{\partial y}\left[\rho_2\frac{\partial\eta}{\partial y}\right] \end{aligned} \tag{D.90}$$

$$\begin{aligned} & \frac{\partial}{\partial t}\left(\rho_1+\rho_2\right)-2\left(b+b^*\right)\left(\rho_1-\rho_2\right) \\ & +\frac{\partial}{\partial x}\left[\left(\frac{2\overline{A}_x}{i}+\frac{\partial\chi}{\partial x}\right)\rho_1+\left(-\frac{2\overline{A}_x}{i}+\frac{\partial\eta}{\partial x}\right)\rho_2\right] \\ & +\frac{\partial}{\partial y}\left[\left(\frac{2\overline{A}_y}{i}+\frac{\partial\chi}{\partial y}\right)\rho_1+\left(-\frac{2\overline{A}_y}{i}+\frac{\partial\eta}{\partial y}\right)\rho_2\right] \\ = & \ 0 \end{aligned} \tag{D.91}$$

There is *no* approximation of the type "close to SD ".

Using the notations introducing so-called velocity fields $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$ we have

$$\begin{aligned} & \frac{\partial}{\partial t}\left(\rho_1+\rho_2\right)-2\left(b+b^*\right)\left(\rho_1-\rho_2\right) \\ & +\frac{\partial}{\partial x}\left[v_x^{(1)}\rho_1+v_x^{(2)}\rho_2\right]+\frac{\partial}{\partial y}\left[v_y^{(1)}\rho_1+v_y^{(2)}\rho_2\right] \\ = & \ 0 \end{aligned} \tag{D.92}$$

Only the *algebraic ansatz* is used.

D.2.1 Approximative form of the equation for $\Xi = \rho_1 + \rho_2$ close to self-duality

We assume that close to the SD we can approximate

$$\chi \approx -\eta \tag{D.93}$$

Then

$$\begin{aligned} & -i\frac{\partial}{\partial x}\left[\rho_1 \frac{\partial \chi}{\partial x}\right] - i\frac{\partial}{\partial x}\left[\rho_2 \frac{\partial \eta}{\partial x}\right] \\ \approx & \; -i\frac{\partial}{\partial x}\left[\left(\rho_1 - \rho_2\right) \frac{\partial \chi}{\partial x}\right] \end{aligned} \tag{D.94}$$

and

$$\begin{aligned} & -i\frac{\partial}{\partial y}\left[\rho_1 \frac{\partial \chi}{\partial y}\right] - i\frac{\partial}{\partial y}\left[\rho_2 \frac{\partial \eta}{\partial y}\right] \\ \approx & \; -i\frac{\partial}{\partial y}\left[\left(\rho_1 - \rho_2\right) \frac{\partial \chi}{\partial y}\right] \end{aligned} \tag{D.95}$$

and G becomes

$$\begin{aligned} & G\left(\Delta; \phi_1, \phi_2\right) \\ \approx & \; -i\frac{\partial}{\partial x}\left[\left(\rho_1 - \rho_2\right) \frac{\partial \chi}{\partial x}\right] - i\frac{\partial}{\partial y}\left[\left(\rho_1 - \rho_2\right) \frac{\partial \chi}{\partial y}\right] \end{aligned} \tag{D.96}$$

$$\begin{aligned} & i\frac{\partial}{\partial t}\Xi - 2i\left(b + b^*\right)\Omega \\ = & \; -i\frac{\partial}{\partial x}\left[\left(\rho_1 - \rho_2\right) \frac{\partial \chi}{\partial x}\right] - i\frac{\partial}{\partial y}\left[\left(\rho_1 - \rho_2\right) \frac{\partial \chi}{\partial y}\right] \\ & -\frac{\partial}{\partial x}\left[\left(a - a^*\right)\Omega\right] - i\frac{\partial}{\partial y}\left[\left(a + a^*\right)\Omega\right] \end{aligned} \tag{D.97}$$

In addition we consider that close to SD

$$b + b^* \approx 0 \tag{D.98}$$

$$\begin{aligned} & i\frac{\partial}{\partial t}\Xi + i\frac{\partial}{\partial x}\left[\Omega \frac{\partial \chi}{\partial x}\right] + \frac{\partial}{\partial x}\left[\left(a - a^*\right)\Omega\right] \\ & +i\frac{\partial}{\partial y}\left[\Omega \frac{\partial \chi}{\partial y}\right] + i\frac{\partial}{\partial y}\left[\left(a + a^*\right)\Omega\right] \\ = & \; 0 \end{aligned} \tag{D.99}$$

$$i\frac{\partial}{\partial t}\Xi + \frac{\partial}{\partial x}\left[\left(2\overline{A}_x + i\frac{\partial \chi}{\partial x}\right)\Omega\right] + \frac{\partial}{\partial y}\left[\left(2\overline{A}_y + i\frac{\partial \chi}{\partial y}\right)\Omega\right] = 0 \tag{D.100}$$

For comparison we place together the two equations

$$\begin{aligned} & i\frac{\partial}{\partial t}\left(\rho_1 - \rho_2\right) \\ & +\frac{\partial}{\partial x}\left[\left(\rho_1 + \rho_2\right)\left(2\overline{A}_x + i\frac{\partial \chi}{\partial x}\right)\right] \\ & +\frac{\partial}{\partial y}\left[\left(\rho_1 + \rho_2\right)\left(2\overline{A}_y + i\frac{\partial \chi}{\partial y}\right)\right] \\ \approx\ & 0 \quad \text{close to SD} \end{aligned} \tag{D.101}$$

and

$$\begin{aligned} & i\frac{\partial}{\partial t}\left(\rho_1 + \rho_2\right) \\ & +\frac{\partial}{\partial x}\left[\left(\rho_1 - \rho_2\right)\left(2\overline{A}_x + i\frac{\partial \chi}{\partial x}\right)\right] \\ & +\frac{\partial}{\partial y}\left[\left(\rho_1 - \rho_2\right)\left(2\overline{A}_y + i\frac{\partial \chi}{\partial y}\right)\right] \\ \approx\ & 0 \quad \text{close to SD} \end{aligned} \tag{D.102}$$

The potential is actually imaginary. Schematically one can write,

$$\begin{aligned} \frac{\partial}{\partial t}\Omega + div\left(\mathbf{v}^{(1)}\Xi\right) &\approx 0 \;\; \text{close to SD} \\ \frac{\partial}{\partial t}\Xi + div\left(\mathbf{v}^{(1)}\Omega\right) &\approx 0 \;\; \text{close to SD} \end{aligned} \tag{D.103}$$

where

$$\begin{aligned} v_x^{(1)} &\equiv \frac{2\overline{A}_x}{i} + \frac{\partial \chi}{\partial x} \\ v_y^{(1)} &= \frac{2\overline{A}_y}{i} + \frac{\partial \chi}{\partial y} \end{aligned} \tag{D.104}$$

D.3 Derivation of the equation for ρ_1

We have obtained equations for the functions

$$\begin{aligned} \Omega &\equiv \rho_1 - \rho_2 \\ \Xi &\equiv \rho_1 + \rho_2 \end{aligned} \tag{D.105}$$

These are

$$\begin{aligned}
& \frac{\partial}{\partial t}\left(\rho_1-\rho_2\right)-2\left(b+b^*\right)\left(\rho_1+\rho_2\right) \\
& +\frac{\partial}{\partial x}\left[\left(\frac{2\overline{A}_x}{i}+\frac{\partial \chi}{\partial x}\right)\rho_1+\left(\frac{2\overline{A}_x}{i}-\frac{\partial \eta}{\partial x}\right)\rho_2\right] \\
& +\frac{\partial}{\partial y}\left[\left(\frac{2\overline{A}_y}{i}+\frac{\partial \chi}{\partial y}\right)\rho_1+\left(\frac{2\overline{A}_y}{i}-\frac{\partial \eta}{\partial y}\right)\rho_2\right] \\
= & \; 0
\end{aligned} \tag{D.106}$$

and

$$\begin{aligned}
& \frac{\partial}{\partial t}\left(\rho_1+\rho_2\right)-2\left(b+b^*\right)\left(\rho_1-\rho_2\right) \\
& +\frac{\partial}{\partial x}\left[\left(\frac{2\overline{A}_x}{i}+\frac{\partial \chi}{\partial x}\right)\rho_1+\left(-\frac{2\overline{A}_x}{i}+\frac{\partial \eta}{\partial x}\right)\rho_2\right] \\
& +\frac{\partial}{\partial y}\left[\left(\frac{2\overline{A}_y}{i}+\frac{\partial \chi}{\partial y}\right)\rho_1+\left(-\frac{2\overline{A}_y}{i}+\frac{\partial \eta}{\partial y}\right)\rho_2\right] \\
= & \; 0
\end{aligned} \tag{D.107}$$

These equations are general, do not contain approximation close to SD.

We will combine them to obtain the equation for ρ_1.

NOTE. If we take as starting point forms of the equations that have been obtained at previous levels, we will repeat some calculations.

We start from the equations for the difference Ω and for the sum Ξ.

For the difference $\rho_1-\rho_2$:

$$\begin{aligned}
& i\frac{\partial}{\partial t}\left(\rho_1-\rho_2\right)-2i\left(b+b^*\right)\left(\rho_1+\rho_2\right)+\frac{\partial}{\partial x}\left[2\overline{A}_x\left(\rho_1+\rho_2\right)\right]+\frac{\partial}{\partial y}\left[2\overline{A}_y\left(\rho_1+\rho_2\right)\right] \\
= & \; F\left(\Delta;\phi_1,\phi_2\right)
\end{aligned} \tag{D.108}$$

where

$$\begin{aligned}
& F\left(\Delta;\phi_1,\phi_2\right) \\
= & -i\frac{\partial}{\partial x}\left[\rho_1\frac{\partial \chi}{\partial x}\right]-i\frac{\partial}{\partial y}\left[\rho_1\frac{\partial \chi}{\partial y}\right]+i\frac{\partial}{\partial x}\left[\rho_2\frac{\partial \eta}{\partial x}\right]+i\frac{\partial}{\partial y}\left[\rho_2\frac{\partial \eta}{\partial y}\right]
\end{aligned} \tag{D.109}$$

For the sum $\rho_1+\rho_2$:

$$\begin{aligned}
& i\frac{\partial}{\partial t}\left(\rho_1+\rho_2\right)-2i\left(b+b^*\right)\left(\rho_1-\rho_2\right)+\frac{\partial}{\partial x}\left[2\overline{A}_x\left(\rho_1-\rho_2\right)\right]+\frac{\partial}{\partial y}\left[2\overline{A}_y\left(\rho_1-\rho_2\right)\right] \\
= & \; G\left(\Delta;\phi_1,\phi_2\right)
\end{aligned} \tag{D.110}$$

where

$$
\begin{aligned}
& G(\Delta;\phi_1,\phi_2) \qquad \text{(D.111)} \\
= \;& -i\frac{\partial}{\partial x}\left[\rho_1\frac{\partial\chi}{\partial x}\right]-i\frac{\partial}{\partial y}\left[\rho_1\frac{\partial\chi}{\partial y}\right]-i\frac{\partial}{\partial x}\left[\rho_2\frac{\partial\eta}{\partial x}\right]-i\frac{\partial}{\partial y}\left[\rho_2\frac{\partial\eta}{\partial y}\right]
\end{aligned}
$$

These equations can be combined to become equations for only ρ_1 and respectively ρ_2, which is not exact since the velocity field depends on both variables and the separation is not possible. **END.**

Adding the two equations we obtain

$$
\begin{aligned}
& 2i\frac{\partial}{\partial t}\rho_1-4i\left(b+b^*\right)\rho_1+\frac{\partial}{\partial x}\left[4\overline{A}_x\rho_1\right]+\frac{\partial}{\partial y}\left[4\overline{A}_y\rho_1\right] \qquad \text{(D.112)} \\
= \;& -2i\frac{\partial}{\partial x}\left[\rho_1\frac{\partial\chi}{\partial x}\right]-2i\frac{\partial}{\partial y}\left[\rho_1\frac{\partial\chi}{\partial y}\right]
\end{aligned}
$$

and can be written as

$$
\frac{\partial}{\partial t}\rho_1-2\left(b+b^*\right)\rho_1+\frac{\partial}{\partial x}\left[\left(\frac{2\overline{A}_x}{i}+\frac{\partial\chi}{\partial x}\right)\rho_1\right]+\frac{\partial}{\partial y}\left[\left(\frac{2\overline{A}_y}{i}+\frac{\partial\chi}{\partial y}\right)\rho_1\right]=0 \qquad \text{(D.113)}
$$

There is no approximation of the type "close to SD ".

This can be written as

$$
\left[\frac{\partial}{\partial t}-2\left(b+b^*\right)\right]\rho_1+\frac{\partial}{\partial x}\left(v_x^{(1)}\rho_1\right)+\frac{\partial}{\partial y}\left(v_y^{(1)}\rho_1\right)=0 \qquad \text{(D.114)}
$$

If we define

$$
\frac{\partial}{\partial t'}\equiv\frac{\partial}{\partial t}-2\left(b+b^*\right) \qquad \text{(D.115)}
$$

and remember that we dispose of the definition

$$
v_x^{(1)}\equiv\frac{2\overline{A}_x}{i}+\frac{\partial\chi}{\partial x}\ ,\ v_y^{(1)}=\frac{2\overline{A}_y}{i}+\frac{\partial\chi}{\partial y} \qquad \text{(D.116)}
$$

we obtain

$$
\frac{\partial}{\partial t'}\rho_1+div\left(\mathbf{v}^{(1)}\rho_1\right)=0 \qquad \text{(D.117)}
$$

At SD, $\partial/\partial t'\rightarrow\partial/\partial t$.

D.4 Derivation of the equation for ρ_2

Now we substract the two equations

$$
\begin{aligned}
& -2i\frac{\partial}{\partial t}\rho_2 - 4i\left(b+b^*\right)\rho_2 + \frac{\partial}{\partial x}\left[4\overline{A}_x\rho_2\right] + \frac{\partial}{\partial y}\left[4\overline{A}_y\rho_2\right] \\
= \; & 2i\frac{\partial}{\partial x}\left[\rho_2\frac{\partial\eta}{\partial x}\right] + 2i\frac{\partial}{\partial y}\left[\rho_2\frac{\partial\eta}{\partial y}\right]
\end{aligned}
\tag{D.118}
$$

or

$$
\frac{\partial}{\partial t}\rho_2 + 2\left(b+b^*\right)\rho_2 + \frac{\partial}{\partial x}\left[\left(-\frac{2\overline{A}_x}{i}+\frac{\partial\eta}{\partial x}\right)\rho_2\right] + \frac{\partial}{\partial y}\left[\left(-\frac{2\overline{A}_y}{i}+\frac{\partial\eta}{\partial y}\right)\rho_2\right] = 0
\tag{D.119}
$$

Now, we can use the definition

$$
v_x^{(2)} \equiv -\frac{2\overline{A}_x}{i} + \frac{\partial\eta}{\partial x} \ , \ v_y^{(2)} = -\frac{2\overline{A}_y}{i} + \frac{\partial\eta}{\partial y}
\tag{D.120}
$$

together with

$$
\frac{\partial}{\partial t''} \equiv \frac{\partial}{\partial t} + 2\left(b+b^*\right)
\tag{D.121}
$$

and write

$$
\frac{\partial}{\partial t''}\rho_2 + \frac{\partial}{\partial x}\left(v_x^{(2)}\rho_2\right) + \frac{\partial}{\partial y}\left(v_y^{(2)}\rho_2\right) = 0
\tag{D.122}
$$

$$
\frac{\partial}{\partial t''}\rho_2 + div\left(\mathbf{v}^{(2)}\rho_2\right) = 0
\tag{D.123}
$$

We know that $\partial/\partial t'' \to \partial/\partial t$ at SD, where $b+b^*=0$. Visibly, at SD, where $\eta=-\chi$ the two velocity fields $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$ are simply opposite.

E Appendix E. The current of the Euler FT

E.1 General expressions for the current's components

The formula for the FT current in the Euler case is

$$
\begin{aligned}
J^0 &= \left[\phi,\phi^\dagger\right] \\
J^i &= -\frac{i}{2m}\left(\left[\phi^\dagger, D_i\phi\right] - \left[\left(D_i\phi\right)^\dagger,\phi\right]\right)
\end{aligned}
\tag{E.1}
$$

$$
\begin{aligned}
J^i &= -\frac{i}{2m}\left(\left[\phi^\dagger, \frac{\partial\phi}{\partial x^i} + [A_i,\phi]\right] - \left[\left(\frac{\partial\phi}{\partial x^i} + [A_i,\phi]\right)^\dagger, \phi\right]\right) \qquad \text{(E.2)}\\
&= -\frac{i}{2m}\left[\phi^\dagger\left(\frac{\partial\phi}{\partial x^i} + [A_i,\phi]\right) - \left(\frac{\partial\phi}{\partial x^i} + [A_i,\phi]\right)\phi^\dagger\right.\\
&\quad \left. - \left(\frac{\partial\phi}{\partial x^i} + [A_i,\phi]\right)^\dagger \phi + \phi\left(\frac{\partial\phi}{\partial x^i} + [A_i,\phi]\right)^\dagger\right]\\
&= -\frac{i}{2m}\left\{\phi^\dagger\frac{\partial\phi}{\partial x^i} + \phi^\dagger\left(A_i\phi - \phi A_i\right) - \frac{\partial\phi}{\partial x^i}\phi^\dagger - \left(A_i\phi - \phi A_i\right)\phi^\dagger\right.\\
&\quad \left. - \left(\frac{\partial\phi^\dagger}{\partial x^i} + \left(\phi^\dagger A_i^\dagger - A_i^\dagger\phi^\dagger\right)\right)\phi + \phi\left(\frac{\partial\phi^\dagger}{\partial x^i} + \left(\phi^\dagger A_i^\dagger - A_i^\dagger\phi^\dagger\right)\right)\right\}
\end{aligned}
$$

Let us collect the part that depends only on ϕ and $\phi^\dagger$ and separately the part that depends on A_i and $A_i^\dagger$.

$$
\begin{aligned}
J^i &= -\frac{i}{2m}\left\{\phi^\dagger\frac{\partial\phi}{\partial x^i} - \frac{\partial\phi}{\partial x^i}\phi^\dagger - \frac{\partial\phi^\dagger}{\partial x^i}\phi + \phi\frac{\partial\phi^\dagger}{\partial x^i}\right. \qquad \text{(E.3)}\\
&\quad +\phi^\dagger A_i\phi - \phi^\dagger\phi A_i - A_i\phi\phi^\dagger + \phi A_i\phi^\dagger\\
&\quad \left. -\phi^\dagger A_i^\dagger\phi + A_i^\dagger\phi^\dagger\phi + \phi\phi^\dagger A_i^\dagger - \phi A_i^\dagger\phi^\dagger\right\}
\end{aligned}
$$

This expression will be used later just as a check for the result of the derivation presented below.

The current for $\mu \equiv k$ (space components) is

$$
\begin{aligned}
J^k &= -\frac{i}{2m}\left\{\phi^\dagger\left(\partial^k\phi\right) - \left(\partial^k\phi\right)\phi^\dagger - \left(\partial^k\phi^\dagger\right)\phi + \phi\left(\partial^k\phi^\dagger\right)\right. \qquad \text{(E.4)}\\
&\quad \left. + \left[\phi^\dagger, \left[A^k,\phi\right]\right] + \left[\phi, \left[\phi^\dagger, A^{k\dagger}\right]\right]\right\}\\
&\equiv \Lambda_1^k + \Lambda_2^k
\end{aligned}
$$

where

$$
\begin{aligned}
\Lambda_1^k &\equiv -\frac{i}{2m}\left\{\phi^\dagger\left(\partial^k\phi\right) - \left(\partial^k\phi\right)\phi^\dagger - \left(\partial^k\phi^\dagger\right)\phi + \phi\left(\partial^k\phi^\dagger\right)\right\} \qquad \text{(E.5)}\\
\Lambda_2^k &\equiv -\frac{i}{2m}\left(\left[\phi^\dagger, \left[A^k,\phi\right]\right] + \left[\phi, \left[\phi^\dagger, A^{k\dagger}\right]\right]\right)
\end{aligned}
$$

E.1.1 The expression of the first part of the current, Λ_1

The terms containing space and time derivatives (here the symbol Ψ is replaced by ϕ)

$$
\Lambda_1^k = -\frac{i}{2m}\left[\phi^\dagger\left(\partial^k\phi\right) - \left(\partial^k\phi\right)\phi^\dagger - \left(\partial^k\phi^\dagger\right)\phi + \phi\left(\partial^k\phi^\dagger\right)\right] \qquad \text{(E.6)}
$$

where we have to insert

$$\begin{aligned} \phi &= \phi_1 E_+ + \phi_2 E_- \\ \phi^\dagger &= \phi_1^* E_- + \phi_2^* E_+ \end{aligned} \tag{E.7}$$

This consists of two commutators.

The first commutator is

$$\begin{aligned} \left[\phi^\dagger, \partial^k \phi\right] &= \phi^\dagger \left(\partial^k \phi\right) - \left(\partial^k \phi\right) \phi^\dagger \\ &= \left(\phi_1^* E_- + \phi_2^* E_+\right) \left(\frac{\partial \phi_1}{\partial x_k} E_+ + \frac{\partial \phi_2}{\partial x_k} E_- \right) \\ &\quad - \left(\frac{\partial \phi_1}{\partial x_k} E_+ + \frac{\partial \phi_2}{\partial x_k} E_- \right) \left(\phi_1^* E_- + \phi_2^* E_+\right) \\ &= \phi_1^* \frac{\partial \phi_1}{\partial x_k} E_- E_+ + \underline{\phi_1^* \frac{\partial \phi_2}{\partial x_k} E_- E_-} + \underbrace{\phi_2^* \frac{\partial \phi_1}{\partial x_k} E_+ E_+} + \phi_2^* \frac{\partial \phi_2}{\partial x_k} E_+ E_- \\ &\quad - \phi_1^* \frac{\partial \phi_1}{\partial x_k} E_+ E_- \underbrace{- \phi_2^* \frac{\partial \phi_1}{\partial x_k} E_+ E_+} \underline{- \phi_1^* \frac{\partial \phi_2}{\partial x_k} E_- E_-} - \phi_2^* \frac{\partial \phi_2}{\partial x_k} E_- E_+ \end{aligned} \tag{E.8}$$

The coefficients of $E_- E_-$ and of $E_+ E_+$ cancel. The result is

$$\begin{aligned} &\left[\phi^\dagger, \partial^k \phi\right] \\ &= \phi_1^* \frac{\partial \phi_1}{\partial x_k} \left[E_-, E_+\right] + \phi_2^* \frac{\partial \phi_2}{\partial x_k} \left[E_+, E_-\right] \end{aligned} \tag{E.9}$$

Here we must use the commutators of the generators of the algebra and obtain

$$\left[\phi^\dagger, \partial^k \phi\right] = -\left(\phi_1^* \frac{\partial \phi_1}{\partial x_k} - \phi_2^* \frac{\partial \phi_2}{\partial x_k} \right) H \tag{E.10}$$

The second commutator in Λ_1^k is

$$\begin{aligned} \left[\phi, \partial^k \phi^\dagger\right] &= \phi \left(\partial^k \phi^\dagger\right) - \left(\partial^k \phi^\dagger\right) \phi \\ &= \left(\phi_1 E_+ + \phi_2 E_-\right) \left(\frac{\partial \phi_1^*}{\partial x_k} E_- + \frac{\partial \phi_2^*}{\partial x_k} E_+ \right) \\ &\quad - \left(\frac{\partial \phi_1^*}{\partial x_k} E_- + \frac{\partial \phi_2^*}{\partial x_k} E_+ \right) \left(\phi_1 E_+ + \phi_2 E_-\right) \\ &= \phi_1 \frac{\partial \phi_1^*}{\partial x_k} E_+ E_- + \underline{\phi_2 \frac{\partial \phi_1^*}{\partial x_k} E_- E_-} + \underbrace{\phi_1 \frac{\partial \phi_2^*}{\partial x_k} E_+ E_+} + \phi_2 \frac{\partial \phi_2^*}{\partial x_k} E_- E_+ \\ &\quad - \phi_1 \frac{\partial \phi_1^*}{\partial x_k} E_- E_+ \underbrace{- \phi_1 \frac{\partial \phi_2^*}{\partial x_k} E_+ E_+} \underline{- \phi_2 \frac{\partial \phi_1^*}{\partial x_k} E_- E_-} - \phi_2 \frac{\partial \phi_2^*}{\partial x_k} E_+ E_- \end{aligned} \tag{E.11}$$

As above, the coefficients of the terms E_+E_+ and respectively E_-E_- cancel. The other represent commutators that can be expressed by H:

$$\begin{aligned} & \left[\phi, \partial^k \phi^\dagger\right] \\ = \; & \phi_1 \frac{\partial \phi_1^*}{\partial x_k} \left[E_+, E_-\right] - \phi_2 \frac{\partial \phi_2^*}{\partial x_k} \left[E_+, E_-\right] \\ = \; & \left(\phi_1 \frac{\partial \phi_1^*}{\partial x_k} - \phi_2 \frac{\partial \phi_2^*}{\partial x_k}\right) H \end{aligned} \tag{E.12}$$

Putting together these results we have

$$\begin{aligned} \Lambda_1^k &= -\frac{i}{2m}\left[\phi^\dagger\left(\partial^k \phi\right) - \left(\partial^k \phi\right)\phi^\dagger - \left(\partial^k \phi^\dagger\right)\phi + \phi\left(\partial^k \phi^\dagger\right)\right] \\ &= -\frac{i}{2m}\left\{\left[\phi^\dagger, \partial^k \phi\right] + \left[\phi, \partial^k \phi^\dagger\right]\right\} \\ &= -\frac{i}{2m}\left[-\left(\phi_1^* \frac{\partial \phi_1}{\partial x_k} - \phi_2^* \frac{\partial \phi_2}{\partial x_k}\right) H + \left(\phi_1 \frac{\partial \phi_1^*}{\partial x_k} - \phi_2 \frac{\partial \phi_2^*}{\partial x_k}\right) H\right] \\ &= -\frac{i}{2m}\left[\phi_1 \frac{\partial \phi_1^*}{\partial x_k} - \phi_1^* \frac{\partial \phi_1}{\partial x_k} - \phi_2 \frac{\partial \phi_2^*}{\partial x_k} + \phi_2^* \frac{\partial \phi_2}{\partial x_k}\right] H \end{aligned} \tag{E.13}$$

The derivatives look like the derivatives of ratios ϕ/ϕ^* if we multiply by the adequet denominator.

$$\begin{aligned} \phi_1 \frac{\partial \phi_1^*}{\partial x_k} - \phi_1^* \frac{\partial \phi_1}{\partial x_k} &= -\left(\phi_1^*\right)^2 \frac{\frac{\partial \phi_1}{\partial x_k}\phi_1^* - \phi_1 \frac{\partial \phi_1^*}{\partial x_k}}{\left(\phi_1^*\right)^2} = \\ &= -\left(\phi_1^*\right)^2 \frac{\partial}{\partial x_k}\left(\frac{\phi_1}{\phi_1^*}\right) \end{aligned} \tag{E.14}$$

$$\begin{aligned} -\phi_2 \frac{\partial \phi_2^*}{\partial x_k} + \phi_2^* \frac{\partial \phi_2}{\partial x_k} &= \left(\phi_2^*\right)^2 \frac{\frac{\partial \phi_2}{\partial x_k}\phi_2^* - \phi_2 \frac{\partial \phi_2^*}{\partial x_k}}{\left(\phi_2^*\right)^2} \\ &= \left(\phi_2^*\right)^2 \frac{\partial}{\partial x_k}\left(\frac{\phi_2}{\phi_2^*}\right) \end{aligned} \tag{E.15}$$

Then this part is

$$\begin{aligned} \Lambda_1^k &= -\frac{i}{2m}\left[\phi^\dagger\left(\partial^k \phi\right) - \left(\partial^k \phi\right)\phi^\dagger - \left(\partial^k \phi^\dagger\right)\phi + \phi\left(\partial^k \phi^\dagger\right)\right] \\ &= -\frac{i}{2m}\left[-\left(\phi_1^*\right)^2 \frac{\partial}{\partial x_k}\left(\frac{\phi_1}{\phi_1^*}\right) + \left(\phi_2^*\right)^2 \frac{\partial}{\partial x_k}\left(\frac{\phi_2}{\phi_2^*}\right)\right] H \end{aligned} \tag{E.16}$$

Postponing a reformulation of this expression, we just represent here the functions ϕ_1 and ϕ_2 as they are defined, we have

$$\begin{aligned} \rho_1 &= |\phi_1|^2 = \exp(\psi_1) \\ \rho_2 &= |\phi_2|^2 = \exp(\psi_2) \end{aligned} \tag{E.17}$$

Then

$$\phi_1 = \exp\left(\frac{\psi_1}{2}\right)\exp(i\chi) \tag{E.18}$$

$$\phi_2 = \exp\left(\frac{\psi_2}{2}\right)\exp(i\eta) \tag{E.19}$$

Then

$$\begin{aligned} \frac{\phi_1}{\phi_1^*} &= \exp(2i\chi) \\ \frac{\phi_2}{\phi_2^*} &= \exp(2i\eta) \end{aligned} \tag{E.20}$$

$$\begin{aligned} (\phi_1^*)^2 &= \rho_1 \exp(-2i\chi) \\ (\phi_2^*)^2 &= \rho_2 \exp(-2i\eta) \end{aligned} \tag{E.21}$$

and

$$\begin{aligned} \Lambda_1 &= -\frac{i}{2m}\left[-(\phi_1^*)^2 \frac{\partial}{\partial x_k}\left(\frac{\phi_1}{\phi_1^*}\right) + (\phi_2^*)^2 \frac{\partial}{\partial x_k}\left(\frac{\phi_2}{\phi_2^*}\right)\right] H \\ &= -\frac{i}{2m}\left[-\rho_1 \exp(-2i\chi) \frac{\partial}{\partial x_k}\exp(2i\chi) + \rho_2 \exp(-2i\eta)\frac{\partial}{\partial x_k}\exp(2i\eta)\right] H \\ &= -\frac{i}{2m}\left[-\rho_1 2i \frac{\partial \chi}{\partial x_k} + \rho_2 2i \frac{\partial \eta}{\partial x_k}\right] H \\ &= \frac{1}{m}\left(-\rho_1 \frac{\partial \chi}{\partial x_k} + \rho_2 \frac{\partial \eta}{\partial x_k}\right) H \end{aligned} \tag{E.22}$$

E.1.2 The expression of the second part of the current, Λ_2

According to the expansion done above we have to calculate

$$\begin{aligned} \Lambda_2 &= -\frac{i}{2m}\left\{\phi^\dagger A_i \phi - \phi^\dagger \phi A_i - A_i \phi \phi^\dagger + \phi A_i \phi^\dagger \right. \\ &\quad \left. -\phi^\dagger A_i^\dagger \phi + A_i^\dagger \phi^\dagger \phi + \phi \phi^\dagger A_i^\dagger - \phi A_i^\dagger \phi^\dagger \right\} \end{aligned} \tag{E.23}$$

Let us replace here

$$\begin{aligned} \phi &\equiv \phi = \phi_1 E_+ + \phi_2 E_- \\ \phi^\dagger &= \phi^\dagger = \phi_1^* E_- + \phi_2^* E_+ \end{aligned} \tag{E.24}$$

and the formulas

$$\begin{aligned} A_x &= \frac{1}{2}\left(a - a^*\right) H \\ A_y &= \frac{i}{2}\left(a + a^*\right) H \end{aligned} \tag{E.25}$$

Calculation of the x component We ignore for the moment the coefficient $(-i/2)$.

First term on the first line

$$\begin{aligned} \phi^\dagger A_x \phi &= \left(\phi_1^* E_- + \phi_2^* E_+\right) \frac{1}{2}\left(a - a^*\right) H \left(\phi_1 E_+ + \phi_2 E_-\right) \\ &= \phi_1^* \phi_1 \frac{1}{2}\left(a - a^*\right) \quad E_- H E_+ \\ &\quad + \phi_2^* \phi_1 \frac{1}{2}\left(a - a^*\right) \quad E_+ H E_+ \\ &\quad + \phi_1^* \phi_2 \frac{1}{2}\left(a - a^*\right) \quad E_- H E_- \\ &\quad + \phi_2^* \phi_2 \frac{1}{2}\left(a - a^*\right) \quad E_+ H E_- \end{aligned} \tag{E.26}$$

The second term on the first line

$$\begin{aligned} -\phi^\dagger \phi A_i &= -\left(\phi_1^* E_- + \phi_2^* E_+\right)\left(\phi_1 E_+ + \phi_2 E_-\right) \frac{1}{2}\left(a - a^*\right) H \\ &= -\phi_1^* \phi_1 \frac{1}{2}\left(a - a^*\right) \quad E_- E_+ H \\ &\quad - \phi_1^* \phi_2 \frac{1}{2}\left(a - a^*\right) \quad E_- E_- H \\ &\quad - \phi_2^* \phi_1 \frac{1}{2}\left(a - a^*\right) \quad E_+ E_+ H \\ &\quad - \phi_2^* \phi_2 \frac{1}{2}\left(a - a^*\right) \quad E_+ E_- H \end{aligned} \tag{E.27}$$

The third term on the first line

$$
\begin{aligned}
-A_i \phi \phi^\dagger &= -\frac{1}{2}\left(a-a^*\right) H \left(\phi_1 E_+ + \phi_2 E_-\right)\left(\phi_1^* E_- + \phi_2^* E_+\right) \quad \text{(E.28)} \\
&= -\frac{1}{2}\left(a-a^*\right)\phi_1\phi_1^* \quad H E_+ E_- \\
&\quad -\frac{1}{2}\left(a-a^*\right)\phi_1\phi_2^* \quad H E_+ E_+ \\
&\quad -\frac{1}{2}\left(a-a^*\right)\phi_2\phi_1^* \quad H E_- E_- \\
&\quad -\frac{1}{2}\left(a-a^*\right)\phi_2\phi_2^* \quad H E_- E_+
\end{aligned}
$$

The fourth term on the first line

$$
\begin{aligned}
\phi A_i \phi^\dagger &= \left(\phi_1 E_+ + \phi_2 E_-\right)\frac{1}{2}\left(a-a^*\right) H \left(\phi_1^* E_- + \phi_2^* E_+\right) \\
&= \phi_1\phi_1^*\frac{1}{2}\left(a-a^*\right) \quad E_+ H E_- \\
&\quad +\phi_1\phi_2^*\frac{1}{2}\left(a-a^*\right) \quad E_+ H E_+ \\
&\quad +\phi_2\phi_1^*\frac{1}{2}\left(a-a^*\right) \quad E_- H E_- \\
&\quad +\phi_2\phi_2^*\frac{1}{2}\left(a-a^*\right) \quad E_- H E_+
\end{aligned}
$$

Now we go to the second line in the detailed expression of Λ_2;

The first term is similar to the first term of the first line, but A_i is now *daggered*:

$$
\begin{aligned}
-\phi^\dagger A_i^\dagger \phi &= -\left(\phi_1^* E_- + \phi_2^* E_+\right)\frac{1}{2}\left(a^*-a\right) H \left(\phi_1 E_+ + \phi_2 E_-\right) \quad \text{(E.29)} \\
&= -\phi_1^*\phi_1\frac{1}{2}\left(a^*-a\right) \quad E_- H E_+ \\
&\quad -\phi_2^*\phi_1\frac{1}{2}\left(a^*-a\right) \quad E_+ H E_+ \\
&\quad -\phi_1^*\phi_2\frac{1}{2}\left(a^*-a\right) \quad E_- H E_- \\
&\quad -\phi_2^*\phi_2\frac{1}{2}\left(a^*-a\right) \quad E_+ H E_-
\end{aligned}
$$

the second term of the second line is $A_i^\dagger \phi^\dagger \phi$, or

$$\begin{aligned} A_i^\dagger \phi^\dagger \phi &= \frac{1}{2}(a^* - a) H (\phi_1^* E_- + \phi_2^* E_+)(\phi_1 E_+ + \phi_2 E_-) \\ &= \frac{1}{2}(a^* - a)\phi_1^*\phi_1 \quad HE_-E_+ \\ &\quad + \frac{1}{2}(a^* - a)\phi_1^*\phi_2 \quad HE_-E_- \\ &\quad + \frac{1}{2}(a^* - a)\phi_2^*\phi_1 \quad HE_+E_+ \\ &\quad + \frac{1}{2}(a^* - a)\phi_2^*\phi_2 \quad HE_+E_- \end{aligned} \tag{E.30}$$

the third term in the second line

$$\begin{aligned} \phi\phi^\dagger A_i^\dagger &= (\phi_1 E_+ + \phi_2 E_-)(\phi_1^* E_- + \phi_2^* E_+)\frac{1}{2}(a^* - a) H \\ &= \frac{1}{2}(a^* - a)\phi_1\phi_1^* \quad E_+E_-H \\ &\quad + \frac{1}{2}(a^* - a)\phi_1\phi_2^* \quad E_+E_+H \\ &\quad + \frac{1}{2}(a^* - a)\phi_2\phi_1^* \quad E_-E_-H \\ &\quad + \frac{1}{2}(a^* - a)\phi_2\phi_2^* \quad E_-E_+H \end{aligned} \tag{E.31}$$

the fourth term in the second line

$$\begin{aligned} -\phi A_i^\dagger \phi^\dagger &= -(\phi_1 E_+ + \phi_2 E_-)\frac{1}{2}(a^* - a) H (\phi_1^* E_- + \phi_2^* E_+) \\ &= -\phi_1\phi_1^*\frac{1}{2}(a^* - a) \quad E_+HE_- \\ &\quad -\phi_1\phi_2^*\frac{1}{2}(a^* - a) \quad E_+HE_+ \\ &\quad -\phi_2\phi_1^*\frac{1}{2}(a^* - a) \quad E_-HE_- \\ &\quad -\phi_2\phi_2^*\frac{1}{2}(a^* - a) \quad E_-HE_+ \end{aligned} \tag{E.32}$$

We now collect the coefficients of the terms

$$\begin{aligned}
&\text{for} \quad \phi_1^* \phi_1 \frac{1}{2} (a - a^*) \quad \text{these are} \\
&+E_- H E_+ \\
&-E_- E_+ H \\
&-H E_+ E_- \\
&+E_+ H E_- \\
&+E_- H E_+ \\
&-H E_- E_+ \\
&-E_+ E_- H \\
&+E_+ H E_-
\end{aligned} \tag{E.33}$$

We can combine these operator products

$$\begin{array}{lll}
E_- (H E_+ - E_+ H) & \text{this is} & E_- (2E_+) \\
-(H E_+ - E_+ H) E_- & \text{this is} & -(2E_+) E_- \\
+(E_- H - H E_-) E_+ & \text{this is} & -(-2E_-) E_+ \\
-E_+ (E_- H - H E_-) & \text{this is} & -E_+ (-) (-2E_-)
\end{array} \tag{E.34}$$

or

$$2 [E_- E_+ - E_+ E_- + E_- E_+ - E_+ E_-] = 2 [-H - H] = -4H \tag{E.35}$$

Finally from this term we obtain

$$(-4) \phi_1^* \phi_1 \frac{1}{2} (a - a^*) \quad H \tag{E.36}$$

The next term

$$\begin{aligned}
&\text{for} \quad \phi_2^* \phi_1 \frac{1}{2} (a - a^*) \quad \text{these are} \\
&E_+ H E_+ \\
&-E_+ E_+ H \\
&-H E_+ E_+ \\
&+E_+ H E_+ \\
&+\ E_+ H E_+ \\
&-H E_+ E_+ \\
&-E_+ E_+ H \\
&+E_+ H E_+
\end{aligned} \tag{E.37}$$

and we combine the product of operators

$$\begin{array}{lll} E_+ (HE_+ - E_+ H) & \text{this is} & E_+ (2E_+) \\ -(HE_+ - E_+ H) E_+ & \text{this is} & -(2E_+) E_+ \\ -(HE_+ - E_+ H) E_+ & \text{this is} & -(2E_+) E_+ \\ +E_+ (HE_+ - E_+ H) & \text{this is} & E_+ (2E_+) \end{array} \tag{E.38}$$

and we find

$$2\,[0]$$

which makes that the term contribute zith *zero*

$$\phi_2^* \phi_1 \frac{1}{2} (a - a^*) \quad \times 2\,[0] = 0 \tag{E.39}$$

The next term

$$\begin{array}{l} \text{for } \phi_1^* \phi_2 \frac{1}{2} (a - a^*) \quad \text{these are} \\ E_- H E_- \\ -E_- E_- H \\ -H E_- E_- \\ +E_- H E_- \\ +E_- H E_- \\ -H E_- E_- \\ -E_- E_- H \\ +E_- H E_- \end{array} \tag{E.40}$$

We combine the porducts of operators

$$\begin{array}{lll} E_- (HE_- - E_- H) & \text{this is} & E_- (-2E_-) \\ -(HE_- - E_- H) E_- & \text{this is} & -(-2E_-) E_- \\ -(HE_- - E_- H) E_- & \text{this is} & -(-2E_-) E_- \\ +E_- (HE_- - E_- H) & \text{this is} & E_- (-2E_-) \end{array} \tag{E.41}$$

which gives finally

$$2\,[0]$$

and this term does not contribute to the final expression

$$\phi_1^* \phi_2 \frac{1}{2} (a - a^*) \; \times \; 2\,[0] = 0 \tag{E.42}$$

The next term is

$$\begin{aligned}
&\text{for}\quad \phi_2^*\phi_2\frac{1}{2}(a-a^*)\quad\text{these are} \qquad \text{(E.43)}\\
&E_+HE_-\\
&-E_+E_-H\\
&-HE_-E_+\\
&+E_-HE_+\\
&+E_+HE_-\\
&-HE_+E_-\\
&-E_-E_+H\\
&+E_-HE_+
\end{aligned}$$

we combine the products of operators

$$\begin{aligned}
&E_+(HE_- - E_-H)\quad\text{this is}\quad E_+(-2E_-) \qquad \text{(E.44)}\\
&-(HE_- - E_-H)E_+\quad\text{this is}\quad -(-2E_-)E_+\\
&-(HE_+ - E_+H)E_-\quad\text{this is}\quad -(2E_+)E_-\\
&+E_-(HE_+ - E_+H)\quad\text{this is}\quad E_-(2E_+)
\end{aligned}$$

which gives

$$2\left[-E_+E_- + E_-E_+ - E_+E_- + E_-E_+\right] = -4\left[E_+E_- - E_-E_+\right] = -4H \qquad \text{(E.45)}$$

and it results that the contribution of this term is

$$\phi_2^*\phi_2\frac{1}{2}(a-a^*)(-4)\quad H \qquad \text{(E.46)}$$

We put together the two terms

$$\begin{aligned}
\Lambda_2^x &= -\frac{i}{2m}\left[(-4)\phi_1^*\phi_1\frac{1}{2}(a-a^*)\quad H + (-4)\phi_2^*\phi_2\frac{1}{2}(a-a^*)\quad H\right]\\
&= \frac{i}{m}(a-a^*)(\rho_1+\rho_2)\ H \qquad \text{(E.47)}
\end{aligned}$$

This will be confirmed by a cross check below.

Now the x-component of the current is

$$\begin{aligned}
J^x/H &= \Lambda_1^x + \Lambda_2^x\ /H \qquad \text{(E.48)}\\
&= \frac{1}{m}\left(-\rho_1\frac{\partial\chi}{\partial x_k} + \rho_2\frac{\partial\eta}{\partial x_k}\right) + \frac{i}{m}(a-a^*)(\rho_1+\rho_2)
\end{aligned}$$

NOTE that we use the symbolic writting J^x/H and similar to denote the coefficient of the H generator in the alegbraic expression of J^x. In other situations we use the notation

$$A_x = \overline{A}_x H \tag{E.49}$$

to separate in A_x the coefficient $\overline{A}_x$ from the algebraic generator H.

Calculation of the y component It differs from the x term by the insertion of

$$\begin{aligned} A_y &= \frac{i}{2}(a + a^*) H \\ A_y^\dagger &= -\frac{i}{2}(a^* + a) H = -A_y \end{aligned} \tag{E.50}$$

It has similar properties as A_x and $A_x^\dagger$. We just need to replace

$$\begin{aligned} a - a^* &\rightarrow a + a^* \\ \frac{1}{2} &\rightarrow \frac{i}{2} \end{aligned} \tag{E.51}$$

in Λ_2^x to obtain

$$\begin{aligned} \Lambda_2^y &= -\frac{i}{2m}\left[(-4)\,\phi_1^*\phi_1\frac{i}{2}(a + a^*) \quad H + (-4)\,\phi_2^*\phi_2\frac{i}{2}(a + a^*) \quad H\right] \\ &= -\frac{1}{m}(a + a^*)(\rho_1 + \rho_2) \; H \end{aligned} \tag{E.52}$$

Now, for the current

$$\begin{aligned} J^y/H &= \Lambda_1^y + \Lambda_2^y \;\; /H \\ &= \frac{1}{m}\left(-\rho_1\frac{\partial\chi}{\partial x_k} + \rho_2\frac{\partial\eta}{\partial x_k}\right) - \frac{1}{m}(a + a^*)(\rho_1 + \rho_2) \end{aligned} \tag{E.53}$$

E.1.3 The time component of the Euler current

This is given by

$$\begin{aligned} J^0 &= \left[\phi, \phi^\dagger\right] \\ &= \left[\phi_1 E_+ + \phi_2 E_-, \phi_1^* E_- + \phi_2^* E_+\right] \\ &= \phi_1\phi_1^*\left[E_+, E_-\right] + \phi_2\phi_2^*\left[E_-, E_+\right] \\ &= \left|\phi_1\right|^2 H - \left|\phi_2\right|^2 H \end{aligned} \tag{E.54}$$

or

$$J^0 = (\rho_1 - \rho_2)\, H \tag{E.55}$$

This is the *charge* and we see that it is the vorticity, since

$$\rho_1 - \rho_2 = -\frac{\kappa\omega}{2} \tag{E.56}$$

E.2 The expression of the EULER current J^μ

Finally

$$J^\mu = \Lambda_1^\mu + \Lambda_2^\mu \tag{E.57}$$

gives

$$\begin{aligned} J^x &= \frac{1}{m}\left[-\rho_1\frac{\partial\chi}{\partial x} + \rho_2\frac{\partial\eta}{\partial x} + i(a-a^*)\,(\rho_1+\rho_2)\right] H \\ J^y &= \frac{1}{m}\left[-\rho_1\frac{\partial\chi}{\partial y} + \rho_2\frac{\partial\eta}{\partial y} - (a+a^*)\,(\rho_1+\rho_2)\right] H \\ J^0 &= (\rho_1-\rho_2)\, H \end{aligned} \tag{E.58}$$

We give a slightly different expression for the components of the current, introducing the potentials $\overline{A}_{x,y}$.

$$\begin{aligned} J^x\ /H &= \frac{1}{m}\left[-\rho_1\frac{\partial\chi}{\partial x} + \rho_2\frac{\partial\eta}{\partial x} + i(a-a^*)\,(\rho_1+\rho_2)\right] \\ &= \frac{1}{m}\left[-\rho_1\frac{\partial\chi}{\partial x} + \rho_2\frac{\partial\eta}{\partial x} - \frac{2\overline{A}_x}{i}\,(\rho_1+\rho_2)\right] \end{aligned} \tag{E.59}$$

$$\begin{aligned} J^y\ /H &= \frac{1}{m}\left[-\rho_1\frac{\partial\chi}{\partial y} + \rho_2\frac{\partial\eta}{\partial y} - (a+a^*)\,(\rho_1+\rho_2)\right] \\ &= \frac{1}{m}\left[-\rho_1\frac{\partial\chi}{\partial y} + \rho_2\frac{\partial\eta}{\partial y} - \frac{2\overline{A}_y}{i}\,(\rho_1+\rho_2)\right] \end{aligned} \tag{E.60}$$

$$J^0\ /H = \rho_1 - \rho_2 \tag{E.61}$$

E.3 Expression of the Euler current at self - duality

At self-duality (and only at self-duality) we can replace the functions a and a^* that define the potentials $A_\pm$ with expressions of the functions $\phi_{1,2}$ and $\phi_{1,2}^*$ coming from the first equation of self-duality, $D_-\phi = 0$.

We will replace the potentials a and a^* using

$$a + a^* = -\frac{1}{2}\frac{\partial \psi}{\partial x} - \frac{\partial \chi}{\partial y} \quad \text{(pure real)} \tag{E.62}$$

$$a - a^* = i\left(\frac{1}{2}\frac{\partial \psi}{\partial y} - \frac{\partial \chi}{\partial x}\right) \quad \text{(pure imaginary)} \tag{E.63}$$

E.3.1 The x component of the current, J^x, at SD

For the x-component we use Eq.(E.63). We have

$$\begin{aligned} \left[mJ^x\right] /H &= -\exp(\psi)\frac{\partial \chi}{\partial x} + \exp(-\psi)\frac{\partial \eta}{\partial x} + i(a - a^*)(\rho_1 + \rho_2) \\ &= -\exp(\psi)\frac{\partial \chi}{\partial x} + \exp(-\psi)\frac{\partial \eta}{\partial x} + i\frac{i}{2}\left[\frac{\partial \psi}{\partial y} - \frac{\partial(2\chi)}{\partial x}\right](\rho_1 + \rho_2) \\ &= -\left[\frac{\partial(\psi/2)}{\partial y} - \frac{\partial \chi}{\partial x}\right](\rho_1 + \rho_2) - \exp(\psi)\frac{\partial \chi}{\partial x} + \exp(-\psi)\frac{\partial \eta}{\partial x} \end{aligned} \tag{E.64}$$

NOTE

Before going further we explore the possibilities of this equation. For this we replace since we are already at SD

$$\begin{aligned} \rho_1 &\rightarrow \exp(\psi) \\ \rho_2 &\rightarrow \exp(-\psi) \end{aligned} \tag{E.65}$$

$$\begin{aligned} \left[mJ^x\right] /H = & -(\rho_1 + \rho_2)\frac{\partial(\psi/2)}{\partial y} \\ & + \underline{\frac{\partial \chi}{\partial x}\exp(\psi)} + \frac{\partial \chi}{\partial x}\exp(-\psi) \\ & - \underline{\exp(\psi)\frac{\partial \chi}{\partial x}} + \exp(-\psi)\frac{\partial \eta}{\partial x} \end{aligned} \tag{E.66}$$

We find the expression

$$\left[mJ^x\right] /H = -(\rho_1 + \rho_2)\frac{\partial(\psi/2)}{\partial y} + \frac{\partial \chi}{\partial x}\exp(-\psi) + \exp(-\psi)\frac{\partial \eta}{\partial x} \tag{E.67}$$

where we can use

$$\chi = -\eta \tag{E.68}$$

and obtain

$$\left[mJ^x\right] /H = -(\rho_1 + \rho_2)\frac{\partial(\psi/2)}{\partial y} \tag{E.69}$$

Finally

$$[mJ^x]\ /H = -\frac{\partial}{\partial y}\frac{1}{2}(\rho_1-\rho_2) \tag{E.70}$$
$$= \frac{\kappa}{4}\frac{\partial}{\partial y}\omega \ \text{ at SD}$$

E.3.2 The y component of the current, J^y at SD

Now the y component of the current. We will use Eq.(E.62) and obtain

$$\begin{aligned}[mJ^y]\ /H &= -\exp(\psi)\frac{\partial\chi}{\partial y}+\exp(-\psi)\frac{\partial\eta}{\partial y}-(a+a^*)(\rho_1+\rho_2) \\ &= -\exp(\psi)\frac{\partial\chi}{\partial y}+\exp(-\psi)\frac{\partial\eta}{\partial y}+\left[\frac{\partial(\psi/2)}{\partial x}+\frac{\partial\chi}{\partial y}\right](\rho_1+\rho_2) \\ &= -\exp(\psi)\frac{\partial\chi}{\partial y}+\exp(-\psi)\frac{\partial\eta}{\partial y}+\left[\frac{\partial(\psi/2)}{\partial x}+\frac{\partial\chi}{\partial y}\right](\rho_1+\rho_2)\end{aligned} \tag{E.71}$$

Expanding

$$\begin{aligned}[mJ^y]\ /H = &\frac{\partial(\psi/2)}{\partial x}(\rho_1+\rho_2)+\underline{\exp(\psi)\frac{\partial\chi}{\partial y}}+\exp(-\psi)\frac{\partial\chi}{\partial y} \\ &-\underline{\exp(\psi)\frac{\partial\chi}{\partial y}}+\exp(-\psi)\frac{\partial\eta}{\partial y}\end{aligned} \tag{E.72}$$

and the two underlined terms cancel each other. The relation

$$\chi=-\eta \tag{E.73}$$

leads to

$$[mJ^y]\ /H=\frac{\partial(\psi/2)}{\partial x}(\rho_1+\rho_2) \tag{E.74}$$

Finally

$$[mJ^y]\ /H = \frac{\partial}{\partial x}\frac{1}{2}(\rho_1-\rho_2) \tag{E.75}$$
$$= -\frac{\kappa}{4}\frac{\partial}{\partial x}\omega \ \text{ at SD}$$

E.3.3 Summary, at SD

We list them again

$$[mJ^x]\ /H = -\left[\frac{\partial(\psi/2)}{\partial y} - \frac{\partial\chi}{\partial x}\right](\rho_1+\rho_2) - \exp(\psi)\frac{\partial\chi}{\partial x} + \exp(-\psi)\frac{\partial\eta}{\partial x} \tag{E.76}$$

$$[mJ^y]\ /H = +\left[\frac{\partial(\psi/2)}{\partial x} + \frac{\partial\chi}{\partial y}\right](\rho_1+\rho_2) - \exp(\psi)\frac{\partial\chi}{\partial y} + \exp(-\psi)\frac{\partial\eta}{\partial y} \tag{E.77}$$

When the phases are replaced as $\chi = -\eta$ it is obtained

$$\begin{aligned}[mJ^x]\ /H &= -\frac{\partial}{\partial y}\frac{1}{2}(\rho_1-\rho_2) \\ &= \frac{\kappa}{4}\frac{\partial}{\partial y}\omega \quad \text{at SD}\end{aligned} \tag{E.78}$$

and

$$\begin{aligned}[mJ^y]\ /H &= \frac{\partial}{\partial x}\frac{1}{2}(\rho_1-\rho_2) \\ &= -\frac{\kappa}{4}\frac{\partial}{\partial x}\omega \quad \text{at SD}\end{aligned} \tag{E.79}$$

To this we have to add

$$J^0 = \rho_1 - \rho_2 = -\frac{\kappa}{2}\omega \quad \text{at SD} \tag{E.80}$$

Then the covariant conservation of the current results

$$D_\mu J^\mu = 0 \tag{E.81}$$

We **NOTE** that the current appears as the rotational of the density of vorticity.

F Appendix F. The current projected along the streamlines and the perpendicular direction

The expressions of the current components are

$$[mJ^x]\ /H = -\left[\frac{\partial(\psi/2)}{\partial y} - \frac{\partial\chi}{\partial x}\right](\rho_1+\rho_2) - \exp(\psi)\frac{\partial\chi}{\partial x} + \exp(-\psi)\frac{\partial\eta}{\partial x} \tag{F.1}$$

$$[mJ^y]\ /H = +\left[\frac{\partial\left(\psi/2\right)}{\partial x}+\frac{\partial\chi}{\partial y}\right](\rho_1+\rho_2)-\exp\left(\psi\right)\frac{\partial\chi}{\partial y}+\exp\left(-\psi\right)\frac{\partial\eta}{\partial y} \tag{F.2}$$

and without the phases, taking into account that at SD $\chi = -\eta$.

$$[mJ^x]\ /H = -\frac{\partial}{\partial y}\frac{1}{2}(\rho_1-\rho_2)\quad \text{at SD} \tag{F.3}$$

$$[mJ^y]\ /H = \frac{\partial}{\partial x}\frac{1}{2}(\rho_1-\rho_2)\quad \text{at SD} \tag{F.4}$$

F.1 Projection formulas

We will make a change of the system of reference in plane

$$(\widehat{\mathbf{e}}_x, \widehat{\mathbf{e}}_y) \rightarrow (\widehat{\mathbf{e}}_\psi, \widehat{\mathbf{e}}_\perp) \tag{F.5}$$

where we have to define the two versors.

The infinitesimal displacement along the streamline is represented by the vector

$$\begin{aligned}\mathbf{dl}_\parallel &= (\delta x, \delta y) \\ &= \delta x\widehat{\mathbf{e}}_x + \delta y\widehat{\mathbf{e}}_y\end{aligned} \tag{F.6}$$

with the length

$$\begin{aligned}\left|\mathbf{dl}_\parallel\right| &= \sqrt{(\delta x)^2+(\delta y)^2} \\ &= \delta x\sqrt{1+\left(\frac{\delta y}{\delta x}\right)^2}\end{aligned} \tag{F.7}$$

and the versor is

$$\widehat{\mathbf{e}}_\psi = \frac{\mathbf{dl}_\parallel}{\left|\mathbf{dl}_\parallel\right|} = \frac{1}{\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2}}\widehat{\mathbf{e}}_x + \frac{\frac{\partial y}{\partial x}}{\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2}}\widehat{\mathbf{e}}_y \tag{F.8}$$

where the streamline is represented in tow forms

$$\begin{aligned}\psi\left(x,y\right) &= \text{const} \\ y &= y\left(x\right)\end{aligned} \tag{F.9}$$

From the theorem of implicit functions we get

$$\begin{aligned}\frac{\partial\psi}{\partial x}+\frac{\partial\psi}{\partial y}\frac{\partial y}{\partial x} &= 0 \\ \frac{\partial y}{\partial x} &= -\frac{\frac{\partial\psi}{\partial x}}{\frac{\partial\psi}{\partial y}}\end{aligned} \tag{F.10}$$

and the versor along the streamline is

$$\widehat{\mathbf{e}}_\psi = \frac{\frac{\partial \psi}{\partial y}}{\sqrt{\left(\frac{\partial \psi}{\partial y}\right)^2 + \left(\frac{\partial \psi}{\partial x}\right)^2}} \widehat{\mathbf{e}}_x + \left(-\frac{\frac{\partial \psi}{\partial x}}{\frac{\partial \psi}{\partial y}}\right) \frac{\frac{\partial \psi}{\partial y}}{\sqrt{\left(\frac{\partial \psi}{\partial y}\right)^2 + \left(\frac{\partial \psi}{\partial x}\right)^2}} \widehat{\mathbf{e}}_y \tag{F.11}$$

We replace

$$\sqrt{\left(\frac{\partial \psi}{\partial y}\right)^2 + \left(\frac{\partial \psi}{\partial x}\right)^2} = |\nabla \psi| \tag{F.12}$$

and we have

$$\widehat{\mathbf{e}}_\psi = \frac{1}{|\nabla \psi|} \frac{\partial \psi}{\partial y} \widehat{\mathbf{e}}_x - \frac{1}{|\nabla \psi|} \frac{\partial \psi}{\partial x} \widehat{\mathbf{e}}_y \tag{F.13}$$

The other versor, perpendicular on the streamline, is defined by a vector product

$$\begin{aligned} \widehat{\mathbf{e}}_\perp &= \widehat{\mathbf{e}}_z \times \widehat{\mathbf{e}}_\psi = \begin{pmatrix} \widehat{\mathbf{e}}_x & \widehat{\mathbf{e}}_y & \widehat{\mathbf{e}}_z \\ 0 & 0 & 1 \\ \frac{1}{|\nabla \psi|} \frac{\partial \psi}{\partial y} & -\frac{1}{|\nabla \psi|} \frac{\partial \psi}{\partial x} & 0 \end{pmatrix} \\ &= \widehat{\mathbf{e}}_x \left(+\frac{1}{|\nabla \psi|} \frac{\partial \psi}{\partial x}\right) + \widehat{\mathbf{e}}_y \left(+\frac{1}{|\nabla \psi|} \frac{\partial \psi}{\partial y}\right) \end{aligned} \tag{F.14}$$

We have the transformation

$$\begin{pmatrix} \widehat{\mathbf{e}}_\psi \\ \widehat{\mathbf{e}}_\perp \end{pmatrix} = \begin{pmatrix} \frac{1}{|\nabla \psi|} \frac{\partial \psi}{\partial y} & -\frac{1}{|\nabla \psi|} \frac{\partial \psi}{\partial x} \\ \frac{1}{|\nabla \psi|} \frac{\partial \psi}{\partial x} & \frac{1}{|\nabla \psi|} \frac{\partial \psi}{\partial y} \end{pmatrix} \begin{pmatrix} \widehat{\mathbf{e}}_x \\ \widehat{e}_y \end{pmatrix} \tag{F.15}$$

The determinant of this matrix is

$$\det \begin{pmatrix} \frac{1}{|\nabla \psi|} \frac{\partial \psi}{\partial y} & -\frac{1}{|\nabla \psi|} \frac{\partial \psi}{\partial x} \\ \frac{1}{|\nabla \psi|} \frac{\partial \psi}{\partial x} & \frac{1}{|\nabla \psi|} \frac{\partial \psi}{\partial y} \end{pmatrix} = \frac{1}{|\nabla \psi|^2} \left(\frac{\partial \psi}{\partial y}\right)^2 + \frac{1}{|\nabla \psi|^2} \left(\frac{\partial \psi}{\partial x}\right)^2 = 1 \tag{F.16}$$

and the inverse transformation

$$\begin{pmatrix} \widehat{\mathbf{e}}_x \\ \widehat{e}_y \end{pmatrix} = \begin{pmatrix} \frac{1}{|\nabla \psi|} \frac{\partial \psi}{\partial y} & \frac{1}{|\nabla \psi|} \frac{\partial \psi}{\partial x} \\ -\frac{1}{|\nabla \psi|} \frac{\partial \psi}{\partial x} & \frac{1}{|\nabla \psi|} \frac{\partial \psi}{\partial y} \end{pmatrix} \begin{pmatrix} \widehat{\mathbf{e}}_\psi \\ \widehat{\mathbf{e}}_\perp \end{pmatrix} \tag{F.17}$$

or

$$\begin{aligned} \widehat{\mathbf{e}}_x &= \frac{1}{|\nabla \psi|} \frac{\partial \psi}{\partial y} \widehat{\mathbf{e}}_\psi + \frac{1}{|\nabla \psi|} \frac{\partial \psi}{\partial x} \widehat{\mathbf{e}}_\perp \\ \widehat{\mathbf{e}}_y &= -\frac{1}{|\nabla \psi|} \frac{\partial \psi}{\partial x} \widehat{\mathbf{e}}_\psi + \frac{1}{|\nabla \psi|} \frac{\partial \psi}{\partial y} \widehat{\mathbf{e}}_\perp \end{aligned} \tag{F.18}$$

Now we rotate the vector

$$\begin{aligned}\mathbf{J} &= \widehat{\mathbf{e}}_x J_x + \widehat{\mathbf{e}}_y J_y \\ &= \widehat{\mathbf{e}}_\psi J_\psi + \widehat{\mathbf{e}}_\perp J_\perp\end{aligned} \tag{F.19}$$

$$\begin{aligned}\mathbf{J} &= J_x \left(\frac{1}{|\nabla\psi|} \frac{\partial\psi}{\partial y} \widehat{\mathbf{e}}_\psi + \frac{1}{|\nabla\psi|} \frac{\partial\psi}{\partial x} \widehat{\mathbf{e}}_\perp \right) + J_y \left(-\frac{1}{|\nabla\psi|} \frac{\partial\psi}{\partial x} \widehat{\mathbf{e}}_\psi + \frac{1}{|\nabla\psi|} \frac{\partial\psi}{\partial y} \widehat{\mathbf{e}}_\perp \right) \\ &= \frac{1}{|\nabla\psi|} \left(J_x \frac{\partial\psi}{\partial y} - J_y \frac{\partial\psi}{\partial x} \right) \widehat{\mathbf{e}}_\psi \\ &\quad + \frac{1}{|\nabla\psi|} \left(J_x \frac{\partial\psi}{\partial x} + J_y \frac{\partial\psi}{\partial y} \right) \widehat{\mathbf{e}}_\perp\end{aligned} \tag{F.20}$$

Now we calculate the two components using the expressions of $J_{x,y}$,

$$\begin{aligned}J_\psi &= \frac{1}{|\nabla\psi|} \left(J_x \frac{\partial\psi}{\partial y} - J_y \frac{\partial\psi}{\partial x} \right) \\ &= \frac{1}{|\nabla\psi|\, m} \left\{ \frac{\partial\psi}{\partial y} \left(-\left[\frac{\partial(\psi/2)}{\partial y} - \frac{\partial\chi}{\partial x} \right] (\rho_1 + \rho_2) - \exp(\psi) \frac{\partial\chi}{\partial x} + \exp(-\psi) \frac{\partial\eta}{\partial x} \right) \right. \\ &\quad - \left. \frac{\partial\psi}{\partial x} \left(-\left[\frac{\partial(\psi/2)}{\partial x} + \frac{\partial\chi}{\partial y} \right] (\rho_1 + \rho_2) - \exp(\psi) \frac{\partial\chi}{\partial y} + \exp(-\psi) \frac{\partial\eta}{\partial y} \right) \right\}\end{aligned} \tag{F.21}$$

and

$$\begin{aligned}J_\perp &= \frac{1}{|\nabla\psi|} \left(J_x \frac{\partial\psi}{\partial x} + J_y \frac{\partial\psi}{\partial y} \right) \\ &= \frac{1}{|\nabla\psi|\, m} \left\{ \frac{\partial\psi}{\partial x} \left(-\left[\frac{\partial(\psi/2)}{\partial y} - \frac{\partial\chi}{\partial x} \right] (\rho_1 + \rho_2) - \exp(\psi) \frac{\partial\chi}{\partial x} + \exp(-\psi) \frac{\partial\eta}{\partial x} \right) \right. \\ &\quad + \left. \frac{\partial\psi}{\partial y} \left(-\left[\frac{\partial(\psi/2)}{\partial x} + \frac{\partial\chi}{\partial y} \right] (\rho_1 + \rho_2) - \exp(\psi) \frac{\partial\chi}{\partial y} + \exp(-\psi) \frac{\partial\eta}{\partial y} \right) \right\} \\ &=\end{aligned} \tag{F.22}$$

F.2 Using the final formulas for the current components

We can use

$$[mJ^x]\ /H = -\frac{\partial}{\partial y} \frac{1}{2} (\rho_1 - \rho_2) \quad \text{at SD} \tag{F.23}$$

$$[mJ^y]\ /H = \frac{\partial}{\partial x} \frac{1}{2} (\rho_1 - \rho_2) \quad \text{at SD} \tag{F.24}$$

or the equivalent forms

$$[mJ^x]\ /H = -\frac{1}{2}(\rho_1+\rho_2)\frac{\partial\psi}{\partial y} \quad \text{at SD} \tag{F.25}$$

$$[mJ^y]\ /H = \frac{1}{2}(\rho_1+\rho_2)\frac{\partial\psi}{\partial x} \quad \text{at SD} \tag{F.26}$$

and obtain

$$\begin{aligned} J_\psi &= \frac{1}{|\nabla\psi|}\left(J_x\frac{\partial\psi}{\partial y} - J_y\frac{\partial\psi}{\partial x}\right) \\ &= \frac{1}{|\nabla\psi|\,m}\frac{1}{2}(\rho_1+\rho_2)\left[-\left(\frac{\partial\psi}{\partial y}\right)^2 - \left(\frac{\partial\psi}{\partial x}\right)^2\right] \\ &= -\frac{1}{2m}(\rho_1+\rho_2)\,|\nabla\psi| \quad \text{at SD} \end{aligned} \tag{F.27}$$

and

$$\begin{aligned} J_\perp &= \frac{1}{|\nabla\psi|}\left(J_x\frac{\partial\psi}{\partial x} + J_y\frac{\partial\psi}{\partial y}\right) \\ &= \frac{1}{|\nabla\psi|\,m}\frac{1}{2}(\rho_1+\rho_2)\left(-\frac{\partial\psi}{\partial y}\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial y}\right) \\ &= 0 \quad \text{at SD} \end{aligned} \tag{F.28}$$

This indeed confirms that the only current is along the streamlines and the current transversal to them vanishes.

References

[1] W.H. Matthaeus, W.T. Stribling, D. Martinez, S. Oughton, and D. Montgomery. Decaying, two-dimensional, navier-stokes turbulence at very long times. *Physica D*, 51:531–538, 1991.

[2] D. Montgomery, W.H. Matthaeus, W.T. Stribling, D. Martinez, and S. Oughton. Relaxation in two-dimensions and the "sinh-poisson" equation. *Phys. Fluids A*, 4:3–6, 1992.

[3] A.C. Ting, H.H. Chen, and Y.C. Lee. Exact solutions of a nonlinear boundary value problem: The vortices of the two-dimensional sinh-poisson equation. *Physica D: Nonlinear Phenomena*, 26(13):37 – 66, 1987.

[4] L. J. Mason and N. M. J. Woodhouse. *Integrability, self - duality and twistor theory*. London Mathematical Society Monographs New Series. Clarendon Press, Oxford, 1996.

[5] F. Spineanu and M. Vlad. Self-duality of the asymptotic relaxation states of fluids and plasmas. *Phys. Rev. E*, 67:046309–1–4, 2003.

[6] R. Jackiew and So-Y. Pi. Classical and quantal nonrelativistic chern-simons theory. *Phys. Rev. Lett.*, 64:2969–2979, 1990.

[7] R H Kraichnan and D Montgomery. Two-dimensional turbulence. *Reports on Progress in Physics*, 43(5):547–619, 1980.

[8] R. Robert and J. Sommeria. Statistical equilibrium states for two dimensional flows. *J. Fluid Mech*, 229:291, 1991.

[9] R. Robert and J. Sommeria. Relaxation towards a statistical equilibrium state in two-dimensional perfect fluid dynamics. *Phys. Rev. Lett.*, 69:2776–2779, 1992.

[10] Pierre-Henri Chavanis. Statistical mechanics of two-dimensional vortices and stellar systems. In Thierry Dauxois, Stefano Ruffo, Ennio Arimondo, and Martin Wilkens, editors, *Dynamics and Thermodynamics of Systems with Long-Range Interactions*, volume 602 of *Lecture Notes in Physics*, pages 208–289. Springer Berlin Heidelberg, 2002.

[11] L. Onsager. Statistical hydrodynamics. *Il Nuovo Cimento Series 9*, 6(2):279–287, 1949.

[12] A.J. Majda and X. Wang. *Non-linear dynamics and statistical theories for basic geophysical flows*. Cambridge University Press, 2006.

[13] D.G Dritschel, M. Lucia, and A.C. Poje. Equilibrium statistics and dynamics of point vortex flows on the sphere. *http://arxiv.org/pdf/1407.3151.pdf*, 2014.

[14] S.F. Edwards and J.B. Taylor. Negative temperatures states of two-dimensional plasmas and vortex fluids. *Proc. R. Soc. Lond. A*, 336:257–271, 1974.

[15] G. Joyce and D.C. Montgomery. Negative temperature states of the two-dimensional guiding-centre plasma. *J. Plasma Phys.*, 10:107–121, 1973.

[16] J.B. Taylor. Filamentation, current profile and transport in a tokamak. *Phys. Fluids B5*, pages 4378–4383, December 1993.

[17] F. Bethuel, G. Orlandi, and D. Smets. Collisions and phase-vortex interactions in dissipative ginzburg-landau dynamics. *Duke Math. J.*, 130:523–614, 2005.

[18] Gerald V. Dunne, R. Jackiw, So-Young Pi, and Carlo A. Trugenberger. Self-dual chern-simons solitons and two-dimensional nonlinear equations. *Phys. Rev. D*, 43:1332–1345, Feb 1991.

[19] G. Dunne. *Self - dual Chern - Simons theories*, volume 36 of *Lecture Notes in Physics*. Springer Verlag, Berlin Heidelberg, 1995.

[20] P.C. Martin, E.D. Siggia, and H.A. Rose. Statistical dynamics of classical systems. *Phys. Rev. A*, 8:423–437, 1973.

[21] R.V. Jensen. Functional integral approach to classical statistical dynamics. *J. Stat. Phys.*, 5:183–210, 1981.

[22] L. Jacobs and C. Rebbi. Interaction energy of superconducting vortices. *Phys. Rev. B*, 19:4486–4494, 1979.

[23] K. Arthur. Interaction energy of chern-simons vortices. *Phys. Lett. B*, 356:509–515, 1995.

[24] W.G. Fuertes and J.M. Guilarte. Low energy vortex dynamics in Abelian Higgs systems. *arxiv.org*, hep-th:9812103, 1998.

[25] N.S. Manton. Statistical mechanics of vortices. *Nucl. Phys. B*, 400[FS]:624–632, 1993.

[26] B.J. Nauta and A. Arrizabalaga. Asymmetric chern-simons number diffusion from cp-violation. *Nucl. Phys. B*, 635:255–285, 2002.

[27] P. Arnold, D. Son, and L.G. Yaffe. The hot baryon violation rate is $o\,(\alpha_W^5 t^4)$. *Phys. Rev. D*, 55:6264–6273, 1997.

[28] R. Jackiw and S.Y. Pi. Finite and infinite symmetries in $2+1$ dimensional field theory. *Nucl. Phys. B (Proc. Suppl)*, 33C:104–113, 1993.

[29] L. Berge, A. de Bouard, and J.C. Saut. Collapse of chern-simons - gauged matter fields. *Phys. Rev. Lett*, 74(20), 1995.

[30] E. Levich. Certain problems in the theory of developed hydrodynamical turbulence. *Physics Reports*, 151(3–4):129 – 238, 1987.

[31] J.B. Taylor. Negative temperatures in two dimensional vortex motion. *Phys. Letters*, 40A:1–2, 1972.

[32] E.J. Hopfinger and G.J.F. van Heijst. Vortices in rotating fluids. *Annu. Rev. Fluid Mech.*, 25:241–289, 1993.

[33] G. M. Corcos and F. S. Sherman. Vorticity concentration and the dynamics of unstable free shear layers. *Journal of Fluid Mechanics*, 73:241–264, 1 1976.

[34] James C. McWilliams. The emergence of isolated coherent vortices in turbulent flow. *Journal of Fluid Mechanics*, 146:21–43, 9 1984.

[35] F. Spineanu and M. Vlad. Stationary vortical flows in two-dimensional plasma and in planetary atmospheres. *Phys. Rev. Lett.*, 94:235003–1–4, 2005.

[36] F. Spineanu and M. Vlad. Relationships between the main parameters of the stationary two-dimensional vortical flows in the planetary atmosphere. *Geophys. Astro. Fluid. Dyn.*, 103:223–244, 2009.

[37] F. Spineanu and M. Vlad. A field theoretical prediction of the tropical cyclone properties . *arxiv.org*, physics:1310.2750, 2013.

[38] F. Spineanu and M. Vlad. The asymptotic quasi-stationary states of the two-dimensional magnetically confined plasma and of the planetary atmosphere. *arxiv.org*, physics:0501020, 2005.