A MHD invariant and the confinement regimes in Tokamak

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Abstract

Fundamental Lagrangian, frozen-in and topological invariants can be useful to explain systematic connections between plasma parameters. At high plasma temperature the dissipation is small and the robust invariances are manifested. We invoke a frozen-in invariant which is an extension of the Ertel's theorem and connects the vorticity of the large scale motions with the profile of the safety factor and of particle density. Assuming ergodicity of the small scale turbulence we consider the approximative preservation of the invariant for changes of the vorticity in an annular region of finite radial extension (i.e. poloidal rotation). We find that the ionization-induced rotation triggered by a pellet requires a reversed-q profile in an off-axis region of the core. In the H-mode, the invariance requires the accumulation of the current density in the rotation layer at the edge. Then this becomes a vorticity-current sheet which may explain experimental observations related to the penetration of the Resonant Magnetic Perturbation and the filamentation during the Edge Localized Modes.

1 Introduction

To understand and predict properties of the tokamak plasma it is necessary to investigate turbulent-induced transport processes and, of equal importance, coherent structures (like zonal flows, H-mode rotation layers and convective cells). The analytical and numerical approaches and the methods are however largely different, from statistical theory to exact integrability. It is then helpful to remember that both physical aspects, even if different in their manifestation, must obey a basic set of constraints which are due to the existence of Lagrangian invariants and of frozen-in invariants. The invariants are constraints that must be verified by any dynamical evolution of the system. In general for real plasma (*i.e.* with finite resistivity and viscosity) the invariance property can only be approximative, but the high temperature (typical for fusion reactor) reduces the collisional effects even in regions close to the plasma edge. Then the persistence of some invariants can still be used as a benchmark for our analytical models and for numerical simulations, similar to the conservation of the energy or the angular momentum.

We will be interested in the invariants that include in their expression the plasma vorticity $\omega = \nabla \times \mathbf{v}$, *i.e.* the sheared plasma rotation, which has constantly been proven an important factor in the confinement. In the specific case of the tokamak, variation of the parameters composing the invariant appear as small perturbations around the high cyclotron frequency $\mathbf{\Omega}_c$, which is a kind of "condensate of vorticity". Although it is a passive component of the "absolute vorticity" $\omega + \mathbf{\Omega}_c = \nabla\times \left(\mathbf{v}+\frac{1}{2}\mathbf{\Omega}_c \times \mathbf{r}\right)$ the high magnitude of Ω_c reduces the practical relevance of the invariant to those situations where the physical vorticity is sufficiently high. We expect that the H-mode, the zonal flows and the internal transport barrier belong to this class.

If the Lagrangian and the frozen-in invariants are difficult to be tested directly, they are however responsible for another class of invariants, of topological nature.

2 Geometry versus plasma dynamics, reflected in types of invariants

2.1 Topological invariants

Below we underline the difference between the topological invariants and those that involve fluid advection.

There is a hierarchy of topological invariants [1] [2], the lowest and most familiar being the Gauss linking of two lines in space, a discrete number showing how many times one line turns around the other. If the Kolmogorov length is a local measure of the chaotic state (*i.e.* the presence of the stochastic instability), the Gauss linking number is a non-local variable which can also be stochastic [3], [4], [5]. It is a quantitative measure that goes beyond the Chirikov criterion of overlapping of chains of magnetic islands developing at nearby rational surfaces, a criterion usually invoked for the onset of magnetic stochasticity [6], [7]. More generally, the lines in space can be magnetic field lines or streamlines of the plasma flow. The fluid quantities that are associated are the magnetic helicity whose volume density is $\mathbf{A} \cdot \mathbf{B}$, kinetic helicity $\mathbf{v} \cdot \omega$ and cross-helicity $\mathbf{v} \cdot \mathbf{B}$ [8]. The connection between the helicity that is defined by the density of a continuous field in the fluid volume and the discrete Gaussian link numbers [9] can be seen in the following way. Consider a set of lines γ_k that are closed, like in tokamak, or possibly after extending to infinity. We take as basic fields defined in the volume of the fluid/plasma, the magnetic potential $A_\mu = (A_k, A_0)$ and the velocity $v_\mu = (v_k, c_0)$ (where c_0 is a constant) and define the circulation as the integrals of the potential or velocity along these lines

$$\oint_{\gamma_k} \mathbf{A} \cdot \mathbf{dl} \quad \text{and} \quad \oint_{\gamma_k} \mathbf{v} \cdot \mathbf{dl}$$

As results from the Stokes theorem, these are the fluxes of the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ respectively of the vorticity $\omega = \nabla \times \mathbf{v}$ through the surface bounded by the curve γ_k. The freezing-in of $\mathbf{B}$ and respectively the Kelvin theorem ensure that, when the line γ_k is carried by the flow (*i.e.* it is a *material* curve) the fluxes remain constant. Consider the variable

$$\exp\left(\sum_k \oint_{\gamma_k} A_\mu dx^\mu\right)$$

(at fixed time x^0 the integrand is identical with that of the previous expression; all calculations can be done for v_μ as well) for a particular configuration of the field A_μ and calculate the average of this quantity over an ensemble of realizations of the field A_μ defined in space $\mathbf{R}^3$. The statistical accessibility of a state with flux $\oint_{\gamma_k} A_\mu dx^\mu$ is given by the Boltzmann-like weight factor, defined as the exponential of the total helicity in the volume

$$\exp\left(-\int d^3x \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho\right)$$

The integrand is the general form of the usual helicity $\mathbf{A} \cdot \mathbf{B} = \mathbf{A} \cdot (\nabla \times \mathbf{A})$. This Boltzmann factor "penalizes" states with high total amount of helicity in the volume by making them less accessible. It can be shown that the functional integral

$$\int \mathcal{D}\left[A_\mu\right] \exp\left(\sum_k \oint_{\gamma_k} A_\mu dx^\mu\right) \exp\left(-\int d^3x \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho\right)$$

which is the statistical average of the exponential of the field fluxes through surfaces bounded by the fixed set of closed spatial curves γ_k $(k = 1, 2, ...)$, over fields of all possible helicities, is the exponential of the sum of Gaussian linking number of pairs of curves γ_k. The *entanglement* of γ_k's is a purely geometric characteristic and for ideal fluids is conserved by the motion. Higher order topological invariants can be found, with more sofisticated definitions

[2]. The nonconservation of the topological content of the magnetic field configuration is determined by reconnection. In the tokamak conditions, only the presence of dissipation allows breaking-up and reconnections of magnetic field lines, thus making possible transitions between classes of states with distinct topological content. When the fluid is slightly non-ideal, low-order topological invariants like the *entanglement* are robust and they still remain relevant, while higher order topological invariants are gradually lost [10], [11]. This is better illustrated by the two-dimensional topology of streamlines of a quasi-ideal fluid. The inverse cascade in $2D$ leads asymptotically to a highly coherent state of the flow, a simple dipole in the double periodic plane domain, after all positive respectively negative vorticity elements have been separated and concentrated [12]. This evolution essentially traverses a large set of distinct topological configuration and this is possible only through break-up and reconnection of the streamlines (*i.e.* violation of the Kelvin invariance), made possible by the finite dissipation. The dissipation is located in very small volumes where the gradients of the magnetic field (respectively vorticity for fluids) are very large and the ensemble of these points is fractal in the total volume [13]. The amount of energy lost at every reconnection is small and for this reason the total energy is still a good invariant while the topology undergoes radical changes [14], [15].

We note from the example given above that a topological invariant does not involve directly the dynamics of fluid motion. The topological properties mentioned above are commonly associated to sets of lines with clear physical meaning (magnetic lines and lines of fluid flow) whose stochastic entanglement is the cause of high rate particle and heat transport. Other sets of lines must be considered, which can be defined according to the same method as for the magnetic field. The Clebsch representation of the magnetic potential $\mathbf{A} = \nabla\theta + \alpha\nabla\beta$ shows that the lines of the magnetic field $\mathbf{B} = \nabla\alpha \times \nabla\beta$ are defined by the intersection of two surfaces $\alpha =$const, $\beta =$const. More generally, the two functions α and β are Lagrangian invariants that define the frozen-in invariant $\mathbf{B}/\rho$. A comprehensive explanation is presented by Tur and Yanovskii [16]. The purpose of the above discussion is to underline the distinction between the topology-preserving constraints and the fluid invariants, the latter being our subject.

2.2 Invariants of the plasma dynamics

The Lagrangian invariants are quantities I that remain constant along the lines of flow

$$\frac{\partial I}{\partial t} + (\mathbf{v} \cdot \nabla)\, I = 0 \tag{1}$$

An elementary example is the density ρ. We can see the flow as a collection of streamlines each carying as unique label the initial point $\mathbf{x}_0 = \mathbf{x}\,(t=0)$. This characteristic remains the same for all moments of displacement of a particle along the streamline and so $\mathbf{x}_0$ is a Lagrangian invariant. The integrals $\mathbf{J}$ defined by the equation

$$\frac{\partial \mathbf{J}}{\partial t} + (\mathbf{v}\cdot\nabla)\,\mathbf{J} = (\mathbf{J}\cdot\nabla)\,\mathbf{v} \tag{2}$$

and are vector fields that are advected by the flow as frozen-in. For example, the magnetic field $\mathbf{B}$ is frozen into the plasma. Frozen-in quantities can be obtained from two Lagrangian invariants I_1 and I_2, as

$$\mathbf{J} = \frac{1}{\rho}\nabla I_1 \times \nabla I_2 \tag{3}$$

which is suggested by the fact that $I_k\,(x,y,z,t) = \text{const}$ along the streamlines defines a surface and the gradient of the scalar invariant is proportional with the vector of the element of area

$$\nabla I_k \sim \rho d\mathbf{S} \tag{4}$$

Then $\nabla I_1 \times \nabla I_2$ is the line of intersection of the two surfaces $I_{1,2} = \text{const}$. More generally, multiplying this expression with another Lagrangian invariant,

$$J_\alpha = \frac{1}{\rho}\varepsilon_{\alpha\beta\gamma}\varepsilon_{ijk} I_i \frac{\partial I_j}{\partial x^\beta}\frac{\partial I_k}{\partial x^\gamma} \tag{5}$$

one obtains a new class of frozen-in invariants. We note each Lagrangian invariant is a scalar I_k while $\rho\mathbf{J}$ is a current in the $(3+1)$ - dimensional geometry. For the two-dimensional ideal fluid the ratio

$$\mathbf{J}^\omega = \frac{\omega}{\rho} \tag{6}$$

is a frozen-in invariant. Since $\omega = \nabla\times\mathbf{v}$ we have $\nabla\cdot(\mathbf{J}^\omega\rho) = 0$. For incompressible fluid $\nabla\cdot\mathbf{v} = 0$, $\rho = \text{const}$ and the equation

$$\frac{d}{dt}\left(\rho\mathbf{J}^\omega\right) = 0 \tag{7}$$

becomes the Euler equation. For the plasma immersed in strong magnetic field (similarly for planetary atmosphere, both being well approximated by a two-dimensional geometry, see [17]) the invariant (Ertel) is

$$\mathbf{J}^E = \frac{\omega + \mathbf{\Omega}_c}{n} \tag{8}$$

where $\mathbf{\Omega}_c = \Omega_c \widehat{\mathbf{e}}_z$ is the cyclotron frequency (in magnetized plasma) or the Coriolis frequency (in planetary atmosphere); $\widehat{\mathbf{e}}_z$ is the unit vector perpendicular on the plane. From this invariant it is derived the Charney-Hasegawa-Mima (quasi-three dimensional) equation, governing the turbulent fluctuations and the cuasi-coherent vortical structures at the level of the ion Larmor radius [18]. If the gyration frequency is far from the frequency scale associated to the plasma motions, a natural extension of this invariant exists for charged particles (separately for electrons and for ions), which includes the absolute vorticity and the magnetic field [19]. It is a frozen-in invariant of the same structure as Eq.(2)

$$\frac{d}{dt}\frac{\omega+\eta\mathbf{B}+\mathbf{\Omega}_c}{\rho} = \left(\frac{\omega+\eta\mathbf{B}+\mathbf{\Omega}_c}{\rho}\cdot\nabla\right)\mathbf{v} \tag{9}$$

where η is a dimensional constant, $\eta = |e|/m$. Since we are interested in plasma motion we will use this invariant with reference to ions. Eq.(9) is the extended form of the Ertel's theorem which includes the magnetic field. The lines of force of the field

$$\mathbf{J} = \frac{\nabla \times \mathbf{v}+\eta\mathbf{B}+\mathbf{\Omega_c}}{n} \tag{10}$$

are strongly influenced by the turbulence. We can choose an arbitrary point $\mathbf{x}_1$ situated on a line of force of $\mathbf{J}$ and the value $\mathbf{J}(\mathbf{x}_1)$ taken in that point. Assuming that the turbulence is ergodic another line of force, originating from a different initial position $\mathbf{x}_2$ will come close to the chosen point. Since the two lines of force are carrying different values of $\mathbf{J}$ (as determined by their initial conditions) the real value that will be "measured" is an average taken over a small spatial patch around $\mathbf{x}_1$, since the viscosity will smooth out the strong differences. The space-time evolution of $\mathbf{J}$ initialized at $\mathbf{x}_1$, requires the fields $(\omega, \mathbf{B}, n)$ that enter its expression to adjust themselves such as to ensure invariance. If in a region of plasma the vorticity is higher than that of $\mathbf{x}_1$ then the magnetic field $\mathbf{B}$ and/or the density n must compensate for this change.

We note finally that in the invariant $\mathbf{J}$ are present the "relative" ω and the "absolute" $\omega + \mathbf{\Omega}_c$ vorticities (in the language of the fluid physics).

2.3 Brief comparison with the TEP invariants

Invariants that explicitely constrain the motions in plasma have been derived in three basic approaches: relabeling invariance within the Hamiltonian formulation for fluids; analytical derivation of frozen-in and Lagrangian invariants as properties of the basic equations; and Turbulent Equipartition (TEP).

Our focus will be on the MHD invariant found by Sagdeev, Moseev, Tur and Yanovskii (SMTY), Eq.(10), which can be derived from the basic two-fluid plasma equations. In increasing complexity, the SMTY invariant can be seen as an extended form of well known frozen-in invariant $\mathbf{B}/\rho$. Or, there is a particularity of the latter invariant, in that it can also be derived, in a similar form, within the concept of turbulent equipartition (TEP). Then one can be tempted to suppose that the SMTY invariant, or other frozen-in invariants, can also be connected somehow with the TEP. We find useful to discuss this aspect since the two possible views on the SMTY invariant would be substantially different and would have impact on the applications. From the ideal MHD model

$$\begin{aligned}
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\,\mathbf{v} &= -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\frac{\nabla\times\mathbf{B}}{\mu_0}\times\mathbf{B} \qquad (11)\\
\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{v}\cdot\nabla)\,\mathbf{B} &= (\mathbf{B}\cdot\nabla)\,\mathbf{v}\\
\frac{\partial \rho}{\partial t} + (\mathbf{v}\cdot\nabla)\,\rho &= 0
\end{aligned}$$

one obtains

$$\frac{\partial}{\partial t}\left(\frac{\mathbf{B}}{\rho}\right) + (\mathbf{v}\cdot\nabla)\left(\frac{\mathbf{B}}{\rho}\right) = \left(\frac{\mathbf{B}}{\rho}\cdot\nabla\right)\mathbf{v} \qquad (12)$$

and this shows that $\mathbf{B}/\rho$ is a frozen-in *fluid* invariant.

Unlike the fluid MHD the TEP essentially relies on the particle dynamics. One starts from the invariants of the equations of motion of a single charged particle: (ϵ,μ,J), which are derived in the particular geometry of the toroidally magnetic confinement. This is much less general than the MHD equations above. Since the free motion of the particle is periodic (gyration, bounce on banana orbit, toroidal precession) one can introduce action-angle variables [20]. The interaction with the turbulence leads to the loss of some of the invariants, if the frequency of periodic motion is less than the typical frequency of the electrostatic turbulence [21]. Due to the high gyration frequency the magnetic momentum μ remains invariant $\mu = v_\perp^2/(2B)$. The phase space can be reduced to $2D$ *i.e.* (x, y, v_x, v_y) where the invariant $\mu=\mu_0$ defines a hypersurface. The velocity space in the laboratory frame (v_x, v_y) is mapped onto (μ,ζ) where ζ is the gyrophase. The turbulence produces uniform distribution on the hypersurface $\mu = \text{const}$ of the probability to find the state of the particle in some infinitesimal $2D$ volume. We need to express the result in the real-space coordinates which means to return from (μ,ζ) to (v_x, v_y). The density which is uniform on $\mu = \text{const}$ is multiplied by the

Jacobian of the transformation and this introduces the dependence of the parameters [22]

$$n(x,y) \sim \frac{\partial(v_x, v_y)}{\partial(\mu, \zeta)} = B \tag{13}$$

or n/B =const. This result is identical to the MHD result B/ρ =const (after reduction to $2D$ there is no need to exhibit the vectorial nature of B). We note that in this TEP derivation the density n occurs as the distribution function integrated over the velocity space. This is because we know the density on the sub-manifold determined in general by constants of motion of the particle, like μ = const., more precisely we know that this is constant. And we want to carry back in the laboratory frame this result. The magnetic field B appears from the Jacobian (which is the ratio between two volume elements) of the transformation between the two systems of coordinates: canonical and laboratory. The physical meaning of TEP restricts the effect to trapped electrons [22] and expects to extend to circlating electrons and to ions through collisions.

The fact that the frozen-in invariant is also derived in TEP suggests to explore similarities for ω/ρ (for fluids) and $(\omega + \mathbf{\Omega}_{ci})/\rho$ (for plasma). The latter invariant is derived for tokamak plasma starting from basic equations [18] and in the general case from relabeling invariance [23].

3 The large scale vorticity in tokamak

Usually one encounters the vorticity as an essential field in the dynamics of the drift wave instabilities and turbulence, *i.e.* at spatial scales of the order of ρ_s (millimeter). In these cases one has $\omega = \Delta\phi$ where ϕ is the fluctuating electric potential of a drift-type instability. The vorticity that is involved in the invariant Eq.(8) is a large scale variable, resulting from the sheared rotation velocity, either poloidal or toroidal. It refers to flows on spatial scales of the order of the minor radius. The vorticity ω, which is the radial derivative of the sheared velocity profile, can be localized if there are fast variation of $v(r)$ on the minor radius, as is the case for the rotation layer in the H-mode or for flows associated to narrow Internal Transport Barriers.

The importance of the Ertel's theorem asserting the invariance in Eq.(8) is well known. The $2D$ compressibility of the ion polarization drift and the assumed Boltzmann distribution of density perturbation along the magnetic field lines lead to a single equation for the electric potential in the two-dimensional (poloidal plane) approximation of the drift waves, *i.e.* the Charney-Hasegawa-Mima equation. The invariant Eq.(10) , which can be seen as an extension of the Ertel's theorem, is particularly important in a plasma where the zonal flows, the H-mode layer of plasma rotation and the Internal Transport Barriers imply spatial variation of the magnitude of plasma velocity, *i.e.* vorticity. When an element of plasma that initially has a set of values $(v_\theta, B_\theta, n)^{(0)}$ is driven by the irregular motions of drift turbulence in a different region, where a different magnitude v'_θ is imposed by the plasma rotation, the magnetic field (or, profile of current density) and/or density must change, to keep J_z invariant.

Physical sources of generation of poloidal and toroidal rotation are Reynolds stress (sustained by turbulence), Stringer mechanism, loss of ions to the limiter, neoclassical intrinsic ambipolarity and, in addition, injection of torque from external sources (NBI, ICRH) or from creation of alpha particles by fusion reactions. Since all have rates with spatial variation, which is reflected in spatial variation of the velocity, the presence of vorticity is ubiquituous. Therefore the invariance Eq.(10) should be examined in the context of the mentioned sources of vorticity.

In the present work we include another source of torque (and rotation), arising from an equally ubiquituous phenomenon in the tokamak plasma, the ionization of neutral atoms [24].

Every event of ionization of a neutral particle in tokamak plasma it followed by the displacement of the newly born charges (electron and ion) towards the equilibrium neoclassical orbits. The motion of the "new" ion on either circulating and trapped (banana) orbit is asymptotically periodic. However there is the first interval, just after the ionization, when the ion evolves to take the periodic trajectory, which is an intrinsically non-periodic transient. This displacement is an effective radial current. At the end of this finite, transitory part, the motion becomes periodic and there is no radial current. The short and unique transient radial current, multiplied by the rate of ionization, can produce a significat torque and plasma rotation, thus influencing the confinement. We introduce the rate of ionization $\dot{n}^{(0)} S(x,t)$ where the first factor is an average magnitude ($ions/m^3/s$) and $S(x,t)$ is a function of order unity that represents the space inhomogeneity and time non-uniformity of the rate of ionization. The total current traversing the point x at time t is obtained by summing over all the individually short currents

$$\begin{aligned} j(x,t) &= \dot{n}^{(0)} \int_0^t dt_0 \int_0^x dx_0 \; S(x_0,t_0) \\ &\quad \times \Theta(t_\Delta - t)\,\Theta(t-t_0) \times |e|\, v_D \delta(x - v_D t) \times \Theta(x - x_0) \end{aligned} \tag{14}$$

Here (x_0, t_0) are the place and time of creation of a new ion, v_D is the neoclassical drift velocity and t_Δ is the time of transit of the ion from its creation to the "center" of the periodic orbit, *i.e.* the time that there is an electric current. The distance on which the elementary contribution is non-zero is $\Delta = v_D t_\Delta$. The Heaviside functions Θ underline the limited time of existence of a single contribution to the current.

The interval of existence of the current of a single ion is temporarly short and is spatially small. Then we can expand $S(x_0, t_0)$ and after integration over the nearby contributions we obtain the expression of the radial current

$$j(x,t) \approx -\frac{1}{2} |e| \dot{n}_0^{ioniz} \left(\frac{\partial S}{\partial x} \right) \rho_i^2 q^2 \varepsilon^{-1/2} \tag{15}$$

The details of this calculation are presented elsewhere ([24]). This radial current interacts with the magnetic field producing a torque that acts on the new ions. Since rough estimations show that, in some particular situations, the magnitude of the torque is comparable or higher with the Transit Time Magnetic Damping we will focus on the poloidal rotation, which has a strong effect on the turbulence.

4 A contribution to the reversed q profile

The pellet enhanced performance (PEP) experiments [25], [26] have shown the formation of strong pressure gradients that drive a bootstrap current peaked off-axis. The bootstrap current depends essentially on collisions but within the known limits: electron-electron collisions are necessary for the current of trapped electrons to be transferred to circulating electrons, and electron-ion collisions saturate the bootstrap current by momentum transfer to the ions [27]. At the high temperature ($\sim 5\ KeV$) collisions are not frequent, $\nu_{ee} \sim n/T_e^{3/2}$. It also depends on the presence of trapped particles [28] which, close to the center, are less numerous $\sim \sqrt{r/R}$. We note in the following the the SMTY invariant suggests the existence of an additional contribution that can enhance $j(r,t)$ beyond the bootstrap current. It comes from any local creation of plasma sheared rotation, *i.e.* from the creation of a local vorticity, $\omega = \partial v_\theta / \partial r$. Several sources are possible, with different degrees of effectiveness. The ICRH modifies the width of the trapped ion banana orbits and the implicit radial excursion of the virtual center of the bouncing ion is a radial current. This produces a sheared rotation [29]. In [24] we draw attention to an unexplored mechanism of generating rotation, through the radial current resulting from transitory events when a particle changes from trapped to circulating or inversely. Changing between two periodic motions, the virtual center of the averaged position in one state is displaced to correspond to the other state, which means a transitory radial displacement of a charge, therefore a radial current. ICRH and also ECRH will show this effect and must be considered sources of ω. Finally, as mentioned before, the ionization of a pellet generates a radial current that can be substantial. None of these sources of ω should be ignored but in the following we refer in principal to the ionization-induced rotation.

Returning to Eq.(10) we will assume that the two-dimensional reduction of $\mathbf{J}$, *i.e.* transversal on the poloidal plane, maintains the invariance. In this case, the invariance of J_z appears as an extension of the Ertel's theorem which yields the Hasegawa Mima equation [18].

The magnetic field is

$$B \approx \frac{B_0}{R} + \frac{1}{2}\frac{B_0}{R}\frac{\varepsilon^2}{q^2} + ... \tag{16}$$

Then the quantity normalized to Ω_c

$$J_z = \frac{\frac{\omega}{\Omega_{ci}} + \frac{\varepsilon^2}{2q^2} + 2}{n} \tag{17}$$

is maintained by turbulence as an approximative constant over the area where one can assume ergodicity.

We consider a case where a pellet reaches a deep region in the plasma and the ionizations produce a significant radial current. Then the plasma response, consisting of the return current generates a poloidal rotation with radial variation, which means with vorticity $\omega = \partial v_\theta / \partial r \sim \partial^2 S / \partial r^2$. The density has a slower response and we neglect its radial nonuniformity. It results

$$\frac{-|\omega|}{\Omega_{ci}} + \frac{\varepsilon^2}{2q^2} \sim \text{const} \times n - 2 \tag{18}$$

When $|\omega|$ increases on a zone which is off-axis q must correspondingly be reduced in order the two terms (of opposite sign) to compensate their change and the sum to remain approximately constant. This means that q will aquire a minimum in that region. For a pellet, the radial variation of the ionization $S(r,t)$ profile can produce $\omega \sim 10^6$ (s^{-1}) [24]. It suggests that a sudden onset of sheared rotation on a radial interval will produce a contribution that can lead to a reversed-q profile. The condition is that ω is opposite to the magnetic field. The neglect of the changes of the density can only be justified by the fast processes related to ionization, the slow changes of $n(r)$ having been noticed in experiments [30].

We note again that only high variations of $|\omega|$ can have a visible effect, due to the high "vorticity background" Ω_{ci}.

5 The density of the current in the edge H-mode layer

In many cases the L to H-mode transition is characterized by the fast formation of a narrow layer of strongly sheared poloidal plasma rotation [31]. It has also been found that the density that invades the border through the X point is ionized and partially trapped producing a strong torque [32]. The torque is enhanced by the loss of some of the trapped ions. The fast generation of high ω suggests to study the possible consequences imposed by the SMTY invariant

$$\frac{\pm|\omega|}{\Omega_{ci}} + \frac{\varepsilon^2}{q^2} \sim \text{const} \times n - 2 \tag{19}$$

We have a large increase in *magnitude* of the vorticity $|\omega|$ and in consequence q must decrease. In general, at a strong modification of ω there follows a strong modification of q.

$$\frac{1}{q(r)} = \frac{R}{rB_T} B_\theta = \frac{R}{rB_T}\frac{\mu_0}{2\pi r}\int_0^r j(r)\, 2\pi r dr \tag{20}$$

We divide the radial interval $[a-\delta, a]$ at the edge, *i.e.* a poloidal shell, and write

$$\begin{aligned} \int_0^a j(r)\, 2\pi r dr &\approx \int_0^{a-\delta} j(r)\, 2\pi r dr + \int_{a-\delta}^a j(r \approx a)\, 2\pi r dr \\ &\approx I_1 + j(r\approx a)\, 2\pi a \delta \end{aligned} \tag{21}$$

Here we have introduced the notation

$$I_1 = \int_0^{a-\delta} j(r)\, 2\pi r dr \tag{22}$$

and we note that, even if the current in the poloidal shell $[a-\delta, a]$ is significant, the current on the surface $[0, a-\delta]$ is the main part of the current and actually I_1 is not too much different than the total current I. It is, of course, smaller, $I_1 \lesssim I$. With this notation we have

$$\frac{1}{q} = \frac{1}{\overline{q}(a)} + \frac{1}{\overline{q}(a)}\frac{2\pi a\delta}{I_1} j(r\approx a) \tag{23}$$

where

$$\overline{q}(a) \equiv \left(\frac{R}{aB_T}\frac{\mu_0}{2\pi a} I_1\right)^{-1} \tag{24}$$

is approximately the safety factor at the edge. It results

$$\frac{1}{q^2} = \frac{1}{\left[\overline{q}\left(a\right)\right]^2} \left[1 + \frac{2\pi a\delta}{I_1} j_a\right]^2$$

We introduce a notation j_I which is the density in the narrow shell $[a-\delta, a]$ in poloidal plane

$$j_I \equiv \frac{I_1}{2\pi a\delta} \tag{25}$$

This is very large since the term looks like, formally, the full current I would be concentrated in the layer $2\pi a\delta$. Then we can expand

$$\Delta\left(\frac{1}{q^2}\right) \equiv \frac{1}{q^2} - \frac{1}{\left[\overline{q}\left(a\right)\right]^2} = 2\frac{1}{\left[\overline{q}\left(a\right)\right]^2}\frac{j_a}{j_I} \tag{26}$$

or

$$j_a \approx j_I \left[\overline{q}\left(a\right)\right]^2 \frac{1}{2}\Delta\left(\frac{1}{q^2}\right) \tag{27}$$

On the other hand we have the equation derived from the invariant

$$\Delta\left(\frac{1}{q^2}\right) \approx \frac{\Delta\omega}{\Omega_{ci}\varepsilon^2} \tag{28}$$

assuming that the density has a slower reaction to the rotation.

$$j_a \approx j_I \frac{\frac{1}{2}\Delta\left(\frac{1}{q^2}\right)}{\left(\frac{1}{\left[\overline{q}(a)\right]^2}\right)} = j_I \frac{1}{2}\frac{\Delta\omega}{\Omega_{ci}\varepsilon^2}\left[\overline{q}\left(a\right)\right]^2 \tag{29}$$

For an order of magnitude-estimate we take a variation of the vorticity $\Delta\omega \sim \delta v_\theta/\delta r \sim 10\times 10^3\ /0.01\ \left(V/Tm^2\right) = 10^6\ \left(s^{-1}\right)$ and $\Omega_{ci}\varepsilon^2 \sim 10^8 \times 0.1 = 10^7$. Then, for $\overline{q}\left(a\right) \approx q\left(r \sim a\right)$ such that $\left[\overline{q}\left(a\right)\right]^2 \approx 20$, we have

$$\frac{1}{2}\frac{\Delta\omega}{\Omega_{ci}\varepsilon^2}\left[\overline{q}\left(a\right)\right]^2 \sim 1 \tag{30}$$

and from Eq.(29)

$$j_a \sim j_I = \frac{I_1}{2\pi a\delta} \tag{31}$$

which is very large, a factor of $a/\delta \sim 100$ larger than the average density of the current, $\overline{j} = \frac{I}{2\pi a^2}$, since $I_1 \approx I$. This suggests that in the narrow layer at the edge of the tokamak, at the onset of the H mode regime where there is a strong sheared poloidal rotation (*i.e.* high vorticity), there is an accumulation of the current density. This appears to be compatible with the experimental observations (see Fig.9 of [33]).

This contribution to the current density in the layer is very large and depends on the vorticity $\omega = \partial v_\theta / \partial r$. This is not unexpected since $\partial \omega / \partial t = (B/\rho) \nabla_\| j_\|$ is the dynamics in the phase of the rise of the laminar sheared flow in the H-mode layer [34]. This current is not transitory and persists as long as there is no substantial change of the vorticity ω. We note that it is not directly connected with the *bootstrap* current since the latter is generated by the gradients in the pedestal. The fact that at the edge in H - mode there is a vorticity-current layer have several consequences: the Resonant Magnetic Perturbations (RMP) have difficulties to penetrate the shell of current; the edge localized modes (ELM) should have a "tearing-mode" component [35].

6 Conclusions

This paper examines a connection between the vorticity, the current density profile and the density (ω, q, n) as suggested by a frozen-in MHD invariant.

We have first briefly contrasted two classes of invariants useful in the study of the plasma (and fluids): the topological invariants, which are non-local; and Lagrangian and frozen-in invariants, which exist as local constrains to the plasma dynamics. The latter can be very useful in applications, as are the conservation of energy or angular momentum. We have noted two difficulties, which still wait for solutions: (1) the dissipation renders the invariance approximative; (2) and, the conversion from Lagrangian to Eulerian requires certain characteristics of the turbulence, essentially the ergodicity that makes that an element of plasma carrying its "charge" (Lagrangian invariant) can come close to any other element of plasma, in a finite region. This problem is known in the evolution to coherent asymptotic states of the $2D$ Euler fluid [36].

We have mentioned the derivation of Lagrangian and frozen-in invariants, to underline the particular distinction between the approach based on MHD equations and respectively the turbulent equipartition. We have examined the frozen-in MHD invariant derived by Sagdeev, Moiseev, Tur and Yanovskii. The version for ions is particularly useful for the effects of changes of the plasma sheared velocity. Especially the fast changes, like the onset of sheard plasma rotation (vorticity) must be accompanied by a change in the other plasma variables: safety factor q and density n. We have mentioned that the connection between (ω, q, n) imposed by the SMTY invariant suggests to consider all sources of sheared rotation (in particular those able to form internal transport barrier) as connected with the q-profile.

ICRH, ECRH become possible sources of changes on the profile of the current density. Conversely, LHCD, by modifying the current profile, can induce transient sheared rotation of plasma in a finite region. A very high vorticity concentrated around the magnetic axis can expell the current density from the center producing a current hole.

One of the objectives was to look for consequences of the invariance when vorticity is changed by the ionization-induced plasma rotation (at pellet fuelling, or gas-puff or impurity seeding). We have illustrated this by two examples: an effect contributing to the reversed-q profile in the plasma core and the effect of accumulation of density of current in the H-mode rotation layer at the edge.

Previous experimental observations that appear to be compatible with the connection expressed by this invariant must be re-examined: pellet enhanced performance, reversed shear (and enhanced reversed shear), current hole, etc. Numerical work will certainly be very helpful.

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