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# STATISTICAL ANALYSIS OF THE LINKING NUMBER IN STOCHASTIC MAGNETIC FIELDS*

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*Abstract*. The Entanglement among magnetic lines in space is specified by topological invariants. The ideal motions (continuous deformations) that do not change the structure by breaking any line are homeomorphisms which preserve the topological properties. In a stochastic field at nonzero resistivity reconnections are possible and the topological characteristics form a statistical ensemble of realizations. The Gauss linking number for two entangled closed lines is the number of times (affected by a sign) that one chain penetrates a surface spanned by the other. We calculate the statistical properties of the Gauss linking number. For this we adopt a functional method also used for random polymers.

*Key words*: stochasticity, magnetic field lines, Gauss linking, functional integral.

## 1. INTRODUCTION

The stochastic component of the magnetic field plays a major role in astrophysical plasma and is also one of the most efficient channels of energy and particle transport in the magnetically confined plasmas. In the latter case the stochasticity actually has an ambiguous role: in both Ohmic and H-mode (high-confinement mode) without Edge Localized Modes (ELM) the losses in case of a stochastic magnetic perturbation are high and require substantial input of energy to keep the regime of reactor [1]. Strangely enough, in the regime dominated by the ELMs it is desirable to produce a zone of magnetic stochasticity since this will prevent the formation of the filaments of ELMs thus allowing a smooth loss of energy mainly by radiation, instead of a thermal load on the first wall. This is made by the Resonant Magnetic Perturbation (RMP) coils [2]. A similar concept,

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"Dynamic Ergodic Divertor" [3] has been studied as a possible solution for the region where the unperturbed confining magnetic field has a $X$ point configuration and the fluxes along the magnetic lines are very high.

The stochastic fluctuations of the magnetic field lines in the magnetically confined plasma (Tokamak) have a severe impact on the quality of the energy and particle confinement. Even for small perturbations $\beta^2 \sim 10^{-4}$ (where $\beta$ is the amplitude of the transversal random component normalized to the confinement field $B_0$) the very high parallel speed of the electrons and ions generates large dispersions of the orbits, which can be characterized by the "mean square displacement" $\left\langle r_\perp (z) r_\perp (z') \right\rangle$. The directions: perpendicular $(\perp)$ and parallel $(\sim z)$ are relative to the main magnetic field. Since the position of the particle is the Lagrangian argument of the fluctuation, the strong nonlinearity generates a wide spectrum of regimes. This complex behavior is systematically reviewed in Ref. [4]. In addition, using the "Decorrelation Trajectory method" [5, 6] it has been possible to study the effects of trapping, sheared flows, etc. on the "running diffusion coefficients". An alternative approach has been developed, based on the path-integral formalism [7, 8]. The dispersion of the particle's trajectories have been calculated from the generating functional of the correlations which has been explicitly calculated using the Martin-Siggia-Rose (MSR) action functional for both collisional and magnetic random perturbations. Various related aspects are discussed in Ref. [9]. This approach belongs to the more general class of methods based on functional formalism and allows to consider the random trapping of the particles [10] and further extends to field theoretical formulations of plasma and fluid problems [11].

The origin of magnetic fluctuations is the random current filamentation in the drift - tearing instability. The studies aiming at finding the transport rates must combine the effects of collisionality, background turbulence and large scale flows on the particle motion with the statistical properties of the random magnetic field. In the present work we place less emphasis on particle orbits and focus on the topological properties of the stochastic magnetic field lines. This allows to derive averages of ideal invariants like the "link number" in view of future investigation of their relevance for the transport rates. For this objective the path-integral formalism [7, 8] is adequate. For topological applications the method must be adapted and we are guided by the similar studies in the theory of self-avoiding random walks, in particular to their application to polymers [12, 13].

## 2. THE ANALYTICAL FRAMEWORK

### 2.1. THE STOCHASTIC LINES AS RANDOM WALKS

In the basic approximation on the spatial dispersion of magnetic field lines, *i.e.* diffusive, the line results from random scattering represented as white noise in a Langevin equation, $\dot{x} = \eta(t)$ [7]. The generating functional of correlations is calculated explicitly since the Euler-Lagrange equations derived for the MSR action are very simple (Eqs. (3.11–12) of Ref. [7]). Then probability for a magnetic line which goes through $\mathbf{r}_0$ at time $t = 0$ to go through the point $\mathbf{r}$ at $t$ results

$$G_0\left(\mathbf{r}-\mathbf{r}_0,t\right)=\left(2\pi lL\right)^{-3/2}\exp\left[-\frac{\left(\mathbf{r}-\mathbf{r}_0\right)^2}{2lL}\right], \tag{1}$$

where $l \equiv a/3$, with $a$ a typical spatial length and $L \equiv$ the length of the line between $\mathbf{r}_0$ and $\mathbf{r}(t)$. This form is convenient in view of further use of technics from the theory of polymers. $G_0$ is the Green function for the problem

$$\left(\frac{\partial}{\partial L}-\mathsf{H}\left(-\nabla,\mathbf{r}\right)\right)G_0\left(\mathbf{r}-\mathbf{r}_0,t\right)=\delta\left(L\right)\delta\left(\mathbf{r}-\mathbf{r}_0\right). \tag{2}$$

The solution is expressed as a functional integral

$$G\left(\mathbf{r},L;\mathbf{r}_0,0\right)=\int_{\mathbf{r}(0)=\mathbf{r}_0}^{\mathbf{r}(L)=\mathbf{r}}\mathsf{D}\left[\mathbf{r}\left(s\right)\right]\exp\left\{-\int_0^L \mathrm{d}s\,\mathsf{L}\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t},\mathbf{r}\right)\right\}. \tag{3}$$

Here $\mathsf{L}\left(\frac{\mathrm{d}r}{\mathrm{d}t},\mathbf{r}\right)$ is a Lagrangian density from which the Hamiltonian is obtained after a Legendre transformation $\mathsf{H}\left(\mathbf{p},\mathbf{r}\right)=\frac{\partial \mathrm{L}}{\partial\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)}\times\mathbf{r}-\mathsf{L}$. Now we have to specify the expression for the Lagrangian density. In the path-integral formalism, the functional integration must penalize states that depart substantially from the equilibrium state, the unperturbed straight line. A similar case is the renormalization of the propagator in velocity space [14] which is translated in the language of path-integral in [15]. Under the same reasoning as in [15] but applied to real space instead of velocity space we find $\mathsf{L}_0$:$(\mathrm{d}\mathbf{r}/\mathrm{d}t)^2$. This is precisely the **Edwards** model for a *free random walk* applied in the case of polymers

$$\mathsf{L}_0=\frac{3}{2a}\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)^2. \tag{4}$$

Actually this similarity has suggested to employ polymer-type technics to the stochastic magnetic field. In Eq. (4) $a$ is the step length (the length of a bead of a polymer, or a straight part of a random line between two scattering, etc.). The basic length is $l \equiv a/3$ [16]. The fact that the Dupree-type renormalization has led in the path-integral formalism (applied to real space) to the Lagrangian which is the same as for the polymers strongly support the idea that the two cases can have common analytical developments. With this Lagrangian the Green function becomes Eq. (1). Since a magnetic field tube cannot have self - intersections we have to follow the similarity with the polymers with self-avoiding property.

### 2.2. THE GREEN FUNCTION FOR AN ENSEMBLE OF RANDOM LINES IN SPACE (CONSTRAINED BY THE LINE-TENSION AND UNDER RANDOM SCATTERING)

The case of random walk but with the restriction of *self-avoidance* (as is the case of magnetic flux tubes) requires to introduce a repulsive potential for two points of the trajectory. The Green function becomes

$$G(\mathbf{r},L;\mathbf{r}_0,0)=\int_{\mathbf{r}(0)=\mathbf{r}_0}^{\mathbf{r}(L)=\mathbf{r}}\mathbf{D}\left[\mathbf{r}(s)\right]\exp\left\{-\int_0^L \mathrm{d}s\mathbf{L}_0-\frac{1}{2a}\int\mathrm{d}s\int\mathrm{d}s' V\left[\mathbf{r}(s)-\mathbf{r}(s')\right]\right\}. \quad (5)$$

One easily recognizes that the double integration at the exponent can also result as the first cumulant after the averaging of an exponential of a random field. This suggests to introduce instead of the repulsive potential $V$ a noise specified by the property that the correlation in two points is equal with the potential in the same two points

$$\left\langle \varphi(\mathbf{r})\varphi(\mathbf{r}')\right\rangle = \frac{1}{a^2}V(\mathbf{r}-\mathbf{r}'). \quad (6)$$

Details for this procedure can be found in [17]. Inserting the stochastic field equal $\mathrm{i}\varphi$ the Green function is written

$$G(\mathbf{r},L;\mathbf{r}_0,0)=\left\langle \int_{\mathbf{r}(0)=\mathbf{r}_0}^{\mathbf{r}(L)=\mathbf{r}}\mathbf{D}\left[\mathbf{r}(s)\right]\exp\left\{-\int_0^L \mathrm{d}s\left[\mathbf{L}_0+\mathrm{i}\varphi\left(\mathrm{r}(s)\right)\right]\right\}\right\rangle_\varphi . \quad (7)$$

A magnetic field line in a turbulent environment is mapped onto a self-avoiding random walk (SAW) which is a random walk under a fluctuating *imaginary* potential.

## 3. THE GAUSS TOPOLOGICAL INVARIANT (LINKING NUMBER) OF MAGNETIC LINES

### 3.1. FUNCTIONAL INTEGRAL REPRESENTATION OF THE TOPOLOGICAL INVARIANTS

The topological invariants have constant value for a fixed *topological* configuration (*i.e.* modulo homotopic deformations of the trajectory). They are functionals of the magnetic lines, with fixed value: $T_\mu\left[\{\mathbf{r}\}\right]=m_\mu$ for a given topology of the field and in the absence of dissipative mechanism that would allow reconnections. There is a class of magnetic lines that have all the same topology relative to a reference axis, for example the same Gauss linking number. This is explored by inserting a functional Dirac $\delta$ in the functional integral which will restrict the space of function to the subset that has a fixed value of the Gauss linking number, $m$. The formalism applies to the calculation of the probability of a functional of a particular configuration within the statistical ensemble, and in particular, for polymers [12, 13]. The Green function becomes

$$\begin{aligned}G_{\{m\}}\left(\mathbf{r},L;\mathbf{r}_0,0\right)&=\left\langle\int_{\mathbf{r}(0)=\mathbf{r}_0}^{\mathbf{r}(L)=\mathbf{r}}\mathbf{D}\left[\mathbf{r}(s)\right]\delta\left(T\left[\{\mathbf{r}\}\right]-m\right)\exp\left\{-\int_0^L\mathrm{d}s\,\mathbf{L}_\varphi\right\}\right\rangle_\varphi=\\&=\int\frac{\mathrm{d}\lambda}{2\pi}\exp\left(-\mathrm{i}\lambda m\right)\left\langle G_{\{\lambda\}}\left(\mathbf{r},L;\mathbf{r}_0,0\,|\,\varphi\right)\right\rangle_\varphi .\end{aligned}\tag{8}$$

The integrals are ordinary integrals over the real variables $\lambda$ coming from the Fourier representation of the $\delta$ functions. The new Green function averaged over $\varphi$ is

$$G_{\{\lambda\}}\left(\mathbf{r},L;\mathbf{r}_0,0\,|\,\varphi\right)\equiv\int_{\mathbf{r}(0)=\mathbf{r}_0}^{\mathbf{r}(L)=\mathbf{r}}\mathbf{D}\left[\mathbf{r}(s)\right]\exp\left\{-\int_0^L\mathrm{d}s\,\mathbf{L}_\varphi+\mathrm{i}\lambda T\left[\{\mathbf{r}\}\right]\right\},\tag{9}$$

where, as before $\mathbf{L}_\varphi\equiv\mathbf{L}_0+\mathrm{i}\varphi$. The simplest topological invariant, the Gauss linking number of two entangled curves is calculated by the double integral

$$I\left(\gamma_1,\gamma_2\right)\equiv\frac{1}{4\pi}\oint_{\gamma_1}\mathrm{d}s_1\oint_{\gamma_2}\mathrm{d}s_2\left(\frac{\mathrm{d}\mathbf{r}_1}{\mathrm{d}s_1}\times\frac{\mathrm{d}\mathbf{r}_2}{\mathrm{d}s_2}\right)\cdot\frac{\mathbf{r}_1-\mathbf{r}_2}{\left|\mathbf{r}_1-\mathbf{r}_2\right|^3}.\tag{10}$$

We note that $\dfrac{\mathbf{r}_1-\mathbf{r}_2}{\left|\mathbf{r}_1-\mathbf{r}_2\right|^3}=\nabla\left(\dfrac{1}{\left|\mathbf{r}_1-\mathbf{r}_2\right|}\right)=\nabla\dfrac{1}{r_{12}}$. According to Edwards [16] the Green function for one trajectory is reducible to the Green function of a charged particle moving in the field produced by the other trajectory seen as an electric current that produces a magnetic field. The trajectory is formally a line of "current"

$$\mathbf{j}_2(\mathbf{r}) \equiv \oint_{\gamma_2} \mathrm{d}s_2 \left(\frac{\mathrm{d}\mathbf{r}_2}{\mathrm{d}s_2}\right) \delta\left(\mathbf{r}-\mathbf{r}_2(s_2)\right), \tag{11}$$

and the magnetic field produced by this current is

$$\mathbf{B}(\mathbf{r}) = \frac{1}{4\pi} \oint \mathrm{d}s_2 \left(\frac{\mathrm{d}\mathbf{r}_2}{\mathrm{d}s_2}\right) \times \frac{\mathbf{r}-\mathbf{r}_2(s_2)}{\left|\mathbf{r}-\mathbf{r}_2(s_2)\right|^3}, \tag{12}$$

with $\nabla \cdot \mathbf{B} = 0$ , $\nabla \times \mathbf{B} = \mathbf{j}_2$. It is possible to introduce the magnetic potential for this magnetic field

$$\mathbf{A}(\mathbf{r}) = \frac{1}{4\pi} \int \mathrm{d}^3 r' \frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|} \mathbf{j}_2(\mathbf{r}') = \frac{1}{4\pi} \oint \mathrm{d}s_2 \left(\frac{\mathrm{d}\mathbf{r}_2}{\mathrm{d}s_2}\right) \frac{1}{\left|\mathbf{r}-\mathbf{r}_2(s_2)\right|}, \tag{13}$$

with the relations: $\nabla \cdot \mathbf{A} = 0$, $\nabla \times \mathbf{A} = \mathbf{B}$, $\nabla^2 \mathbf{A} = -\mathbf{j}_2$. Then the Gauss linking number is obtained as the integral over the volume $I(\gamma_1,\ \gamma_2) = \int \mathrm{d}^3 r\, \mathbf{j}_1(\mathbf{r}) \cdot \mathbf{B}(\mathbf{r})$. This can be reformulated as the integral over a surface spanned by the trajectory $\gamma_1$:

$$I(\gamma_1,\gamma_2) = \oint \mathrm{d}s_1 \left(\frac{\mathrm{d}\mathbf{r}_1}{\mathrm{d}s_1}\right) \cdot \nabla \times \mathbf{A} = \int\int \mathrm{d}\mathbf{S}_1 \oint \mathrm{d}s_2 \left(\frac{\mathrm{d}\mathbf{r}_2}{\mathrm{d}s_2}\right) \delta\left(\mathbf{r}_1 - \mathbf{r}_2(s_2)\right). \tag{14}$$

The Gauss linking number appears as the number of times the second trajectory $\mathbf{r}_2(s_2)$ penetrates the surface $\mathbf{S}_1$ spanned by the first trajectory $\gamma_1$. This is explained in more detail in Ref. [18]. It is used the fact this second trajectory $\gamma_2$ produces a magnetic field $\mathbf{B}(\mathbf{r})$ and this formally interacts with the current flowing in the "test" $\gamma_1$ trajectory, giving

$$\begin{aligned} T\left[\{\mathbf{r}\}\right] &= I(\gamma_1,\gamma_2) = \int \mathrm{d}^3 r\, \mathbf{j}_1(\mathbf{r}) \cdot \mathbf{B}(\mathbf{r}) = \\ &= \int \mathrm{d}^3 r \left[\oint_{\gamma_2} \mathrm{d}s_2 \left(\frac{\mathrm{d}\mathbf{r}_2}{\mathrm{d}s_2}\right) \delta\left(\mathbf{r}-\mathbf{r}_2(s_2)\right)\right] \cdot \mathbf{B}(\mathbf{r}). \end{aligned} \tag{15}$$

After integrating over the volume with the $\delta$ function, we will get $\mathbf{B}$ acting on only the points of the *test* trajectory

$$T\left[\{\mathbf{r}\}\right]=\oint_{\gamma_2}\mathrm{d}s_2\left(\frac{\mathrm{d}\mathbf{r}_2}{\mathrm{d}s_2}\right)\cdot\mathbf{B}\left(\mathbf{r}\left(s_2\right)\right) \tag{16}$$

and this is introduced in the expression at the exponent

$$\begin{aligned}-\int_0^L\mathrm{d}s\mathbf{L}_\varphi+\mathrm{i}\lambda T\left[\{\mathbf{r}\}\right]&=-\int_0^L\mathrm{d}s\mathbf{L}_\varphi+\mathrm{i}\oint_{\gamma_2}\mathrm{d}s_2\left(\frac{\mathrm{d}\mathbf{r}_2}{\mathrm{d}s_2}\right)\cdot\mathbf{B}\left(\mathbf{r}\left(s_2\right)\right)=\\&=-\int_0^L\mathrm{d}s_2\left[\mathbf{L}_\varphi-\mathrm{i}\left(\frac{\mathrm{d}\mathbf{r}_2}{\mathrm{d}s_2}\right)\cdot\mathbf{B}\left(\mathbf{r}\left(s_2\right)\right)\right].\end{aligned} \tag{17}$$

Then the Green function now modified by the presence of a link constraint with the second trajectory, expressed by the presence of a “magnetic” field $\mathbf{B}$ produced by the latter becomes $G_{\{\lambda\}}\left(\mathbf{r},L;\ \mathbf{r}_0,0\,|\,\phi\right)\to G_{\{\lambda\}}\left(\mathbf{r},L;\ \mathbf{r}_0,0\,|\,\phi,\mathbf{B}\right)$

$$\begin{aligned}&G_{\{\lambda\}}\left(\mathbf{r},L;\mathbf{r}_0,0\,|\,\varphi,\mathbf{B}\right)=\\&=\int_{\mathbf{r}(0)=\mathbf{r}_0}^{\mathbf{r}(L)=\mathbf{r}}\mathbf{D}\left[\mathbf{r}\left(s\right)\right]\exp\left\{-\int_0^L\mathrm{d}s\left[\mathbf{L}_\varphi-\mathrm{i}\lambda\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}\right)\cdot\mathbf{B}\left(\mathbf{r}\left(s\right)\right)\right]\right\}.\end{aligned} \tag{18}$$

This Green function satisfies the differential equation

$$\left[\frac{\partial}{\partial L}-\frac{l}{2}\left(\nabla-\mathrm{i}\lambda\mathbf{B}\right)^2+\mathrm{i}\varphi\right]G_{\{\lambda\}}\left(\mathbf{r},L;\mathbf{r}_0,0\,|\,\varphi,\mathbf{B}\right)=\delta\left(L\right)\delta\left(\mathbf{r}-\mathbf{r}_0\right). \tag{19}$$

The Green function contains the *random* field $\varphi$ previously introduced to represent the self-avoidance constraint. The average over $\varphi$ is done in the Appendix.

### 3.2. LINK OF A STOCHASTIC MAGNETIC FIELD LINE WITH A FIXED STRAIGHT REFERENCE AXIS

The fixed, reference, line is taken as the source of the unique magnetic field $\mathbf{B}\left(\mathbf{r}\right)$ in the problem. Another line generated by a random walk can have various degrees of linking with the reference line. The Gauss linking number is an integer, however for the statistical average we approximate by taking integrals instead of sums over discrete values. The second order moment of the random linking number $m$ is then

$$\left\langle m^2 \right\rangle = \frac{1}{N} \int_{-\infty}^{\infty} \mathrm{d}m\, m^2 G_m\left(\mathbf{r},\mathbf{r};L\right) \tag{20}$$

(where $N = \int_{-\infty}^{\infty} \mathrm{d}mG$ ) and it can be obtained from the functional integral expression for the Green function $G$. For polymers the entanglement between the stochastic line and a fixed reference line (the axis $z$) has been calculated. The reference line (or straight fixed polymer) is taken to be the $z$ axis and its "magnetic" field is

$$\mathbf{B}\left(\mathbf{r}\right) = \frac{1}{2\pi}\left( \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0 \right) \tag{21}$$

and the origin of the polymer is taken on the plane $z=0$, $\mathbf{r}_0 \equiv \left(x_0, y_0, 0\right)$. The following quantity with dimensions of lengths is introduced, combining the elementary step $l$ of the random walk and the length $L$ of the line $\xi \equiv \sqrt{lL}$ . The spatial variables are scaled to $\xi$: $\mathbf{R} \equiv \mathbf{r} / \xi$ , $\mathbf{R}_0 \equiv \mathbf{r}_0 / \xi$ . It will be supposed that the line closes to itself at infinite distance relative to the region of stochasticity. Then

$$\left\langle m^2 \right\rangle = \frac{1}{2\pi^2}\left[ -\ln\left(\frac{l}{\xi}\right)\right] \exp\left(-R_0^2\right) K_0\left(R_0^2\right) \tag{8}$$

and the argument of the ln introduces $-\ln\left(\frac{l}{\sqrt{lL}}\right) = \frac{1}{2}\ln\left(\frac{L}{l}\right) = \frac{1}{2}\ln\left(N\right)$. The Bessel function gives another singularity when $\left|\mathbf{r}_0\right| \ll \xi = \sqrt{lL}$ . This is due to the fact that $\left|\mathbf{r}_0\right|$ is comparable to $l$ but much smaller than $L$. Then $\left|\mathbf{R}_0\right|$ is a very small quantity and the approximation can be used

$$K_0\left(R_0^2\right) \sim -\ln\left(R_0^2\right) \sim -\ln\left(\frac{l}{L}\right) = \ln\left(N\right). \tag{23}$$

Then, since the exponential is close to $1$,

$$\left\langle m^2 \right\rangle \sim \frac{1}{4\pi^2}\left[\ln\left(N\right)\right]^2, \tag{24}$$

where $N = L / l$ measures the number of steps along the line (polymer).

The mean square average of the linking number of a line subject to random Brownian displacements relative to a reference straight axis diverges as the square of the logarithm of the length of the line.

## 4. DISCUSSION

We would have been tempted, in place of this analysis, to use directly the known Spitzer's law, which is formulated as follows [19]. Consider a reference point in the plane $O$ (this would be the equivalent of the point where our reference straight line punctures the transversal plane), and a random walk starting at a point close to $O$. In successive positions the walk accumulates an angle $\theta(t)$ around the point $O$. Spitzer's law says that the random variable $x(t) \equiv \theta(t)/\ln(t)$ has a probability distribution function (PDF) which is Cauchy $\text{PDF}(x) = \frac{1}{1+x^2}$. For large $t$ ($\sim N$ in our case) we would have $\text{PDF} \approx \left[\frac{\ln(t)}{\theta(t)}\right]^2$ and since the angle is $2\pi \times$(winding number or Gauss link $m$), we have an integral over the statistical ensemble of realizations of the angle $\theta$, while time $t$ is just a parameter.

$$\left\langle \theta(t)^2 \right\rangle \sim \int \frac{\mathrm{d}\theta}{2\pi} \left[\theta(t)\right]^2 \left[\frac{\ln(t)}{\theta(t)}\right]^2 \sim \left[\ln(t)\right]^2 . \tag{25}$$

This is compatible with the result from the previous approach since $t \sim N$. However the Spitzer’s law results from an approach that is – in our opinion – less general than the functional approach developed above. The probability distribution function of the winding angle of a self-avoiding random walk in plane can also be calculated using Coulomb gas technique and conformal invariance [20]. However for the stochastic magnetic lines the parallel (to $\mathbf{B}_0$) excursion is an essential part of the model.

The statistical result (dispersion of the Gauss linking $\left\langle m^2 \right\rangle$ along the length of the line) refers to only one aspect of the geometry of the lines: it only refers to stochastic scattering of the trajectory and does not include the helical turns around the axis at the $O$-point of a magnetic island. This may be described in terms of stochastic filamentation of the current density $\widehat{j}$, as

$$\left\langle \left(q^{-1}\right)^2 \right\rangle^{1/2} \sim \frac{\mu_0 R}{r^2 B_0} \left\langle \left(\hat{j}\right)^2 \right\rangle^{1/2}, \tag{26}$$

where $B_0$ is the unperturbed field, $r$ is the radius of a small magnetic island (of drift tearing modes), $R$ is its curvature radius [18]. The stochasticity that has been discussed in this work arises from the random scattering of the line from one island to another. This would give the change of the topological winding around a reference axis. The statistical process of transport associated with this winding is more important than the intrinsic structure of the micro-islands of drift-tearing modes since the line excursions can be large for repetitive scatterings in the same direction, followed by returns, also random. However the number of particles that are carried along these lines is small compared with the particle and current density involved in drift-kinetic instability. Some of these aspects should be verified numerically, however events of rapid, cuasi-ballistic transport have been observed in Tokamak plasma. Besides the conventional approach based on percolation, one should consider the fluctuations of the topological properties.

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## APPENDIX: CALCULATION OF THE AVERAGE OVER THE IMAGINARY FIELD USING THE REPLICA METHOD

This calculation is standard procedure that works for our structure of functional "action". It is introduced here for consistency of the analytical method [12, 13].

It was assumed that the line (or the polymer) has a finite length $L$. If the functions of $L$ would be represented in Fourier space then the components should combine such as to cancel any contribution from $L < 0$. It is more natural to use Laplace transformation on the variable $L$ of the differential equation for $G_{\{\lambda\}}\left(\mathbf{r}, L; \mathbf{r}_0, 0 \mid \varphi, \mathbf{B}\right)$ The function itself becomes

$$G_{\{\lambda\}}\left(\mathbf{r}, \mathbf{r}_0; z \mid \varphi, \mathbf{B}\right) = \int_0^\infty \mathrm{d}L \exp\left\{-zL\right\} G_{\{\lambda\}}\left(\mathbf{r}, L; \mathbf{r}_0, 0 \mid \varphi, \mathbf{B}\right). \tag{27}$$

The equation (eq. 28) becomes $\left(z - H\right) G_{\{\lambda\}}\left(\mathbf{r}, \mathbf{r}_0; z \mid \varphi, \mathbf{B}\right) = \delta\left(\mathbf{r} - \mathbf{r}_0\right)$ with the Hamiltonian $H \equiv \frac{l}{2}\left(\nabla - \mathrm{i}\lambda\mathbf{B}\right)^2 - \mathrm{i}\phi$. The solution is then represented as (suppressing other indices) the matrix element of the inverse of the operator $z - H$, *i.e.* the resolvent

$$G(\mathbf{r},\mathbf{r}_0)=\left(\mathbf{r}\,|\,\frac{1}{z-H}\,|\,\mathbf{r}_0\right). \tag{28}$$

Using the fact that the inverse of a determinant of an operator can be represented as a functional integration over a space of functions $\psi$ , $\psi^*$ with the measure given by the exponential of the integral of the inverse of the operator, we write

$$\begin{aligned}G(\mathbf{r},\mathbf{r}_0)=\frac{1}{Z}\int\mathsf{D}[\psi]\,\mathsf{D}\left[\psi^*\right]\psi(\mathbf{r})\psi^*(\mathbf{r}_0)\times\\ \times\exp\left\{-\int\mathrm{d}\mathbf{r}'\int\mathrm{d}\mathbf{r}''\psi^*\left(\mathbf{r}''\right)\left(\mathbf{r}''\,|\,z-H\,|\,\mathbf{r}'\right)\psi\left(\mathbf{r}'\right)\right\}.\end{aligned} \tag{29}$$

The normalization factor is the same functional integral without the integrand of the product of the two functions $\psi$

$$Z\equiv\int\mathsf{D}[\psi]\,\mathsf{D}\left[\psi^*\right]\exp\left\{-\int\mathrm{d}\mathbf{r}'\int\mathrm{d}\mathbf{r}''\psi^*\left(\mathbf{r}''\right)\left(\mathbf{r}''\,|\,z-H\,|\,\mathbf{r}'\right)\psi\left(\mathbf{r}'\right)\right\}. \tag{30}$$

The exponent is noted $F$ and we take into account that the operator $z-H$ does not depend on two variables $\mathbf{r}''$ and $\mathbf{r}'$. Then the matrix element $\left(\mathbf{r}\,|\,\mathbf{r}_0\right)=\delta\left(\mathbf{r}''-\mathbf{r}'\right)$

$$\begin{aligned}F[\psi]&\equiv\int\mathrm{d}\mathbf{r}'\int\mathrm{d}\mathbf{r}''\psi^*\left(\mathbf{r}''\right)\left(\mathbf{r}''\,|\,z-H\,|\,\mathbf{r}'\right)\psi\left(\mathbf{r}'\right)=\\ &=\int\mathrm{d}\mathbf{r}'\int\mathrm{d}\mathbf{r}''\delta\left(\mathbf{r}''-\mathbf{r}'\right)\left\{\psi^*\left(\mathbf{r}''\right)(-1)\frac{l}{2}(\nabla-\mathrm{i}\lambda\mathbf{B})^2\psi\left(\mathbf{r}'\right)+\right.\\ &\left.+\psi^*\left(\mathbf{r}''\right)(z+\mathrm{i}\varphi)\psi\left(\mathbf{r}'\right)\right\}\end{aligned} \tag{31}$$

after an integration by parts

$$F[\psi]=\int\mathrm{d}\mathbf{r}\left\{\frac{l}{2}\left|(\nabla-\mathrm{i}\lambda\mathbf{B})\psi\right|^2+(z+\mathrm{i}\varphi)|\psi|^2\right\}. \tag{32}$$

In order to make the average over the random field $\varphi$ it is used the *replica method*.

We take $n$ exemplars of this system. In the system with indice $\sigma$ the fields are $\psi_\sigma$ and $\psi_\sigma^*$ and the functional is $F[\psi]\to F\left[\psi_\sigma,\psi_\sigma^*\right]$ . The functional integration is extended over $n$ spaces of functions, the space $\sigma$ being spanned by the pair $\psi_\sigma$ and $\psi_\sigma^*$ . The functional measure is a direct product of the functional measures on each space $\sigma$ . As it is defined the functional integral does not mix the variables $\psi_\sigma$ and $\psi^*_\sigma$. However it is the *averaging* over the random field $\varphi$ (which has been introduced to ensure the condition of self-avoidance of the path) that will mix the fields $\left(\psi_\sigma,\psi^*_\sigma\right)_{\sigma=1,n}$ The Green function will be obtained at the limit $n\to 0$

$$G\left(\mathbf{r},\mathbf{r}_0\right)=\lim_{n\to 0}\int\prod_{\sigma=1}^{n}\mathsf{D}\left[\psi_\sigma\right]\mathsf{D}\left[\psi^*_\sigma\right]\times\psi_\alpha\left(\mathbf{r}\right)\psi^*_\alpha\left(\mathbf{r}_0\right)\times\prod_{\sigma=1}^{n}\exp\left\{-F\left[\psi_\sigma,\psi^*_\sigma\right]\right\}. \quad (33)$$

After making the average over $\varphi$ in the exponential, it results

$$\left\langle\prod_{\sigma=1}^{n}\exp\left\{-F\left[\psi_\sigma,\psi^*_\sigma\right]\right\}\right\rangle_\varphi=\exp\left\{-F\left[\Psi\right]\right\}, \quad (34)$$

where $\Psi\equiv\left(\psi_1,\psi_2,...,\psi_n\right)$ and $F\left[\Psi\right]\equiv\int\mathrm{d}^3r\left\{\frac{l}{2}\left|\left(\nabla-\mathrm{i}\lambda\mathbf{B}\right)\Psi\right|^2+z\left|\Psi\right|^2+v_0\left|\Psi\right|^4\right\}$.

The last term comes from the potential of self-repulsion $V\left(\mathbf{r}-\mathbf{r}'\right)$ which has been simplified to a $\delta$ function $v\left(\mathbf{r}\right)=a^2v_0\delta\left(\mathbf{r}\right)$ . The average over $\varphi$ is

$$\left\langle\exp\left(-\int\mathrm{d}\mathbf{r}\ \mathrm{i}\varphi\left|\psi\right|^2\right)\right\rangle_\varphi=\exp\left(-\int\mathrm{d}\mathbf{r}\int\mathrm{d}\mathbf{r}'\left\langle\varphi\left(\mathbf{r}\right)\varphi\left(\mathbf{r}'\right)\right\rangle\left|\Psi\left(\mathbf{r}\right)\right|^2\left|\Psi\left(\mathbf{r}'\right)\right|^2\right) \quad (35)$$

and taking into account that $\varphi\left(\mathbf{r}\right)$ has $\delta$ spatial correlation, one obtains $\exp\left(-v_0\int\mathrm{d}\mathbf{r}\left|\Psi\left(\mathbf{r}\right)\right|^4\right)$ which is to be added to the rest of the exponential of the functional $F$ . Now the operator $\left|\left(\nabla-\mathrm{i}\lambda\mathbf{B}\right)\Psi\right|^2$ is expanded. By analogy with the quantum current it is introduced

$$\mathbf{j}\left(\mathbf{r}\right)\equiv\frac{l}{2\mathrm{i}}\left(\Psi^*\nabla\Psi-\Psi\nabla\Psi^*\right)=\sum_{\sigma=1}^{n}\frac{l}{2\mathrm{i}}\left(\psi^*_\sigma\nabla\psi_\sigma-\psi_\sigma\nabla\psi^*_\sigma\right). \quad (36)$$

The functional $F$ can be written as

$$F\left[\Psi\right]=F_0\left[\Psi\right]+\int\mathrm{d}^3r\ \mathbf{j}\left(\mathbf{r}\right)\cdot\lambda\mathbf{B}+\frac{l}{2}\int\mathrm{d}^3r\left(\lambda\mathbf{B}\right)^2\left|\Psi\right|^2. \quad (37)$$

The *free* part of the functional is $F_0\left[\Psi\right] \equiv \frac{l}{2}\left|\nabla\Psi\right|^2 + z\left|\Psi\right|^2 + \nu_0\left|\Psi\right|^4$. Now the functional integration variables remain $\Psi$ and $\Psi^*$ since $\mathbf{j}$ is expressed in terms of them. The expression at the exponent is quadratic in $\lambda$. The Fourier integral over $\lambda$ in Eq. (8) can be done with the result

$$\begin{aligned} G_{\{m\}}\left(\mathbf{r},\mathbf{r}_0;z\,|\,\mathbf{B}\right) = \int \mathsf{D}\left[\Psi\right]\mathsf{D}\left[\Psi^*\right]\psi_\alpha\left(\mathbf{r}\right)\psi_\alpha^*\left(\mathbf{r}_0\right)\exp\left\{-F_0\left[\Psi\right]\right\}\times \\ \times\frac{1}{\left(K\right)^{1/2}}\exp\left\{-\frac{1}{2}\left(m - \mathrm{i}\int \mathrm{d}^3 r\ \mathbf{j}\cdot\mathbf{B}\right)\left(K^{-1}\right)\left(m - \mathrm{i}\int \mathrm{d}^3 r\ \mathbf{j}\cdot\mathbf{B}\right)\right\} \end{aligned} \tag{38}$$

The scalar factor is $K \equiv l\int \mathrm{d}^3 r\left(\mathbf{B}\cdot\mathbf{B}\right)\left|\Psi\right|^2$.

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