Notes on a possible phenomenology of internal transport barriers in tokamak

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Abstract

We propose a new phenomenology of the generation of internal transport barriers, based on the exact periodic solution of the Flierl-Petvishvili equation. We examine the stability of this solution and compare the late stages of the flow with the ensemble of vortices.

Keywords: Internal Transport Barriers, Flierl-Petviashvili equation, zonal flows.

Contents

1 Introduction

The aim of these notes is to contribute to the clarification of the dynamical processes leading to the formation, confinement charactersitics and intermittent behavior of Internal Transport Barriers (ITB). We focus here on the role of the stationary periodic structure of the flow appearing in the intermediate spatial range between the highly radially elongated eddies of the Ion Temperature Gradient (ITG) instability and the Larmor - radius scale vortices described by the Hasegawa-Mima equation. In this "mesoscopic" range the structures are described by the ion dynamics equation where the scalar nonlinearity (in contrast to the polarisation drift nonlinearity) is dominating.

The current physical understanding of the formation of the transport barriers is based on the effect of the sheared poloidal plasma rotation on the ITG potential structures. The ITG potential structures are in general of large radial extension. They may persist even in the turbulent regime and are a major agent of transport since the plasma convection inside the eddies is an efficient support for temperature advection between high (small r) and low (high r) temperature regions. It is well known that this geometry of the flow generates velocity stress and that the divergence of this tensor induces a substantial contribution in the momentum balance equation for ions. When this momentum drive is sufficiently high, the plasma begins to rotate, and a space distribution of the poloidal velocity is established. The shear of the poloidal velocity has a strong effect on the turbulent structures with comparable space extension and the radial correlation length is reduced. This correspondingly reduces the transport leading to the formation of a transport barrier. A theory that explains this in quantitative terms has been proposed by Diamond and coworkers, as a "predator-pray" model, in general formalized with Fischer-type equations. It consists of the description, in terms of the density of the energy quanta, of the balance in phase space of two distinct populations, associated with the turbulent waves and respectively with the zonal flows. The model naturally leads to the self-organised state consisting of mutual control of these two physical entities, found in a permanent relative adjustement. This model is essentially a thermodynamical model, which probably remains true for a wide variety of dynamical details.

A picture of the dynamics is allways useful, even if the scaling laws (of threshold and transport rate) are in general based on thermodynamical balance and conservation properties. The later is a more general (and safer) level of description, leading to a picture where the invariance properties of the dynamical equations govern the scaling. The dynamical evolution of the system is a detailed description and its validity is strongly dependent on a clear separation of space-time scales and a correct choice of initial and boundary conditions. However, the physical content can only revealed by the detailed description.

In this work we develop a possible phenomenological model of generation of the Internal Transport Barriers in tokamak. This model relies on the role of the stationary periodic exact solution of the Flierl-Petviashvili equation as an attracting state for the ion turbulence, when the regularity of the eddies is still pronounced. We also analyse analytically and numerically the stability of this solution. Some properties are shown to be connected with the fact that this equation is closely related with other, exactly integrable models.

2 Possible phenomenology of the ITB formation

Numerical simulations show that the structure of the potential perturbation in the ion temperature gradient (ITG) instability are elongated and rather thin on the direction that is transversal to their longest dimension. The Reynolds stress arising from the broken symmetry configuration induces the rotation of the plasma. It may be useful here to recall the similar situation arising in the thermal convection associated with the Rayleigh-Benard (RB) instability. Experiments carried out by Howard and Krishnamurti in typical RB geometry at increasing Rayleigh number have revealed that the classical conduction-convection bifurcation is only the first of a series of transitions generating new system 's behaviors. The RB geometry of convective rolls is sensitive to perturbations and this leads to a symmetry breaking consisting of deformations of the rolls in direction parallel with the two plates, in one direction for the low (high temperature) plate and opposite direction for the upper (low temperature) plate. This is accompanied by a nonvanishing statistical average of the velocity stress tensor, whose nonzero divergence (Reynolds stress) is equivalent to a momentum drive. The lower part of the convective fluid is entrained in a flow in one direction (parallel to the plates) while the upper part of the convective field aquires a flow motion in the opposite direction. These flows are called "winds" and they are seen, together with the strong deformation of the rolls, on the experimental picture of the fluid section. In addition, intermittent processes consisting of rising of finite volumes of cold fluid (from the bottom region) toward the top plate and fall of warm fluid from the top region into the cold region have been observed. These events are called "plumes". One may consider from this picture only the part relating to the effect of the Reynolds stress as particularly relevant for the plasma convection in the eddies of the ITG instability.

The result of the strong Reynolds stress arising in the ITG instability is the deformation ("tilting") of the convection eddies and the generation of the corresponding winds. The convective structures are strongly elongated and tend to align with the poloidal direction, as imposed by the momentum from the Reynolds stress. At a certain moment the deformed ITG pattern of flow evolves to the solution consisting simply of parallel layers of flow, a structure which is essentially poloidally oriented and periodic along the radial direction. We have proved that this is an exact and robust solution of the ion dynamics on mesoscopic space scales, (approx. ρ_s/ε) where it is dominated by the scalar nonlinearity. Attaining this state, the plasma has practically vanishing transport in the radial direction. We have to note that it is not necessary to invoke the tearing appart of the ITG eddies and the destruction of the radial correlation via sheared flow. The solution intrinsically has a radial scale and no transport because there is no phase mismatch between particles and potential.

The flow represented by this solution may last indefinitely except for the collisional decay which however may be considered low in high temperature regimes. Furthermore, in the ideal stationary case, the Reynolds stress vanishes due to the poloidal symmetry. However there is a limitation of the persistance of this periodically layered flow and this arises from the stability against perturbations. We can suppose that this may take place in two different ways. The first of them is essentially a reversed process as that which led to the periodic layers, and consists of closing the flow lines at finite poloidal wavelength, with suppression of the zonal flow and reformation of the tilted ITG pattern. The finite radial projection leads to a sudden increase of transport. The other way is a strong qualitative change of the flow.

We have found that the stability of the periodic flow pattern is determined both by the amplitude and by the geometry of the initial perturbation. Some perturbations (like, for example, the monopolar vortices embedded inside a layer) have a very long stability time since they accomodate with the background flow by reshaping the distribution of local velocities. However, perturbations that have a high amplitude relative to the background flow and/or do not conform to the flow geometry lead to the destabilisation and eventually destruction of the flow pattern. It is important however to note that the numerical simulations (presently with limitted precision) and analytical considerations suggest that the destruction of the flow pattern does not immediately lead to an arbitrary random field: a fundamental process is the generation of small ($\sim \rho_s$) space scale vortices. We can show that the regular flow structure is replaced by a lattice of vortices which is reminescent of the exact solution of a closely related nonlinear equation. This solution evolves, due to the weak interaction between the vortices, to an ensemble of quasi-independent vortices that collide inelastically and become of various amplitudes plus a surrounding drift wave radiation. From this random field the ITG instability may regenerate the structure of eddies and reinstate the transport. The time scale for doing this is γ_{ITG}, followed by saturation, as shown by many simulations.

The successive steps of this qualitative scenario can repeate themselves over and over but the senstivity to perturbation of the time scales involved may lead to an intermittent (bursty) behavior rather than to a limit cicle. From purely theoretical point of view it is hard to expect that this evolution can be lumped into a single equation whose phase space could provide the successive transitions.

We note that the transport is low in the phase where the flow is described by the stationary periodic solution of the Flierl-Petviashvili equation and in the state where the ensemble of vortices has replaced it (since the elementary space scale is very small, of the order of few Larmor radius, as we will show below. The large eddies have a substantial transport.

3 Derivation of the equation at the intermediate space scales

3.1 The equation for the two-dimensional ion dynamics

Consider the equations for the ITG model in two-dimensions with adiabatic electrons:

$$\begin{aligned}\frac{\partial n_i}{\partial t}+\nabla\cdot\left(\mathbf{v}_i n_i\right) &= 0\\ \frac{\partial \mathbf{v}_i}{\partial t}+\left(\mathbf{v}_i\cdot\nabla\right)\mathbf{v}_i &= \frac{e}{m_i}\left(-\nabla\phi\right)+\frac{e}{m_i}\mathbf{v}_i\times\mathbf{B}\end{aligned}$$

We assume the quasineutrality

$$n_i\approx n_e$$

and the Boltzmann distribution of the electrons along the magnetic field line

$$n_e=n_0\exp\left(-\frac{|e|\,\phi}{T_e}\right)$$

In general the electron temperature can be a function of the radial variable

$$T_e\equiv T_e\left(x\right)$$

The velocity of the ion fluid is perpendicular on the magnetic field and is composed of the diamagnetic, electric and polarization drift terms

$$\begin{aligned}\mathbf{v}_i &= \mathbf{v}_{\perp i}\\ &= \mathbf{v}_{dia,i}+\mathbf{v}_E+\mathbf{v}_{pol,i}\\ &= \frac{T_i}{|e|\,B}\frac{1}{n_i}\frac{dn_i}{dr}\widehat{\mathbf{e}}_y\\ &\quad+\frac{-\nabla\phi\times\widehat{\mathbf{n}}}{B}\\ &\quad-\frac{1}{B\Omega_i}\left(\frac{\partial}{\partial t}+\left(\mathbf{v}_E\cdot\nabla_\perp\right)\right)\nabla_\perp\phi\end{aligned}$$

Introducing this velocity into the continuity equation, one obtains an equation for the electrostatic potential ϕ.

In the **Appendix** it is presented a derivation of three versions of the nonlinear differential equations that may govern, in certain conditions, the two-dimensional ion dynamics at intermediate scales.

3.2 The stationary equation at intermediate space scales (Flierl-Petviashvili equation)

The ion drift wave equation has a distinct dynamical behavior according to the space-time scales involved. The Hasegawa-Mima-Charney (HMC) equation is obtained for small scales and the stationary states exhibit dipolar structures of the order of the Larmor radius. On larger scales (intermediate between the HMC dipole and the large ITG eddies) the scalar (or KdV-type) nonlinearity is prevailing and the only known structures are monopolar [4]. Both are not solitonic but very robust and long lived. In the latter case a one-dimensional version of the equation has been derived by Petviashvili [5] who also used it in the study of the Jupiter's Red Spot. The equation has been rederived along with a careful analysis of the scales involved [6], [8], resolving a controversy on the role of the temperature gradient [9], [7]. In the study of ocean flows, Flierl [10] has independently formulated an equation with the same structure. The one dimensional equation has been solved on an infinite domain [11], [9], obtaining as solution the KdV soliton. The two dimensional equation has vortical monopolar solutions, well studied numerically [16], [17]. Other applications include long waves on thin liquid films and Rossby waves in rotating atmosphere.

The Flierl-Petviashvili equation (derived several times in the Appendix) is

$$\Delta\phi = \alpha\phi - \beta\phi^2$$

where

$$\begin{aligned} \alpha &= \frac{1}{\rho_s^2}\left(1-\frac{v_*}{u}\right) \qquad (1) \\ \beta &= \frac{T_e}{2u^2 e B_0^2 \rho_s^2}\frac{\partial}{\partial x}\left(\frac{1}{L_n}\right) = \frac{e}{2m_i u^2}\frac{\partial}{\partial x}\left(\frac{1}{L_n}\right) \end{aligned}$$

By defining a functional of the solution expressed in one-dimensional (radial) geometry and taking the extremum of the functional under the condition ofasymptotic decay, Petviashvili and Pokhotelov [13] have found the solution

$$\phi(r) = \frac{4.8\alpha}{2\beta}\left[\operatorname{sech}\left(\frac{3}{4}\sqrt{x^2+(y-ut)^2}r\right)\right]^{4/3} \qquad (2)$$

This is only an approximate solution and the reason to look for such form resides in the previous indication that the one dimensional version of this equation, with generalized nonlinearity, presents exact solutions as powers of the sech function.

3.3 The monopolar vortex and the similarity with integrable equations

It is well known that the Flierl-Petviashvili equation has long been studied in relation with the possibility that it has as solution an isolated vortex with nontrivial stability properties. This vortex has never been determined analytically and the explanation seems to be that the equation is not integrable. However, analytical approximations exists Eq.(2) and also a very well documented study of numerically-calculated vortex solution is available [17], [16].

The monopolar vortex has been considered in relation with the Great Red Spot, a vortical structure with a long time life (more than three hundread years), which is observed to be embedded in the zonal flow of the atmosphere of Jupiter. The condition imposed to the Flierl-Petviashvili equation is to provide an isolated, finitely extended vortical solution, obeying boundary conditions at infinity (on the $2D$) of smooth decaying.

We consider that the insuccess in determining the analytical vortex solution has a fundamental motivation.

First of all the equation does not pass the Painlevé test, so it is very likely that it is not integrable. This, in the definition that is suggested by the Inverse Scattering Transform method, means that there is no Lax pair of operators for this equation. In other words, one cannot find a system of two linear differential equations, whose condition of compatibility to be expressed as the nonlinear FP equation.

This equation has close ressemblance with some approximative form of other nonlinear equations. For example, consider the equation

$$\Delta\phi = \exp\left(-\phi\right)$$

This is the Liouville equation, with numerous applications. In particular it has been invoked in the description of the distribution of current in the magnetic confinement systems (tokamak). This equation is integrable and exact solutions are known. In addition, exact analytical solutions can be constructed on periodic domains. The expansion of the right hand side of this equation takes a form close to the FP equation.

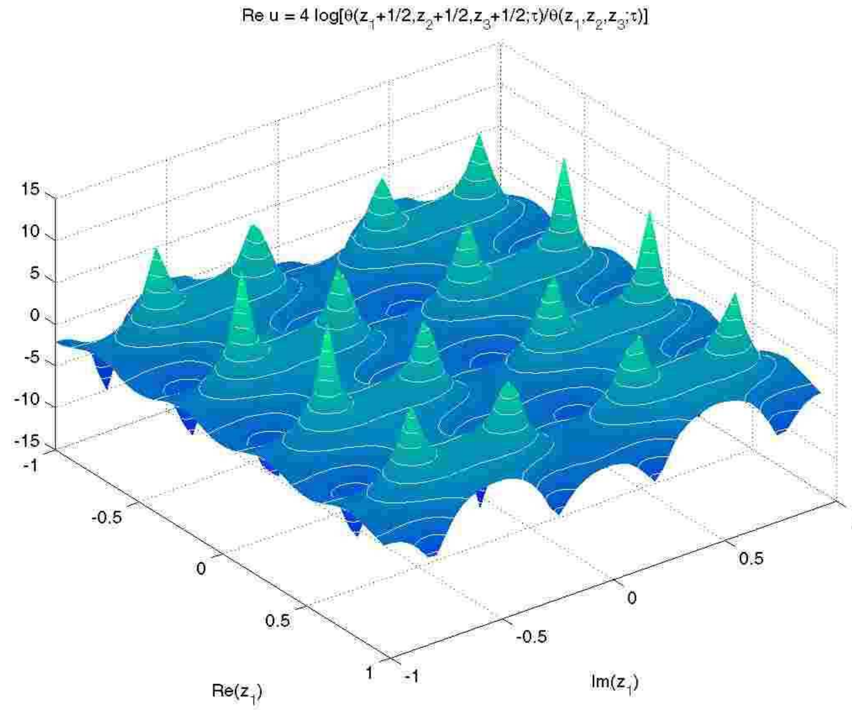

Figure 1: The exact solution of the Jacobs-Rebbi (Abelian-Higgs) equation for ideal fluids obtained analytically as a ratio of Riemann Θ functions.

Even more close is the differential equation

$$\Delta\phi = \exp(\phi) - 1$$

sometimes known as Abelian-Higgs equation (AH), governing the vortices of the superconducting media. This equation has also been derived by Jacobs-Rebbi. It is clear that an expansion of the right hand side leads to something close to the FP equation, especially because the FP equation has actually been derived under the neglect of the third order powers of ϕ.

$$\begin{aligned} \Delta\phi &= \exp(\phi) - 1 \\ &\approx \phi - \frac{1}{2}\phi^2 \end{aligned}$$

Or, this equation is exactly integrable on periodic domains. We have proved [18] that it possesses a Lax pair of operators and we have constructed exact, analytical solutions, expressed as ratios of Riemann Θ functions.

The solutions of the AH equation, as shown in the Fig.1 have the configuration of a lattice of vortices. This solution can be regarded in two different ways: as a function with a periodic set of maxima and minima; or, in a more physical way, as an ensemble of vortices in plane that interact weakly. Then one can easily assume that under the influence of perturbations, the precise periodic geometry can be lost and the vortices begin to move rather freely in plane and the interactions take the form of inelastic collisions.

The transformation of ridges having as section the *sechyperbolic* function, into an array of vortices has been known previously and is well documented numerically [17], [16]. We claim that the actual reason for this evolution, in the numerical simulation of the FP equation, is not the attracting nature of the monopolar vortex (which actually it is not even an exact solution) but the fact that the initial function evolves to the nearest soliton-like solution of the closely related equation, the AH equation. Or, this solution is a lattice of monopolar vortices.

We can also explain why the monopolar vortex of the FP equation, obtained only numerically and studied under a certain numerical imprecision, has an amazing stability: it actually represents one only exemplar of vortex from the lattice, isolated and extended to $2D$ infinity by simply ignoring its appartenence to the full, periodic solution. This is only numerically acceptable, as an approximation, if the distance between neighboring vortices is large enough.

We note, in addition, that the FP equation has also ressemblance with the nonlinear differential equation

$$(\Delta + \lambda)\,\phi = \sinh\phi$$

which describes the stationary states of the Hasegawa-Mima equation. This equation, again, is exactly integrable on periodic domains, and its solution can be analytically expressed as ratios of Riemann Θ function. A comparison between different functions and the exact solution, represented by $\wp$ is shown in Fig.2.

4 Coherent structures and drift wave radiation

In previous studies [24], [25] we have developed a theoretical approach to describe a coherent structure (the monopolar vortex of the FP equation) immersed in a turbulent environment [40], [42], [43]. The theoretical instruments are inspired by the quantum theory of fields, in particular the calculation of field correlations from the effective action [50], [47]. We have defined the action functional according to the Martin-Siggia-Rose description

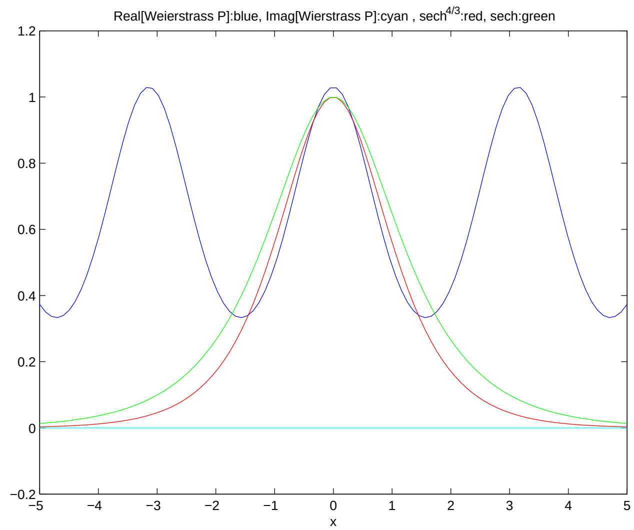

Figure 2: Comparison between the $sech^{4/3}$ with the Weierstrass function, which is the exact solution of the FP equation

[48], [45], and calculated explicitely the generating functional of correlations

$$Z_J = \exp\left(iS_{Js}\right) \frac{1}{2^n i^n} \left(2\pi\right)^{n/2} \left(\det \left. \frac{\delta^2 \widehat{O}}{\delta\varphi\delta\chi} \right|_{\varphi_{Js},\chi_{Js}} \right)^{-1/2} \tag{3}$$

where the determinant is obtained from the product of the eigenvalues of the operator

$$\det \left(\left. \frac{\delta^2 \widehat{O}}{\delta\varphi\delta\chi} \right|_{\varphi_{Js},\chi_{Js}} \right) = \prod_n \lambda_n \tag{4}$$

Then the final expression has been derived in the form

$$\begin{aligned} Z_J &= \exp\left(iS_J\right) \left(\prod_n \frac{(2\pi)}{i} \right) \left[\det \left(\left. \frac{\delta^2 S_J}{\delta\varphi\delta\chi} \right|_{\varphi_{Js},\chi_{Js}} \right) \right]^{-1/2} \\ &= const\ \exp\left(iS_J\right) \left[\frac{\beta/2}{\sinh\left(\beta/2\right)} \right]^{1/4} \left[\frac{\sigma/2}{\sin\left(\sigma/2\right)} \right]^{1/2} \end{aligned} \tag{5}$$

and the correlation results from functional derivatives.

$$\left\langle \varphi(y_2)\varphi(y_1) \right\rangle = Z_J^{-1} \left. \frac{\delta^2 Z_J}{i\delta J(y_2)\, i\delta J\left(y_1\right)} \right|_{J=0}$$

This theoretical approach may be very useful in the study of the late stages of the perturbed flow, where the potential consists of both coherent vortices and drift wave radiation.

5 The analytical solution of the FP eq.

In a previous work [23] we have determined an exact solution of the FP equation. The method consisted in looking to the trajectories of the one-dimensional solution singularities, in the complex plane of the spatial variable. The form of the solution has been assumed as

$$\phi(x,y) = -2\phi_0 \sum_{n=1}^{N} \sum_{l=-\infty}^{\infty} \frac{1}{\left[\gamma\left(y-y_n\right)-ilD\right]^2} \tag{6}$$

and differential equations (plus a constraint) have been derived for the positions of the complex poles $y_n(x)$. An exact solution to the Petviashvilli equation with *constant* coefficients α and β is

$$\phi(x,y) = \frac{\alpha}{2\beta} + s\wp\left(iay + ibx + \omega | g_2 = \frac{3\alpha^2}{(s\beta)^2}\right) \tag{7}$$

with the condition

$$a^2 + b^2 = \frac{s\beta}{6} \tag{8}$$

where $\wp$ is the doubly periodic elliptic Weierstrass function.

6 Comparison with experiments

Experimental measurements of the characteristics of the zonal flow have been performed on Doublet III-D tokamak ([27]). In Ohmic and L-mode plasmas it has been found a perturbed potential $\widetilde{\varphi}_{rms} > 10\,V$ and a flow shear $\omega_{E\times B} \sim 2\times 10^5\, s^{-1}$. The value for the radial wavelength is in the range $k_r\rho_s \in [0.1, 0.6]$ which means $\lambda_r \in (15...30)\,\rho_s$. Due to the different sensitivity of the solution ϕ_s to the parameters, this set is sufficiently restrictive to determine its form. We have to take v_*/u very close to unity and $g_3 \sim -1500$ which gives: $\widetilde{\varphi}_{rms} > 17\,V$, $\lambda_r \simeq 17.4\rho_s$, $\omega_{E\times B} \sim 2.2\times 10^5\, s^{-1}$. The relative amplitude of the perturbation results $\sim 4\%$. An important experimental result is the radial spectrum of the perturbation. We have calculated $S(k_r)$ from the Fourier transform of the correlation of $\phi_s(x)$. The result is very close to Fig.3 of Ref.[27], with two symmetric peaks at $k_{0r} \sim 4\,cm^{-1}$. The same

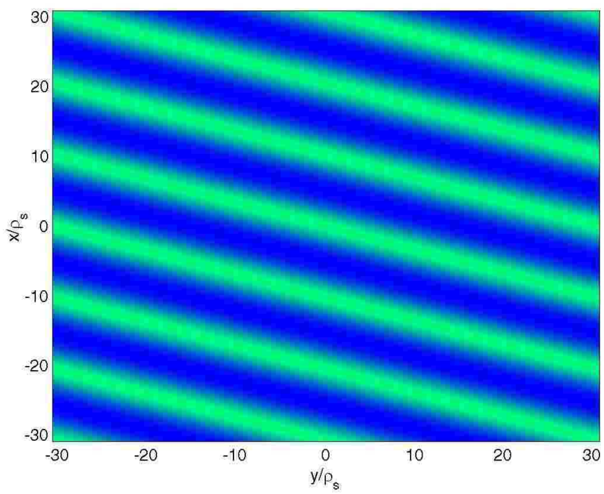

Figure 3: Exact solution of the FP equation.

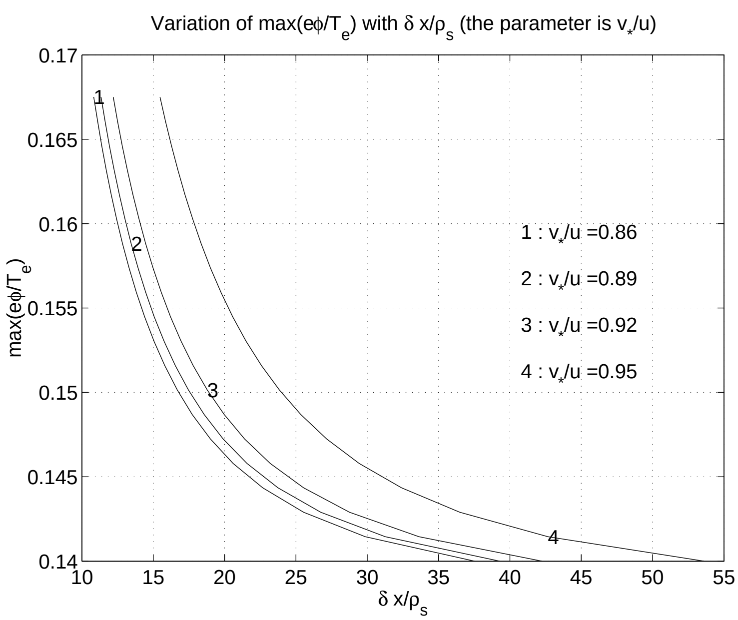

Figure 4: The realation between the amplitude of the potential of the flow and the width of the periodicity layer, parametrised by the ratio v_*/u.

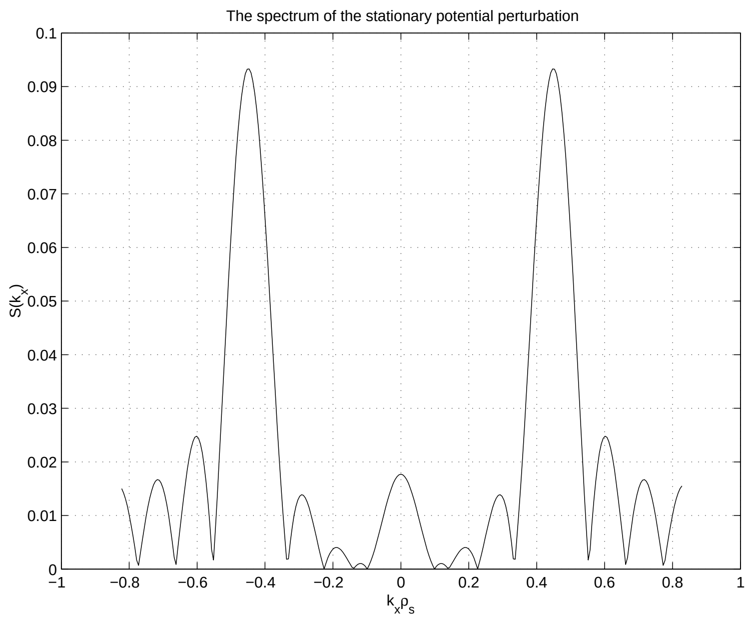

Figure 5: The spectrum obtained from the Fourier transform of the correlation of the solution of the FP equation.

sharp decay for $|k_r| < k_{0r}$ is observed, as described in Ref.[27]. We also note that the analytic spectrum is structured with very narrow dips on its range $k_r \in [1, 10]\ cm^{-1}$ which, according to Ref.[27] cannot be resolved in the measurements. Again, a small fluctuation of the parameters generate an average spectrum very close to Fig.3 of the mentioned work. .

7 Comparisons with results from numerical simulations

7.1 Numerical simulations at Lausanne

The conditions are the following:

$$\begin{aligned} L_{T_i} &= a/3 \ (m) \\ B_T &= 2.5 \ (T) \\ \Omega_i &= 120 \ (MHz) \\ \rho_s &= 1.8 \times 10^{-3} \ (m) \\ T_e &= T_i = 5 \ (KeV) \end{aligned}$$

In these conditions the results obtained were presented in several plots in

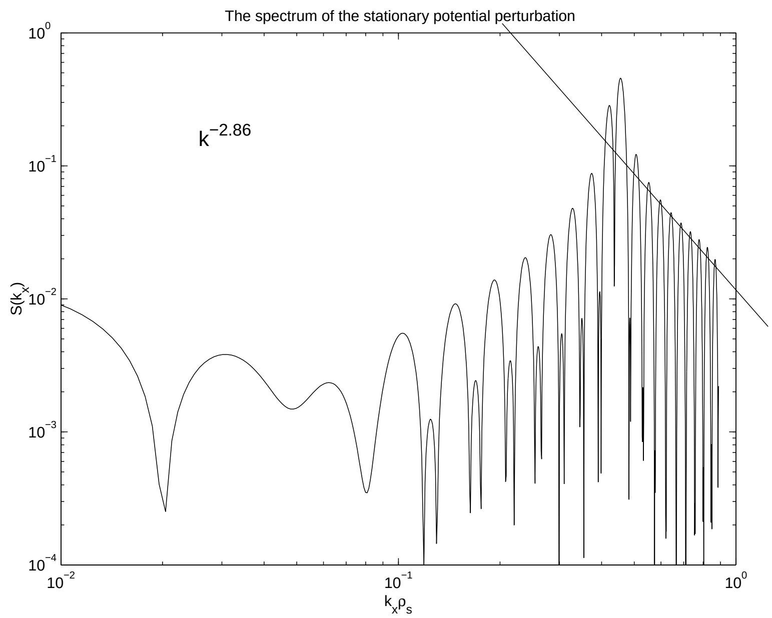

Figure 6: Graph of the spectrum in $log - log$ scale and estimation of the decay.

Ref.[30].

7.1.1 Geometry of the zonal flow

The radial electric field is shown in Figure 4 of this reference. The values are in the range

$$\begin{aligned} E_r &\sim (-0.02 \cdots + 0.02) \times 1.2 \times 10^6 \ (V/m) \\ &\sim (-24 \cdots + 24) \times 10^3 \ (V/m) \end{aligned}$$

Another characteristic of the flow obtained numerically in this reference is the periodicity width, δx. This may be estimated on the basis of the Fig.4.

$$\delta x \sim (0.06 \cdots 0.1) \, a$$

and a in this calculation has been taken

$$a = 135 \rho_L$$

which gives

$$\delta x \sim (8.1 \cdots 13.5) \, \rho_L$$

Using our analytical solution we can compute the radial electric field on the basis of the analytical solution, $\phi_s(x)$. This gives

$$\begin{aligned} E_x^{phys} &= -\frac{d\phi_s^{phys}}{dx^{phys}} \\ &= -\frac{T_e}{|e|}\frac{1}{\rho_s}\left(\frac{d\phi_s}{dx}\right) \quad (V/m) \end{aligned}$$

From this we can calculate the flow velocity

$$v_y^{phys} = \frac{E_x^{phys}}{B} \;\; (m/s)$$

7.1.2 The shearing rate

In the reference [30], the shearing rate is computed as function of the radial coordinate $s \equiv r/a$ and represented in Fig.5 and in Fig.11.

$$\omega_{E\times B} \equiv r\frac{d}{dr}\left(\frac{1}{r}\frac{E_r}{B}\right)$$

The shearing rate is oscillatory as expected for the quasi-periodic geometry of the flow, and a typical value at maximum is

$$\omega_{E\times B} \sim 2\times 10^{-3}\Omega_i$$

We can calculate this quantity

$$\omega_{E\times B}^{phys} = -\frac{v_y^{phys}}{r^{phys}} + \frac{dv_y^{phys}}{dr^{phys}}$$

In the slab geometry assumed in our model,

$$\omega_{E\times B}^{phys} = \frac{dv_y^{phys}}{dr^{phys}} \;\; \left(s^{-1}\right)$$

which can be normalised to Ω_i.

The results and the particular choice of parameters compatible with the simulation Lausanne1 are in figures.

```
Physical parameters
         rhos (m)                              = 0.0018
         cs (m/sec)                          = 0.21600E+06
         Omega_i (sec**(-1))       = 0.12000E+09
         L_n (m)                             = 243.0
         L_T (m)                               = 0.073
```

```
        d(Ln)/dx ( )                     = 6.5
        Vstar (m/sec)                  = 0.160E+01
Parameters in the equations (normalised)
        ekn (rhos/Ln)                          = 0.0000074
        ekt (rhos/LT)                          = 0.0246914
        eknp (rhos**2*d/dx(1/Ln)) = 0.0000000
        v* (v*/Omegai*rhos)             = 0.741E-05
        u (u/Omegai*rhos)                 = 0.100E-04
        cs (cs/Omegai*rhos)             = 0.100E+01
Parameters in the Flierl-Petviashvili equation
        alpha             = 0.259259
        beta              = 1.783265
        s                 = 0.005000
        g2 = 2536.4
        g3 = -2704.2
        Period omega      = 0.265228
        Period omega_prim  = 0.257673
        Orientation of the stationary flow pattern
                    acoef = 0.0000
                    bcoef = 0.0385
        Width on x of the periodicity layer
                    deltax (in units of rhos) = 13.3685
Space domain :
        y (poloidal) ymin = 0.0000   ymax = 100.0000
        x (radial)   xmin = -2.0000  xmax = 106.9479
The minimum of the solution PHIM = 0.727002E-01
the maximum of the solution PHIX = 0.107068
        The difference of min/max amplitude, s1 = 0.343680E-01
        The average amplitude,              s0 = 0.898842E-01
        eddy turn-over time tau (sec) = 0.433340E-04
        Electric field on x, in V/m, minimum :  -23237.9
                                    maximum :  23238.0
        Shear rate normalized at OMCI, min:  -0.272078E-01
                                    max:  0.136097E-01
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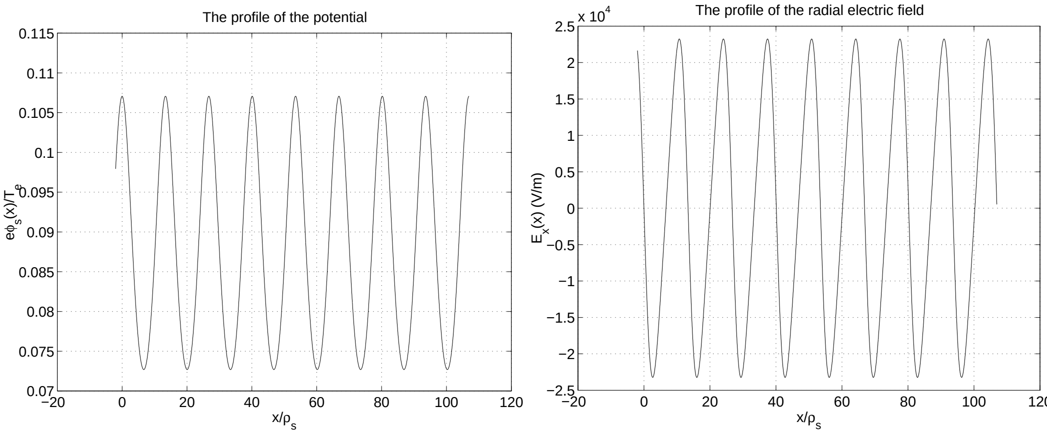

Figure 7: The profile of the potential and of the radiale electric field

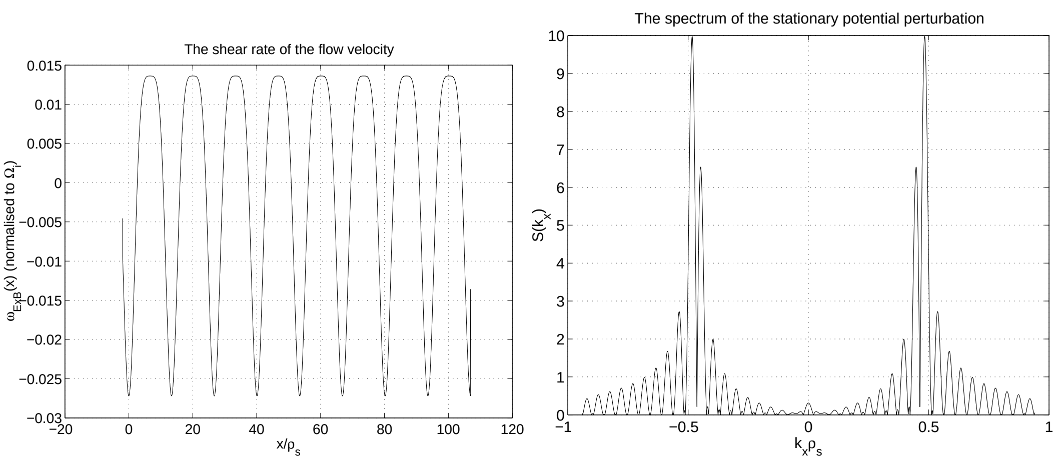

Figure 8: The flow shear rate and the spectrum $S(k_x)$

8 Linear stability of the stationary periodic solution

8.1 Linear dispersion relation

We start from the equation derived before, with the presence of a temperature gradient

$$\frac{\partial}{\partial t}\left(1-\nabla_{\perp}^{2}\right)\phi=\frac{\partial}{\partial y}\left[\left(-\nabla_{\perp}^{2}+4\eta^{2}\right)\phi\right]-\phi\frac{\partial\phi}{\partial y}$$

To study the stability we will perturb the stationary solution by a small function $\varepsilon(x,y,t)$

$$\phi \to \phi_s(x,y) + \varepsilon(x,y,t)$$

where $\phi_s(x,y)$ is the periodic flow solution (expressed in terms of the Weierstrass function). We obtain

$$\frac{\partial}{\partial t}(1-\mathbf{\Delta})\varepsilon = \frac{\partial}{\partial y}\left(-\Delta\varepsilon + 4\eta^2\varepsilon - \phi_s\varepsilon - \frac{1}{2}\varepsilon^2\right)$$

after taking into account the fact that ϕ_s is a solution of the stationary equation.

This equation must be solved starting from an initial condition $\varepsilon(x,y,t=0)$.

If we neglect the quadratic term we have the equation

$$\frac{\partial}{\partial t}(1-\mathbf{\Delta})\varepsilon + \frac{\partial}{\partial y}\Delta\varepsilon - \left(4\eta^2 - \phi_s\right)\frac{\partial\varepsilon}{\partial y} + \varepsilon\frac{\partial\phi_s}{\partial y} = 0 \tag{9}$$

If we consider the stationary periodic solution as being aligned along the poloidal direction, y, then the function ϕ_s has no y dependence and the last term can be discarded

$$\frac{\partial}{\partial t}(1-\mathbf{\Delta})\varepsilon + \frac{\partial}{\partial y}\Delta\varepsilon - 4\eta^2\frac{\partial\varepsilon}{\partial y} + \phi_s\frac{\partial\varepsilon}{\partial y} = 0 \tag{10}$$

In (k_y,ω) space

$$\varepsilon(x,y,t) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk_y d\omega \exp\left(-i\omega t + ik_y y\right)\widetilde{\varepsilon}(x,k_y,\omega)$$

the equation is

$$\left(1 + k_y^2 - \frac{\partial^2}{\partial x^2}\right)(-i\omega)\widetilde{\varepsilon} + ik_y\left(-k_y^2 + \frac{\partial^2}{\partial x^2}\right)\widetilde{\varepsilon} - ik_y\left(4\eta^2 - \phi_s\right)\widetilde{\varepsilon} = 0 \tag{11}$$

Some rearrangement yields the equation

$$\frac{\partial^2\widetilde{\varepsilon}}{\partial x^2} + \frac{ik_y}{i\omega + ik_y}\left(-\frac{\omega}{k_y} - \omega k_y - k_y^2 - 4\eta^2 + \phi_s\right)\widetilde{\varepsilon} = 0$$

Let us introduce new notations

$$q_1 \equiv \frac{1}{1 + \frac{\omega}{k_y}}$$

$$a_1 \equiv -\frac{\omega}{k_y} - \omega k_y - k_y^2 - 4\eta^2$$

and the form of the equation is

$$\frac{\partial^2 \widetilde{\varepsilon}}{\partial x^2} + (q_1 a_1 + q_1 \phi_s)\, \widetilde{\varepsilon} = 0$$

The stationary periodic solution is expressed as

$$\phi_s (x, y) = \phi_0 + s \wp (x)$$

where ϕ_0 is

$$\phi_0 = \frac{3\alpha}{2\beta}$$

and s is a parameter related to the amplitude of the potential perturbation. The equation becomes

$$\frac{\partial^2 \widetilde{\varepsilon}}{\partial x^2} + [q_1 a_1 + q_1 \phi_0 + q_1 s \wp (x)]\, \widetilde{\varepsilon} = 0 \tag{12}$$

This is a Calogero-Moser problem but for a single particle and the analysis of this Schrodinger-type problem is, to our knowledge, not available (see *arxiv.org/hep-th/9903002*). In order to obtain an estimation for physical cases we will retain the periodic character of the Weierstrass function by the replacement

$$\wp (x) \rightarrow s_0 + s_1 \cos (px)$$

where

$$p \equiv (\delta x)^{-1}$$

$$s_0 \equiv \frac{1}{2} [\wp (x) |_{\max} + \wp (x) |_{\min}]$$

$$s_1 \equiv \wp (x) |_{\max} - \wp (x) |_{\min}$$

The parameter p is the inverse of the periodicity length accross the layer in the x direction. The latter is obtained as δx from the imaginary-axis semi-period of the Weierstrass function and the physical parameters. The parameters s_0 and s_1 represent respectively the average amplitude of the function $\wp (x)$ and respectively the variation of its amplitude on a period.

This approximation transform the equation (12) in the Mathieu equation

$$\frac{\partial^2 \widetilde{\varepsilon}}{\partial x^2} + [a_2 + b_2 \cos (px)]\, \widetilde{\varepsilon} = 0$$

with the notations

$$a_2 \equiv q_1 a_1 + q_1 \phi_0 + q_1 s s_0$$

$$b_2 \equiv q_1 s s_1$$

We then change the space variable

$$x \to x' \equiv \frac{px}{2} = \frac{x}{2\,(\delta x)}$$

and arrive at the final form of the equation

$$\frac{\partial^2 \widetilde{\varepsilon}}{\partial x^2} + [a - 2q\cos(2x)]\,\widetilde{\varepsilon} = 0 \tag{13}$$

with the new notations

$$a \equiv 4\,(\delta x)^2\, q_1\,(a_1 + \phi_0 + s s_0)$$

$$q \equiv -2\,(\delta x)^2\, q_1 s s_1$$

In detailed form

$$a = 4\,(\delta x)^2\,\frac{1}{1+\omega/k_y}\left(-\frac{\omega}{k_y} - \omega k_y - k_y^2 - 4\eta^2 + \phi_0 + s s_0\right) \tag{14}$$

$$q = -2\,(\delta x)^2\, s s_1 \frac{1}{1+\omega/k_y} \tag{15}$$

In order $\widetilde{\varepsilon}(x)$ to be a periodic function on x it must be fulfilled one of a discrete set of dispersion relations. We take the simplest approximative form

$$a_0(q) \approx -\frac{q^2}{2} \tag{16}$$

This leads to the following relation between the parameters

$$-\frac{\omega}{k_y} - \omega k_y - k_y^2 + P = -Q\frac{1}{1+\omega/k_y} \tag{17}$$

where

$$\begin{aligned} P &\equiv -4\eta^2 + \phi_0 + s s_0 \\ Q &\equiv \frac{1}{2}\,(\delta x)^2\, s^2 s_1^2 \end{aligned} \tag{18}$$

Eliminating the numitor and grouping the terms

$$-(k_y + \omega)^2 + \frac{\omega}{k_y}(P-1) - \frac{\omega^2}{k_y^2} + P + Q = 0$$

We note

$$w \equiv \frac{\omega}{k_y}$$

and have the equation

$$-k_y^2 (1+w)^2 + w(P-1) - w^2 + P + Q = 0$$

or

$$w^2 + 2t_1 w + t_2 = 0$$

$$\begin{aligned} 2t_1 &\equiv \frac{-2k_y^2 + P - 1}{-k_y^2 - 1} \\ t_2 &\equiv \frac{-k_y^2 + P + Q}{-k_y^2 - 1} \end{aligned}$$

8.2 Approximations based on evaluations of terms

Since the spatial variables are measured in ρ_s we can already say that the wavelength of the y-perturbations will be of the order of several units

$$\lambda_\perp \geq (\text{few units})\, \rho_s$$

We will consider that this is a sufficient indication to take

$$\frac{1}{k_y^2} \ll 1$$

Then we can simplify the expressions

$$\begin{aligned} 2t_1 &= \left(2 - \frac{P-1}{k_y^2}\right)\left(1 - \frac{1}{k_y^2}\right) \\ &\approx 2 - \frac{P-1}{k_y^2} \end{aligned}$$

$$\begin{aligned} t_2 &= \left(1 - \frac{P+Q}{k_y^2}\right)\left(1 - \frac{1}{k_y^2}\right) \\ &\approx 1 - \frac{P+Q+1}{k_y^2} \end{aligned}$$

and the discriminant of the second order equation is

$$\begin{aligned} \Delta &\equiv \left(1 - \frac{P-1}{2k_y^2}\right)^2 - \left(1 - \frac{P+Q+1}{k_y^2}\right) \\ &\approx \frac{Q}{k_y^2} \end{aligned}$$

The solutions are

$$
\begin{aligned}
w &= -\left(1-\frac{P-1}{2k_y^2}\right)\pm\sqrt{\frac{Q}{k_y^2}} \\
&= -1+\frac{P+1\pm 2k_y Q^{1/2}}{2k_y^2}
\end{aligned}
$$

The dispersion relation is at this moment

$$
\frac{\omega}{k_y}=-1+\frac{P+1}{k_y^2}\pm\frac{Q^{1/2}}{k_y} \tag{19}
$$

Let

$$
u\equiv\frac{1}{k_y}
$$

The equation is

$$
\omega u=-1+\frac{P+1}{2}u^2\pm Q^{1/2}u
$$

or

$$
u^2+2u\left[\frac{-\omega\pm Q^{1/2}}{(P+1)/2}\right]-\frac{2}{P+1}=0
$$

This leads to

$$
\frac{1}{k_y}=\frac{\omega\pm Q^{1/2}}{P+1}\left\{1\mp\left[1+\frac{2(P+1)}{\left(-\omega+Q^{1/2}\right)^2}\right]^{1/2}\right\} \tag{20}
$$

We can evaluate

$$
\begin{aligned}
P &= -4\eta^2+\phi_0+ss_0 \\
&= -4\eta^2+\frac{1}{2}\phi_0+\frac{1}{2}s\wp(x)|_{\max}+\frac{1}{2}\phi_0+\frac{1}{2}s\wp(x)|_{\min} \\
&= -4\eta^2+\frac{1}{2}\phi_s|_{\min}+\frac{1}{2}\phi_s|_{\max} \\
&= -4\eta^2+\overline{\phi}
\end{aligned}
$$

where we can have an estimation for the average level of fluctuation

$$
\overline{\phi}\sim\frac{\widetilde{n}}{n_0}
$$

In the numerical experiments the field that clearly evolved to breaking into discrete monopolar vortices had an amplitude of the order of

$$
\overline{\phi}\sim 0.05...0.1
$$

We chose a case with

$$4\eta^2 \equiv 1 - \frac{v_*}{u} \sim 1 - 0.85 = 0.15$$

The two terms are of similar magnitude so we can expect P to be positive or negative. In this case

$$P = -0.15 + 0.1 = -0.05$$

$$\begin{aligned} Q &= (\delta x)^2 s^2 s_1^2 \\ &= (\delta x)^2 \left(\phi_s|_{\max} - \phi_s|_{\min}\right)^2 \\ &\equiv (\delta x)^2 (\delta\phi_s)^2 \end{aligned}$$

We take data from one of the characteristic runs

$$\begin{aligned} \delta\phi_s &= \phi_s|_{\max} - \phi_s|_{\min} \\ &\sim 0.088 - 0.019 \\ &= 0.0\,69 \end{aligned}$$

and

$$\delta x \sim 14.17$$

then

$$\begin{aligned} Q &\sim (14.17)^2 \times (0.069)^2 \\ &= 0.956 \end{aligned}$$

Since

$$\begin{aligned} P + 1 &\sim 1 \\ Q &\lesssim 1 \end{aligned}$$

We note that Q can be smaller for narrower strips of the stationary solution.

We can do the following approximations

$$\begin{aligned} \left[1 + \frac{2(P+1)}{Q}\right]^{1/2} &\sim \left(1 + 2\frac{1}{Q}\right)^{1/2} \\ &\sim \frac{\sqrt{2}}{\sqrt{Q}} \end{aligned}$$

The dispersion relation is

$$\frac{1}{k_y} \simeq \frac{\omega \mp \sqrt{Q}}{P+1}\left(1 \mp \frac{\sqrt{2}}{\sqrt{Q}}\right)$$

The frequency ω is in the range given by the inverse of the typical time of the change of the stationary solution, after being perturbed. The units which connect an interval of time in physical and respectively adimensional form are

$$\Delta t|_{phys} = \frac{\rho_s}{u|_{phys}} \Delta t$$

with

$$\begin{aligned} \rho_s &\sim 0.001 \ (m) \\ u|_{phys} &\sim v_* \sim 0.3 \times 10^3 \ (m/s) \end{aligned}$$

$$\begin{aligned} \Delta t|_{phys} &\sim \frac{0.001}{300} \Delta t \\ &\sim 3 \times 10^{-6} \Delta t \end{aligned}$$

In the numerical simulations the first changes took place after about

$$\Delta t|_{phys} \sim 4000 \Omega_i^{-1} \, (s) \sim \frac{4000}{0.335 \times 10^9} = 11940 \times 10^{-9} = 11.94 \times 10^{-6}$$

which means

$$\Delta t \sim \frac{\Delta t|_{phys}}{3 \times 10^{-6}} = \frac{11.94 \times 10^{-6}}{3 \times 10^{-6}} \sim 4$$

Then the frequency associated with the change is

$$\omega \sim 1/4 = 0.25$$

If we can neglect ω compared with $\sqrt{Q}$ we have

$$\begin{aligned} \frac{1}{k_y} &\simeq \frac{\omega \mp \sqrt{Q}}{P+1} \left(1 \mp \frac{\sqrt{2}}{\sqrt{Q}} \right) \\ &\simeq \mp \sqrt{Q} \left(1 \mp \frac{\sqrt{2}}{\sqrt{Q}} \right) \\ &= \sqrt{2} \mp \sqrt{Q} \end{aligned}$$

or, chosing the $-$ sign

$$\frac{1}{k_y} \simeq \sqrt{2} - (\delta x)(\delta \phi)$$

which has the character of an *eigenvalue* selection, relating the spatial extension of the vortices into which the periodic solution decays to the amplitude and radial width of periodicity.

Taking from above

$$(\delta x)(\delta\phi) \sim 1$$

we obtain

$$\begin{aligned}\lambda_\perp &\sim 2\pi \times 0.41 \\ &\sim 2.6\end{aligned}$$

Our numerical simulations only allow to identify the process of generation of a chain of vortices, but we cannot pursue too far the simulations due to accumulation of numerical errors. It apparently supports a dimension of about 6.

The stability depends also on the type of perturbation, not only simply of the amplitude. A monopolar solution is very stable. The HM term renders the equation fragile to decay into lattice of vortices.

9 Numerical studies

9.1 Equation with the scalar nonlinearity (time dependent Flierl-Petviashvili equation)

9.1.1 Introduction

The equations (A.44), (A.47), (A.46), (A.45), help us to define the space and time domain of physical parameters for the numerical simulation.

For the numerical treatment we start form the time-dependent equation

$$\frac{\partial}{\partial t}\left(1-\nabla_\perp^2\right)\phi = \frac{\partial}{\partial y}\left(-\nabla_\perp^2 + 4\eta^2\right)\phi - \phi\frac{\partial\phi}{\partial y}$$

and notice that it may be preferable to use two fonctions

$$\psi \equiv \left(1-\nabla_\perp^2\right)\phi$$

and ϕ. The equation is transformed into a system of two differential equations for two unknown functions ψ and ϕ

$$\begin{aligned}-\nabla_\perp^2\phi &= -\phi + \psi \\ \frac{\partial\psi}{\partial t} - \frac{\partial\psi}{\partial y} &= \left(-1+4\eta^2\right)\frac{\partial\phi}{\partial y} - \phi\frac{\partial\phi}{\partial y}\end{aligned}$$

or

$$\begin{aligned}-\nabla_\perp^2\phi + \phi &= \psi \\ \frac{\partial\psi}{\partial t} &= \frac{\partial\psi}{\partial y} + \left(-1+4\eta^2\right)\frac{\partial\phi}{\partial y} - \phi\frac{\partial\phi}{\partial y}\end{aligned}$$

This version is one of the form taht will be used in the numerical integration of the equation.

9.1.2 Initial monopolar perturbation

We start by initializing the potential with a monopolar vortex localised in one of the layers of periodicity. The amplitude of the potential at this run was $\phi_s|_{\max} = 0.03$ and of the perturbation is $\phi_{pert}|_{\max} = 5 \times 10^{-3}$ (normalised as $|e|\,\phi/T_e$). The evolution is done with a time step of $10\Omega_i$.

We note a very stable evolution of the total flow, consisting of a slow ($v_*/u \lesssim 1$) displacement of the monopole along the layer. In the first part of the evolution, the profile of the monopolar vortex is adapted to the local speed of the flow, but afterwards the evolution is amazingly stable. The duration of this period of stability is greater than $12 \times 10^3\Omega_i$ the limit where the run was stopped.

This conclusion must be useful in relation with the long stability of the Red Spot of the Jupiter atmosphere. There, the configuration is precisely the same as in this run: a localised vortex in a structure of zonal flow, as shown by the observations of Voyager 1 and 2 (see Busse [26]).

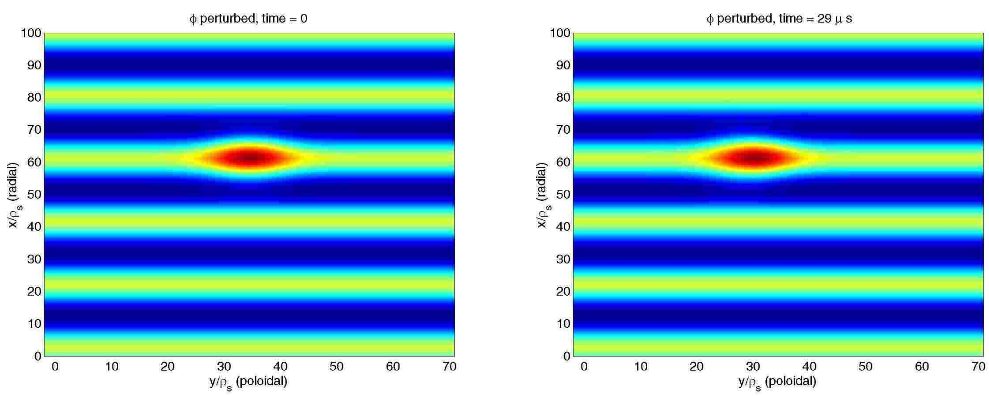

Figure 9: The initial perturbation is monopolar and located symmetrically in a flow layer. The initial ($t = 0$) and final ($t = 29\mu s$) potentials are shown.

9.1.3 Initial dipolar perturbation

The initial field is ϕ_s perturbed by a dipolar vortex placed with maxima approximately on a line of maximum. The amplitude of ϕ_s is 0.03 and that of the dipole is 3×10^{-3}. The evolution in this case is very stable for all the interval of the run ($29\,\mu s$, corresponding to $12 \times 10^3\Omega_i$). We notice the weak change of the shape of the dipole due to the interaction with the background flow, in the first part of the run.

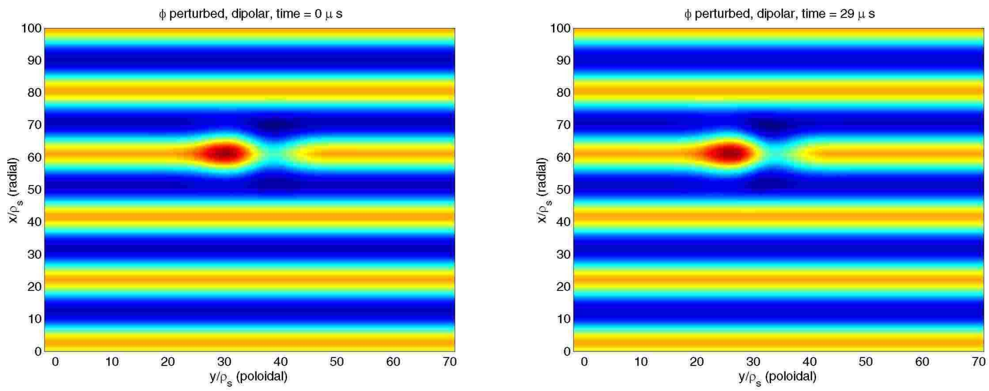

Figure 10: A dipolar perturbation is initialised. The initial ($t = 0$) and final ($t = 29\mu s$) potentials are shown.

In another run, the initial field ϕ_s of amplitude 0.03 has been perturbed by a dipolar vortex with the same localisation as before but with amplitude comparable to the background field, 0.03. The run evolves without destruction of the configuration.

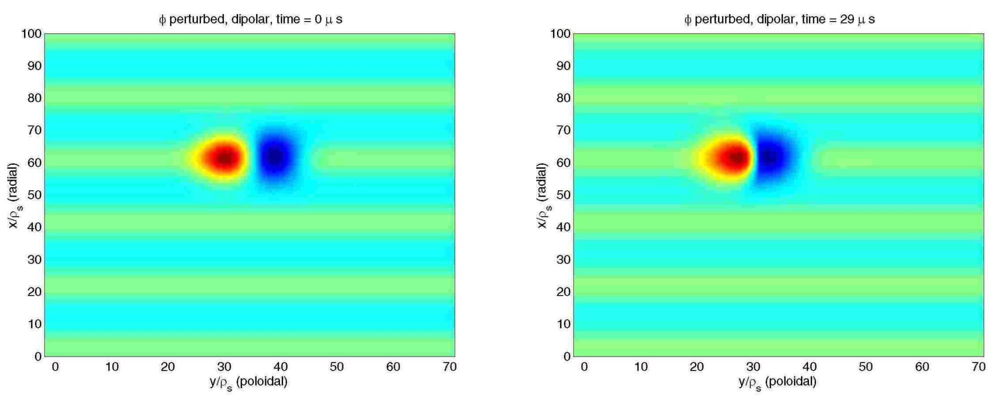

Figure 11: An initial stationary flow of higher amplitude is perturbed with a dipolar vortex of comparable amplitude. Only the scalar nonlinearity is retained.

From the numerical experiments with the dipolar perturbations it seems that low amplitude potential ϕ_s has weak interaction with the perturbation and major changes of the flow may probably arise later than several thousends Ω_i.

9.2 Equation with scalar and polarisation drift nonlinearities (time dependent Flierl-Petviashvili equation with Hasegawa-Mima term)

9.2.1 Initial monopolar perturbation

In a series of run we take a higher initial background flow ϕ_s, with amplitude of 0.06 and perturb it with a monopolar structure strongly elongated in the radial direction. The amplitude of the parturbation is very high, 0.12. We note that the evolution leads first to strong changes of the perturbation structure which is rotated and deformed. But in the same time the flow itself tends to be broken into isolated vortices. These runs require higher numerical precision so that only short period of time can be afforded at this momant. The evolution was stopped at $6\,\mu s$.

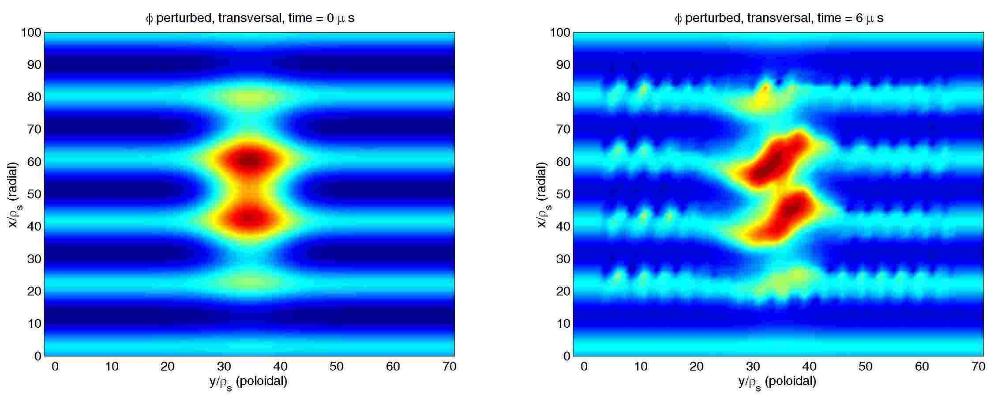

Figure 12: A run with both scalar and polarisation nonlinearities retained. The flow is initialised with a monopolar form perturbing the staionary periodic solution.

The late stages of these runs are useful for the ilustration of the breaking of the background flow into a lattice of vortices.

A particular importance takes the examination of the evolution of the flow when the initial perturbation is monopolar and placed approximately in a layer of periodicity. This should be directly compared with the previous runs with the pure scalar nonlinearity.

We notice that the stability is much stranger for this type of geometrically "compatible" perturbation. Neither the background flow nor the vortex is disturbed for more than $25\,\mu s$. The monopole is only advected by the flow. At late stage, the monopole is deformed and destroyed, while the flow does not show the previously observed tendency to break into a lattice of vortices.

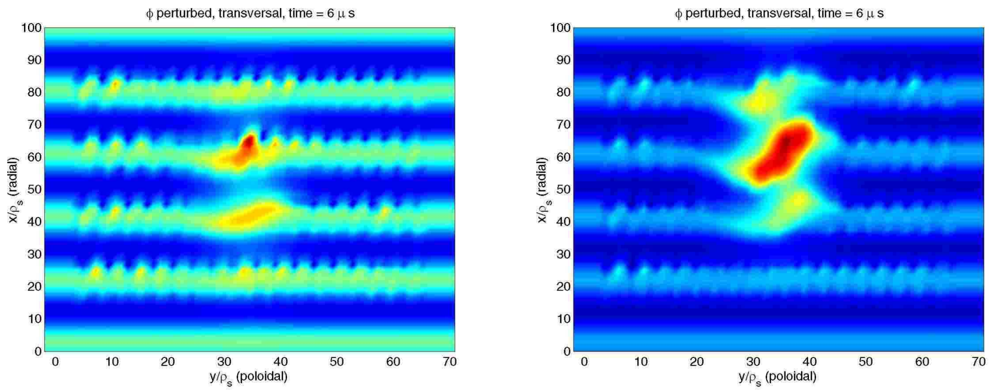

Figure 13: The late stage of a run with monopolar pertubation.The breaking of the flow into vortices is apparent.

However, the region of the very high amplitude where the monopole existed tends to break into vortices.

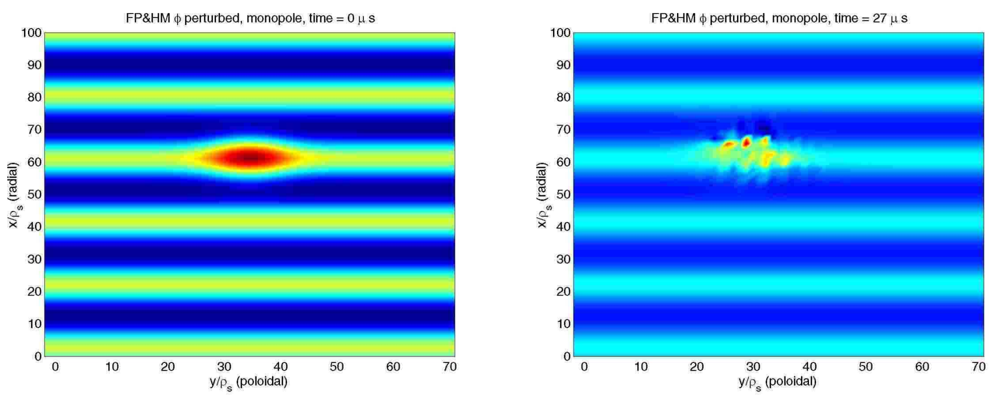

Figure 14: The late stage of a run with monopolar pertubation of high amplitude.The breaking into vortices affects first the strongest perturbation.

This shows that the Hasegawa-Mima nonlinearity has the property of inducing a tandency of the flow to break up, under strong perturbation, into a lattice of monopolar vortices.

9.2.2 Initial dipolar perturbation

A dipolar perturbation shows essentially the same evolution as for monopolar case. The profile is not changed for rather long period, $20\mu s$ but the later stages consist of a clear decay into a set of distinct formations with monopolar structure. We cannot follow too far the simulation due to accumulation of errors, but it is clear that space scale of the new vortical structures is close to those found in preceeding cases. This strengthens the idea that the polarisation drift nonlinearity is the cause of this evolution toward lattices of vortices.

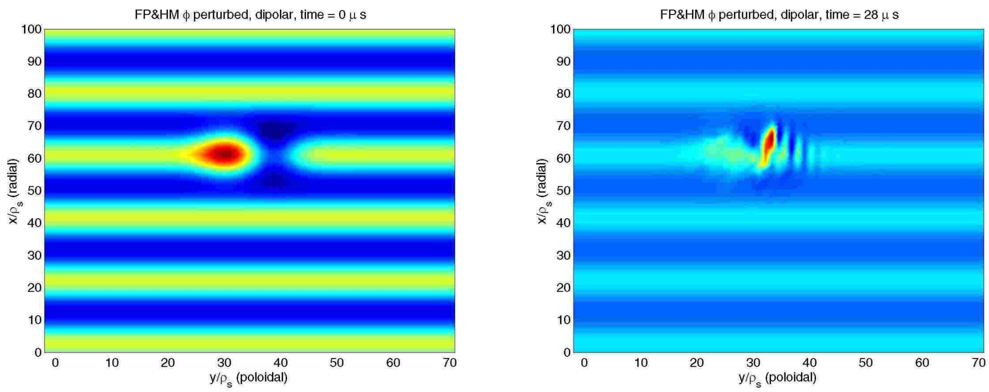

Figure 15: Initial dipolar perturbation

9.2.3 Initial oscillatory perturbation

In order to examine the processes of destruction of the flow and decay into isolated vortices, we initialize with an oscillatory perturbation with a period much larger than that which seems to be chosen by the system as extension of the final monopolar vortices. The flow clearly evolves to the lattice of vortices.

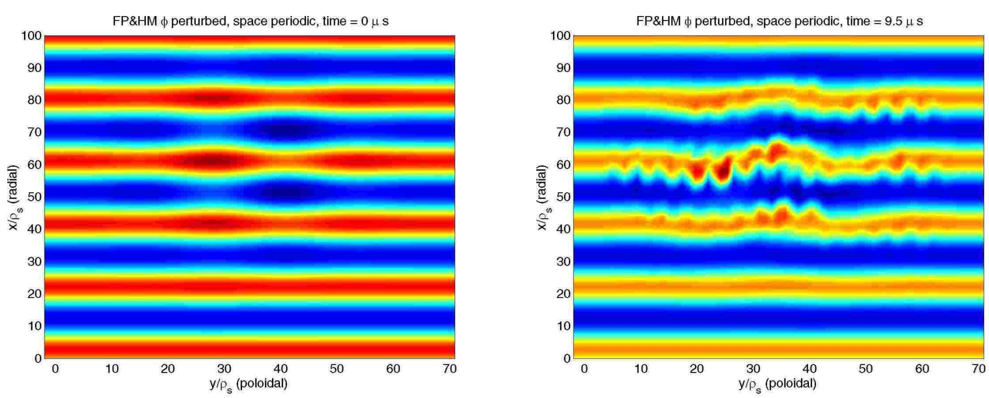

Figure 16: Perturbation initialised as a spacial oscillation of large wavelength.

For a higher amplitude of the initial perturbation the evolution is qualitatively similar.

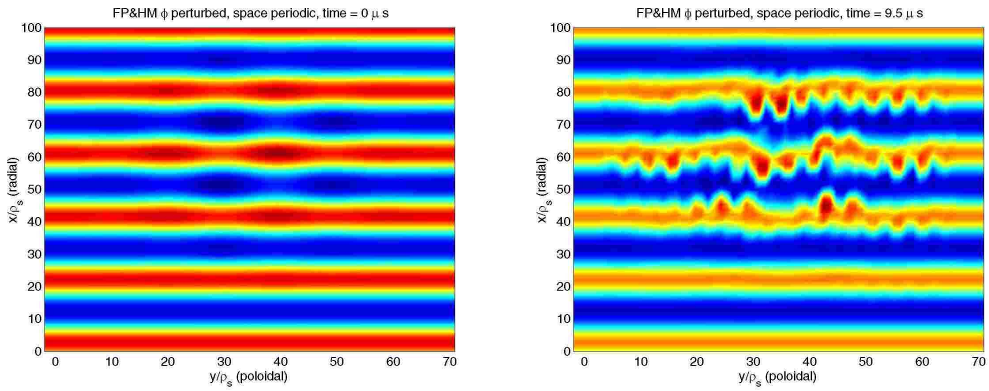

Figure 17: Initial oscillatory perturbation at higher amplitude.

In a series of runs, we initialise with a perturbation consisting of oscillations with higher spatial wavenumber, *i.e.* slightly closer to the final vortices. The result shows clearly that the initial conditions do not influence significantly the space extension of the final monopolar vortices, which means that the system chooses this dimension according to a dispersion relation which has the character of an space-eigenvale problem. This is also suggested by the fact thet the final stages appears to be similar for rather different time duration which is needed by the system to reach the stage of breaking into vortices.

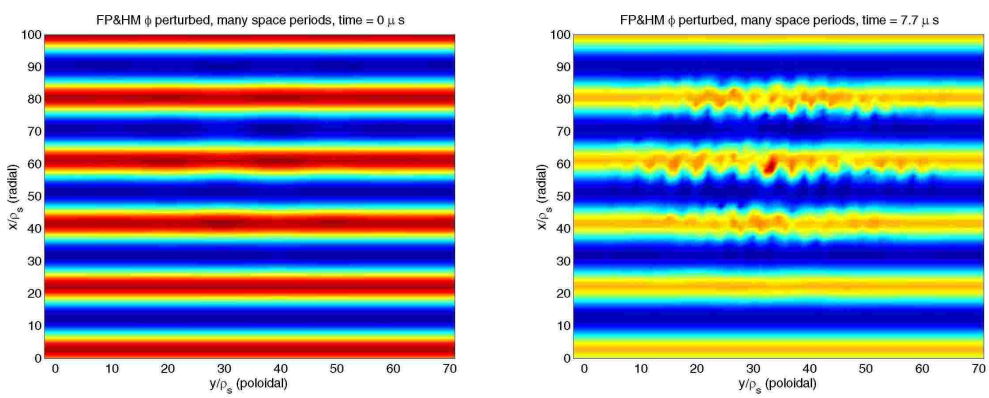

Figure 18: Initial perturbation taken as a space oscillation of relatively small wavelength.

9.3 Conclusions from the numerical studies

Extensive numerical simulations (which however must be considered preliminary) have been done for studying the stability of ϕ_s. In a series of runs the time dependent equations with pure scalar nonlinearity has been used with ϕ_s perturbed by ϕ_p, taken first as a monopolar vortex . For small amplitudes (*e.g.* $\phi_s \sim 0.03$ and $\phi_p \sim 0.005$) the total flow is stable for more than $12 \times 10^3 \Omega_i$. The monopole is reshaped at early stages for equality of the tangential flows and then is stably advected ($v_*/u \lesssim 1$). This result may be relevant for the stability of the Red Spot, the long-lived vortex embedded in the zonal flows of the atmosphere of Jupiter, a geometry very similar to ours. A small amplitude dipolar initial perturbation is also stable for similar durations.

The structural stability of ϕ_s is investigated using the time dependent equation with both scalar and polarisation drift nonlinearities retaind. The flow is stable on only shorter duration and a new effect arises: at late stages, the flow tends to break up into a set of monopolar vortices. Taking ϕ_p of high amplitude ($\phi_s \sim 0.06$, $\phi_p \sim 0.12$) monopolar and strongly elongated on r there is a clear tendency to rotate it such as to align with the background flow. At $6\,\mu s$ the process of generation of vortices is pronounced. Similar conclusions can be drawn from runs with ϕ_p taken as dipolar and as spatial periodic structures. Examination of many runs leads to the conclusion that the spatial extension of the generated vortices depends only weakly on the initial conditions and of the time duration of quasi-stability and is $\lambda \sim 6\rho_s$. Higher numerical precision is required to study their evolution.

10 Appendix

10.1 Derivation of the equation with time dependence

The aim of this Appendix is to review the derivation of the various forms of the equations used in this work. It will be reviewed the stationary and the time dependent equations, with scalar and polarisation drift nonlinearities, etc. The main practical purpose is to compare the units used and to facilitate the present numerical computation and the future possible changes. For this reason some calculations are trivially simple and may be skiped.

Consider the equations for the ITG model in two-dimensions with adiabatic electrons:

$$\begin{aligned}
\frac{\partial n_i}{\partial t} + \nabla \cdot (\mathbf{v}_i n_i) &= 0 \\
\frac{\partial \mathbf{v}_i}{\partial t} + (\mathbf{v}_i \cdot \nabla)\, \mathbf{v}_i &= \frac{e}{m_i} (-\nabla \phi) + \frac{e}{m_i} \mathbf{v}_i \times \mathbf{B}
\end{aligned}$$

We assume the quasineutrality

$$n_i \approx n_e$$

and the Boltzmann distribution of the electrons along the magnetic field line

$$n_e = n_0 \exp\left(-\frac{|e|\, \phi}{T_e} \right)$$

In general the electron temperature can be a function of the radial variable

$$T_e \equiv T_e (x)$$

The velocity of the ion fluid is perpendicular on the magnetic field and is composed of the diamagnetic, electric and polarization drift terms

$$\begin{aligned}
\mathbf{v}_i &= \mathbf{v}_{\perp i} \\
&= \mathbf{v}_{dia,i} + \mathbf{v}_E + \mathbf{v}_{pol,i} \\
&= \frac{T_i}{|e|\, B} \frac{1}{n_i} \frac{dn_i}{dr} \widehat{\mathbf{e}}_y \\
&\quad + \frac{-\nabla \phi \times \widehat{\mathbf{n}}}{B} \\
&\quad - \frac{1}{B \Omega_i} \left(\frac{\partial}{\partial t} + (\mathbf{v}_E \cdot \nabla_\perp) \right) \nabla_\perp \phi
\end{aligned}$$

The diamagnetic velocity will be neglected. Introducing this velocity into the continuity equation, one obtains an equation for the electrostatic potential ϕ.

Before writting this equation we introduce new dimensional units for the variables.

$$\phi^{phys} \rightarrow \phi' = \frac{|e|\,\phi^{phys}}{T_e} \tag{A.1}$$

$$\left(x^{phys}, y^{phys}\right) \rightarrow (x', y') = \left(\frac{x^{phys}}{\rho_s}, \frac{y^{phys}}{\rho_s}\right) \tag{A.2}$$

$$t^{phys} \rightarrow t' = t^{phys}\Omega_i \tag{A.3}$$

The new variables (t, x, y) and the function ϕ are non-dimensional. In the following the *primes* are not written. With these variables the equation obtained is

$$\begin{aligned} & \frac{\partial}{\partial t}\left(1 - \nabla_\perp^2\right)\phi \\ & - \left(-\nabla_\perp \phi \times \widehat{\mathbf{n}}\right) \cdot \mathbf{v}_* \\ & + \left(-\nabla_\perp \phi \times \widehat{\mathbf{n}}\right) \cdot \mathbf{v}_T \phi \\ & + \left[\left(-\nabla_\perp \phi \times \widehat{\mathbf{n}}\right) \cdot \nabla_\perp\right]\left(-\nabla_\perp^2 \phi\right) \\ = \; & 0 \end{aligned}$$

where

$$\begin{aligned} \mathbf{v}_* &\equiv -\nabla_\perp \ln n_0 - \nabla_\perp \ln T_e \\ \mathbf{v}_T &\equiv -\nabla_\perp \ln T_e \end{aligned} \tag{A.4}$$

(This is Eq.(8) from the paper [7]).

NOTE

Before continuing we compare this equation with the equation of paper [29], Eq.(16). Here taking still the units to be physical, the form of the latter equation is (Eq.(12) from that paper)

$$\begin{aligned} & \frac{\partial}{\partial t}\frac{|e|\,\phi}{T_e} - \frac{\partial}{\partial t}\frac{1}{B\Omega_i}\nabla_\perp^2 \phi \\ & + \frac{-\nabla_\perp \phi \times \widehat{\mathbf{n}}}{B} \cdot \nabla_\perp \ln n_0 \\ & + \frac{-\nabla_\perp \phi \times \widehat{\mathbf{n}}}{B} \cdot \nabla_\perp \frac{|e|\,\phi}{T_e} \\ & + \frac{1}{B^2\Omega_i}\left[\left(-\nabla_\perp \phi \times \widehat{\mathbf{n}}\right) \cdot \nabla_\perp\right]\left(-\nabla_\perp^2 \phi\right) \\ = \; & 0 \end{aligned} \tag{A.5}$$

The term containing the gradient of the equilibrium density comes from the continuity equation, as convection of the equilibrium density by the fluctuating $E \times B$ velocity. The term containing the gradient of the equilibrium temperature comes from the continuity equation by the divergence of the fluctuating flow of particles, where the adiabaticity has been assumed,

$$\frac{\widetilde{n}}{n_0} = \frac{|e|\,\phi}{T_e\,(x)}$$

The equation of [29] remains expressed in these terms. However, this is strange in relation with the derivation made by Lakhin et al, where the temperature term is canceled. See below a discussion of the Eq.(8) from Laedke and Spatschek 1986.

For the second term we have

$$\begin{aligned}
\frac{1}{B\Omega_i}\nabla_\perp^2\phi &= \frac{1}{B\Omega_i}\frac{T_e}{|e|}\nabla_\perp^2\frac{|e|\,\phi}{T_e} = \frac{1}{\Omega_i}\frac{1}{\frac{|e|B}{m_i}}\frac{T_e}{m_i}\nabla_\perp^2\frac{|e|\,\phi}{T_e} \\
&= \frac{1}{\Omega_i^2}c_s^2\nabla_\perp^2\frac{|e|\,\phi}{T_e} = \rho_s^2\nabla_\perp^2\frac{|e|\,\phi}{T_e}
\end{aligned}$$

This will become (with its sign)

$$-\frac{\partial}{\partial t}\nabla_\perp'^2\phi' \tag{A.6}$$

in the new variables Eqs.(A.1)-(A.3).

The third term is

$$\begin{aligned}
\frac{-\nabla_\perp\phi\times\widehat{\mathbf{n}}}{B}\cdot\nabla_\perp\ln n_0 &= \frac{1}{B}\frac{T_e}{|e|}\left(-\nabla_\perp\frac{|e|\,\phi}{T_e}\times\widehat{\mathbf{n}}\right)\cdot\nabla_\perp\ln n_0 \\
&= \frac{1}{\frac{|e|B}{m_i}}\frac{T_e}{m_i}\left(-\nabla_\perp\frac{|e|\,\phi}{T_e}\times\widehat{\mathbf{n}}\right)\cdot\nabla_\perp\ln n_0 \\
&= \Omega_i\frac{c_s^2}{\Omega_i^2}\left(-\nabla_\perp\frac{|e|\,\phi}{T_e}\times\widehat{\mathbf{n}}\right)\cdot\nabla_\perp\ln n_0 \\
&= \Omega_i\rho_s^2\left(\nabla_\perp\frac{|e|\,\phi}{T_e}\times\widehat{\mathbf{n}}\right)\cdot\left(-\nabla_\perp\ln n_0\right)
\end{aligned}$$

This will become

$$-\Omega_i\left(-\rho_s\nabla_\perp\phi\times\widehat{\mathbf{n}}\right)\cdot\left(-\rho_s\nabla_\perp\ln n_0\right)$$

in the new variables

$$-\Omega_i\left(-\nabla_\perp'\phi'\times\widehat{\mathbf{n}}\right)\cdot\left(-\nabla_\perp'\ln n_0\right) \tag{A.7}$$

The fourth term is

$$
\begin{aligned}
\frac{-\nabla_\perp \phi \times \widehat{\mathbf{n}}}{B} \cdot \nabla_\perp \frac{|e|\,\phi}{T_e} &= \frac{1}{\frac{|e|B}{m_i}} \frac{T_e}{m_i} \left(-\nabla_\perp \frac{|e|\,\phi}{T_e} \times \widehat{\mathbf{n}} \right) \cdot \nabla_\perp \frac{|e|\,\phi}{T_e} \\
&= \Omega_i \frac{c_s^2}{\Omega_i^2} \left(-\nabla_\perp \frac{|e|\,\phi}{T_e} \times \widehat{\mathbf{n}} \right) \cdot \nabla_\perp \frac{|e|\,\phi}{T_e} \\
&= \Omega_i \rho_s^2 \left(-\nabla_\perp \frac{|e|\,\phi}{T_e} \times \widehat{\mathbf{n}} \right) \cdot \nabla_\perp \frac{|e|\,\phi}{T_e} \\
&= \Omega_i \rho_s^2 \left(-\nabla_\perp \frac{|e|\,\phi}{T_e} \times \widehat{\mathbf{n}} \right) \cdot \left[\frac{|e|}{T_e} \nabla_\perp \phi - \frac{|e|\,\phi}{T_e} \frac{\nabla_\perp T_e}{T_e} \right] \\
&= \Omega_i \rho_s^2 \left(-\nabla_\perp \frac{|e|\,\phi}{T_e} \times \widehat{\mathbf{n}} \right) \cdot \frac{|e|\,\phi}{T_e} \left(-\frac{\nabla_\perp T_e}{T_e} \right)
\end{aligned}
$$

This term will become

$$\Omega_i \left(-\nabla'_\perp \phi' \times \widehat{\mathbf{n}} \right) \phi' \cdot \left(-\nabla'_\perp \ln T_e \right) \tag{A.8}$$

after introducing the new units.

The last term (with the polarization nonlinearity) is in physical units

$$\frac{1}{B^2 \Omega_i} \left[\left(-\nabla_\perp \phi \times \widehat{\mathbf{n}} \right) \cdot \nabla_\perp \right] \left(-\nabla_\perp^2 \phi \right)$$

This is converted to non-dimensional variables

$$\frac{1}{B^2 \Omega_i} \left[\left(-\frac{1}{\rho_s} \frac{T_e}{|e|} \rho_s \nabla_\perp \frac{|e|\,\phi}{T_e} \times \widehat{\mathbf{n}} \right) \cdot \frac{1}{\rho_s} \rho_s \nabla_\perp \right] \frac{1}{\rho_s^2} \frac{T_e}{|e|} \left(-\rho_s^2 \nabla_\perp^2 \frac{|e|\,\phi}{T_e} \right)$$

Collecting the physical coefficient we have

$$
\begin{aligned}
\frac{1}{B^2 \Omega_i} \left(\frac{T_e}{|e|} \right)^2 \frac{1}{\rho_s^4} &= \frac{1}{\left(\frac{|e|B}{m_i} \right)^2} \frac{1}{\Omega_i} \left(\frac{T_e}{m_i} \right)^2 \frac{1}{\rho_s^4} \\
&= \Omega_i \frac{c_s^4}{\Omega_i^4} \frac{1}{\rho_s^4} \\
&= \Omega_i
\end{aligned}
$$

Then, in the new variables, this term becomes

$$\Omega_i \left[\left(-\nabla'_\perp \phi' \times \widehat{\mathbf{n}} \right) \cdot \nabla'_\perp \right] \left(-\nabla'^2_\perp \phi' \right) \tag{A.9}$$

Then the Eqs.(A.5) with the new form of its terms (A.6), (A.7), (A.8) and (A.9) becomes

$$\begin{aligned}
& \frac{\partial}{\partial t}\phi' - \frac{\partial}{\partial t}\nabla_\perp'^2\phi' \\
& -\Omega_i\left(-\nabla_\perp'\phi' \times \widehat{\mathbf{n}}\right)\cdot\left(-\nabla_\perp' \ln n_0\right) \\
& +\Omega_i\left(-\nabla_\perp'\phi' \times \widehat{\mathbf{n}}\right)\phi'\cdot\left(-\nabla_\perp' \ln T_e\right) \\
& +\Omega_i\left[\left(-\nabla_\perp'\phi' \times \widehat{\mathbf{n}}\right)\cdot\nabla_\perp'\right]\left(-\nabla_\perp'^2\phi'\right) \\
= & \; 0
\end{aligned} \tag{A.10}$$

Introducing the time unit Ω_i^{-1}, and eliminating the *primes*

$$\begin{aligned}
& \frac{\partial}{\partial t}\left(1-\nabla_\perp^2\right)\phi \\
& -\left(-\nabla_\perp\phi \times \widehat{\mathbf{n}}\right)\cdot\left(-\nabla_\perp \ln n_0\right) + \left(-\nabla_\perp\phi \times \widehat{\mathbf{n}}\right)\cdot\left(-\nabla_\perp \ln T_e\right)\phi \\
& +\left[\left(-\nabla_\perp\phi \times \widehat{\mathbf{n}}\right)\cdot\nabla_\perp\right]\left(-\nabla_\perp^2\phi\right) \\
= & \; 0
\end{aligned} \tag{A.11}$$

However this is not the equation (16) of Laedke and Spatschek 1988. This is discussed in the following.

END OF THE NOTE

We return to the *physical* equation, Eq.(A.5) or Eq.(12) from [29] (1988)

$$\begin{aligned}
& \frac{\partial}{\partial t}\frac{|e|\,\phi}{T_e} - \frac{\partial}{\partial t}\frac{1}{B\Omega_i}\nabla_\perp^2\phi \\
& +\frac{-\nabla_\perp\phi \times \widehat{\mathbf{n}}}{B}\cdot\nabla_\perp \ln n_0 \\
& +\frac{-\nabla_\perp\phi \times \widehat{\mathbf{n}}}{B}\cdot\nabla_\perp\frac{|e|\,\phi}{T_e} \\
& +\frac{1}{B^2\Omega_i}\left[\left(-\nabla_\perp\phi \times \widehat{\mathbf{n}}\right)\cdot\nabla_\perp\right]\left(-\nabla_\perp^2\phi\right) \\
= & \; 0
\end{aligned} \tag{A.12}$$

We make a change of variables

$$\eta = y - ut$$

which means

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - u\frac{\partial}{\partial \eta}$$

The equation becomes, after removing the *primes* from t

$$\begin{aligned} & \frac{\partial}{\partial t'}\frac{|e|\,\phi}{T_e} - \frac{\partial}{\partial t'}\frac{1}{B\Omega_i}\nabla_\perp^2\phi - u\frac{\partial}{\partial\eta}\frac{|e|\,\phi}{T_e} + u\frac{\partial}{\partial\eta}\frac{1}{B\Omega_i}\nabla_\perp^2\phi \\ & +\frac{-\nabla_\perp\phi\times\widehat{\mathbf{n}}}{B}\cdot\nabla_\perp\ln n_0 \\ & +\frac{-\nabla_\perp\phi\times\widehat{\mathbf{n}}}{B}\cdot\nabla_\perp\frac{|e|\,\phi}{T_e} \\ & +\frac{1}{B^2\Omega_i}\left[\left(-\nabla_\perp\phi\times\widehat{\mathbf{n}}\right)\cdot\nabla_\perp\right]\left(-\nabla_\perp^2\phi\right) \\ = & \ 0 \end{aligned}$$

The first term is

$$\begin{aligned} \frac{\partial}{\partial t}\frac{|e|\,\phi}{T_e} &= \frac{\partial}{\partial t}\frac{1}{B}\frac{|e|\,B}{m_i}\frac{m_i}{T_e}\phi \\ &= \frac{\partial}{\partial t}\frac{1}{B}\frac{\Omega_i}{\Omega_i}\frac{\Omega_i}{c_s^2}\phi \\ &= \frac{1}{B\Omega_i}\frac{\partial}{\partial t}\frac{1}{\rho_s^2}\phi \end{aligned} \tag{A.13}$$

The third term is

$$-u\frac{\partial}{\partial\eta}\frac{|e|\,\phi}{T_e} = -\frac{1}{B\Omega_i}u\frac{\partial}{\partial\eta}\frac{1}{\rho_s^2}\phi$$

the second and fourth terms remains unchanged

$$-\frac{\partial}{\partial t}\frac{1}{B\Omega_i}\nabla_\perp^2\phi + u\frac{\partial}{\partial\eta}\frac{1}{B\Omega_i}\nabla_\perp^2\phi$$

Since the product $1/\left(\Omega_i B\right)$ arises systematically, we will extract it from the third term

$$\frac{-\nabla_\perp\phi\times\widehat{\mathbf{n}}}{B}\cdot\nabla_\perp\ln n_0 = -\frac{1}{B\Omega_i}u\frac{\Omega_i}{u}\kappa_n\frac{\partial\phi}{\partial\eta}$$

where we define

$$\kappa_n\widehat{\mathbf{e}}_x \equiv \nabla_\perp\ln n_0$$

with κ_n measured in m^{-1}.

The term with the gradient of temperature

$$\frac{-\nabla_\perp\phi\times\widehat{\mathbf{n}}}{B}\cdot\nabla_\perp\frac{|e|\,\phi}{T_e} = \frac{-\nabla_\perp\phi\times\widehat{\mathbf{n}}}{B}\,|e|\,\phi\cdot\left(-\frac{1}{T_e^2}\nabla_\perp T_e\right)$$

after introducing the definition

$$\kappa_T\widehat{\mathbf{e}}_x \equiv \nabla_\perp\ln T_e$$

we obtain

$$\begin{aligned}
\frac{-\nabla_{\perp}\phi \times \widehat{\mathbf{n}}}{B} \cdot \nabla_{\perp} \frac{|e|\,\phi}{T_e} &= \frac{\kappa_T}{B\Omega_i} \frac{1}{B} \frac{\Omega_i\,|e|\,B}{\frac{T_e}{m_i} m_i} \phi \frac{\partial \phi}{\partial \eta} \\
&= \frac{1}{B\Omega_i} \frac{1}{\rho_s^2} \frac{1}{B} \kappa_T \phi \frac{\partial \phi}{\partial \eta}
\end{aligned}$$

The term with the polarisation nonlinearity is

$$\frac{1}{B^2\Omega_i} \left[\left(-\nabla_{\perp}\phi \times \widehat{\mathbf{n}} \right) \cdot \nabla_{\perp} \right] \left(-\nabla_{\perp}^2 \phi \right)$$

The form of the equation is

$$\begin{aligned}
& \frac{1}{B\Omega_i} \frac{\partial}{\partial t} \frac{1}{\rho_s^2} \phi - \frac{\partial}{\partial t} \frac{1}{B\Omega_i} \nabla_{\perp}^2 \phi \\
& - \frac{1}{B\Omega_i} u \frac{\partial}{\partial \eta} \frac{1}{\rho_s^2} \phi - \frac{1}{B\Omega_i} u \frac{\Omega_i}{u} \kappa_n \frac{\partial \phi}{\partial \eta} \\
& + u \frac{\partial}{\partial \eta} \frac{1}{B\Omega_i} \nabla_{\perp}^2 \phi \\
& + \frac{1}{B\Omega_i} \frac{1}{\rho_s^2} \frac{1}{B} \kappa_T \phi \frac{\partial \phi}{\partial \eta} \\
& + \frac{1}{B^2\Omega_i} \left[\left(-\nabla_{\perp}\phi \times \widehat{\mathbf{n}} \right) \cdot \nabla_{\perp} \right] \left(-\nabla_{\perp}^2 \phi \right) \\
= \ & 0
\end{aligned}$$

Multiplying by Ω_i

$$\begin{aligned}
& \frac{\partial}{\partial t} \left[\left(\frac{1}{\rho_s^2} - \nabla_{\perp}^2 \right) \frac{\phi}{B} \right] \\
& - u \frac{\partial}{\partial \eta} \left[\left(\frac{1}{\rho_s^2} + \frac{\Omega_i \kappa_n}{u} \right) \frac{\phi}{B} \right] \\
& + u \frac{\partial}{\partial \eta} \nabla_{\perp}^2 \frac{\phi}{B} \\
& + \frac{1}{\rho_s^2} \frac{\kappa_T}{B^2} \phi \frac{\partial \phi}{\partial \eta} \\
& + \frac{1}{B^2} \left[\left(-\nabla_{\perp}\phi \times \widehat{\mathbf{n}} \right) \cdot \nabla_{\perp} \right] \left(-\nabla_{\perp}^2 \phi \right) \\
= \ & 0
\end{aligned} \tag{A.14}$$

NOTE ON THE COMPARISON WITH Su and Horton

We can compare this equation with the corresponding one, Eq. (45), from Ref. [**?**]. This equation is

$$\begin{aligned} & \left(\frac{1}{T(x)} - \nabla_\perp^2\right)\frac{\partial \varphi}{\partial t} + \\ & + \left(v_{d0} + v'_{d0}x - \kappa_T \varphi\right)\frac{\partial \varphi}{\partial y} \\ & - \left[\left(-\nabla_\perp \varphi \times \widehat{\mathbf{n}}\right)\cdot \nabla_\perp\right]\left(-\nabla_\perp^2 \varphi\right) \\ = & \ 0 \end{aligned}$$

where

$$T(x) = \frac{T_e(x)}{T_0}$$

$$\varphi = \frac{L_n}{\rho_{s0}}\frac{|e|\,\Phi}{T_0}$$

$$\varepsilon_n = \frac{\rho_{s0}}{L_n}$$

This is the small parameter of the expansion.

$$v_d = 1 + v'_{d0}x$$

$$v'_{d0} = \rho_{s0}\frac{dv_d}{dx} \sim \varepsilon_n v_{d0}$$

END OF THE NOTE

In Ref.[29] the potential is redefined as

$$\psi \equiv \frac{\phi}{B}$$

and the equation becomes

$$\begin{aligned} & \frac{\partial}{\partial t}\left[\left(\frac{1}{\rho_s^2} - \nabla_\perp^2\right)\psi\right] - u\frac{\partial}{\partial \eta}\left[\left(\frac{1}{\rho_s^2} + \frac{\Omega_i \kappa_n}{u}\right)\psi\right] \\ & + u\frac{\partial}{\partial \eta}\nabla_\perp^2 \psi + \frac{1}{\rho_s^2}\frac{\kappa_T}{B^2}\psi\frac{\partial \psi}{\partial \eta} \\ & + \left[\left(-\nabla_\perp \psi \times \widehat{\mathbf{n}}\right)\cdot \nabla_\perp\right]\left(-\nabla_\perp^2 \psi\right) \\ = & \ 0 \end{aligned}$$

In Ref.[29] the following operator is introduced

$$P \equiv \frac{1}{\rho_s^2} - \nabla_\perp^2$$

and the equation is expressed as

$$P\frac{\partial}{\partial t}\psi = \left[\nabla_{\perp}\left(\psi - ux\right)\times\widehat{\mathbf{n}}\right]\cdot\nabla_{\perp}\left(P\psi + \Omega_i \ln n_0\right)$$

In this way the term containing the temperature gradient κ_T is hidden in the application of the operator $\nabla_{\perp}$ on $1/\rho_s^2$ contained in P.

10.2 The stationary form of the equation

The most general form of the solution at stationarity is

$$\begin{aligned}\frac{\partial\psi}{\partial t} &= 0 \\ P\psi + \Omega_i \ln n_0 &= \Omega_i \ln\left[n_0\left(x - \frac{\psi}{u}\right)\right]\end{aligned}$$

In the natural logarithm we expand the density at x and then parform an expansion

$$\ln\left(1+\varepsilon\right) \approx \varepsilon - \frac{\varepsilon^2}{2}$$

$$\begin{aligned}\Omega_i \ln\left[n_0\left(x - \frac{\psi}{u}\right)\right] &= \Omega_i \ln\left[n_0\left(x\right) - \frac{\psi}{u}\frac{dn_0\left(x\right)}{dx} + \frac{1}{2}\frac{\psi^2}{u^2}\frac{d^2 n_0\left(x\right)}{dx^2}\right] \\ &= \Omega_i \ln\left\{n_0\left(x\right)\left[1 - \frac{\psi}{u}\frac{1}{n_0}\frac{dn_0\left(x\right)}{dx} + \frac{1}{2}\frac{\psi^2}{u^2}\frac{1}{n_0}\frac{d^2 n_0\left(x\right)}{dx^2}\right]\right\} \\ &= \Omega_i \ln\left[n_0\left(x\right)\right] \\ &\quad + \Omega_i \ln\left[1 - \frac{\psi}{u}\frac{1}{n_0}\frac{dn_0\left(x\right)}{dx} + \frac{1}{2}\frac{\psi^2}{u^2}\frac{1}{n_0}\frac{d^2 n_0\left(x\right)}{dx^2}\right]\end{aligned}$$

$$\begin{aligned}&\Omega_i \ln\left[n_0\left(x - \frac{\psi}{u}\right)\right] \\ = \; &\Omega_i \ln\left[n_0\left(x\right)\right] \\ &+\Omega_i\left[-\frac{\psi}{u}\frac{1}{n_0}\frac{dn_0\left(x\right)}{dx} + \frac{1}{2}\frac{\psi^2}{u^2}\frac{1}{n_0}\frac{d^2 n_0\left(x\right)}{dx^2} - \frac{1}{2}\frac{\psi^2}{u^2}\frac{1}{n_0^2}\left(\frac{dn_0}{dx}\right)^2\right]\end{aligned}$$

The last two terms are

$$\begin{aligned}\frac{1}{2}\frac{1}{n_0}\frac{d^2 n_0\left(x\right)}{dx^2} - \frac{1}{2}\frac{1}{n_0^2}\left(\frac{dn_0}{dx}\right)^2 &= \frac{1}{2}\frac{d}{dx}\kappa_n \\ &\equiv \frac{1}{2}\kappa_n'\end{aligned}$$

The final form is

$$\begin{aligned}
& \Omega_i \ln\left[n_0\left(x-\frac{\psi}{u}\right)\right] \\
= \; & \Omega_i \ln\left[n_0(x)\right] \\
& +\Omega_i\left(-\frac{\psi}{u}\kappa_n+\frac{1}{2}\frac{\psi^2}{u^2}\kappa_n'\right)
\end{aligned}$$

In this expression the units are physical:

$$\begin{aligned}
& u \;\; (m/s) \\
& \kappa_n \;\; \left(m^{-1}\right) \\
& \kappa_n' \;\; \left(m^{-2}\right) \\
\psi \;\; \equiv \;\; & \frac{\phi}{B} \;\; \left(m^2/s\right)
\end{aligned}$$

The equation is then

$$\begin{aligned}
P\psi+\Omega_i \ln n_0 &= \Omega_i \ln\left[n_0\left(x-\frac{\psi}{u}\right)\right] \\
&= \Omega_i \ln\left[n_0(x)\right]+\Omega_i\left(-\frac{\psi}{u}\kappa_n+\frac{1}{2}\frac{\psi^2}{u^2}\kappa_n'\right)
\end{aligned}$$

or

$$\left(\frac{1}{\rho_s^2}-\nabla_\perp^2\right)\psi=-\frac{\psi}{u}\Omega_i\kappa_n+\frac{1}{2}\frac{\psi^2}{u^2}\Omega_i\kappa_n' \tag{A.15}$$

10.3 The equation with time dependence and polarisation nonlinearity retained

In order to write an equation for the potential with time dependence, we return to Eq.(A.14). We multiply by B

$$\begin{aligned}
& \frac{\partial}{\partial t}\left[\left(\frac{1}{\rho_s^2}-\nabla_\perp^2\right)\phi\right] \qquad \text{(A.16)} \\
& -u\frac{\partial}{\partial \eta}\left[\left(\frac{1}{\rho_s^2}+\frac{\Omega_i\kappa_n}{u}\right)\phi\right] \\
& +u\frac{\partial}{\partial \eta}\nabla_\perp^2\phi \\
& +\frac{1}{\rho_s^2}\frac{\kappa_T}{B}\phi\frac{\partial \phi}{\partial \eta} \\
& +\frac{1}{B}\left[\left(-\nabla_\perp\phi\times\widehat{\mathbf{n}}\right)\cdot\nabla_\perp\right]\left(-\nabla_\perp^2\phi\right) \\
= \;\; & 0
\end{aligned}$$

Then we derivate Eq.(A.15) with respect to η, multiply by u and return from ψ to $\phi \equiv \psi B$ which will be noted ϕ_s (stationary solution)

$$u\frac{1}{\rho_s^2}\frac{\partial}{\partial\eta}\phi_s - u\nabla_\perp^2\frac{\partial}{\partial\eta}\phi_s + \Omega_i\kappa_n\frac{\partial}{\partial\eta}\phi_s - \frac{\Omega_i\kappa_n'}{uB}\frac{\partial}{\partial\eta}\frac{\phi_s^2}{2} = 0 \tag{A.17}$$

Now we add this equation from Eq.(A.16) and combine the two functions ϕ_s and ϕ in order to obtain an equation for their difference.

We first examine the last term

$$\begin{aligned}
& \left\{\left[-\nabla_\perp\left(\phi-\phi_s\right)\times\widehat{\mathbf{n}}\right]\cdot\nabla_\perp\right\}\left[-\nabla_\perp^2\left(\phi-\phi_s\right)\right] \\
= \; & \left[\left(-\nabla_\perp\phi\times\widehat{\mathbf{n}}\right)\cdot\nabla_\perp\right]\left(-\nabla_\perp^2\phi\right) - \\
& -\left[\left(-\nabla_\perp\phi_s\times\widehat{\mathbf{n}}\right)\cdot\nabla_\perp\right]\left(-\nabla_\perp^2\phi\right) \\
& -\left[\left(-\nabla_\perp\phi\times\widehat{\mathbf{n}}\right)\cdot\nabla_\perp\right]\left(-\nabla_\perp^2\phi_s\right) \\
& +\left[\left(-\nabla_\perp\phi_s\times\widehat{\mathbf{n}}\right)\cdot\nabla_\perp\right]\left(-\nabla_\perp^2\phi_s\right)
\end{aligned}$$

The last term is zero. Expressing the term we need (the first one)

$$\begin{aligned}
& \left[\left(-\nabla_\perp\phi\times\widehat{\mathbf{n}}\right)\cdot\nabla_\perp\right]\left(-\nabla_\perp^2\phi\right) \\
= \; & \left\{\left[-\nabla_\perp\left(\phi-\phi_s\right)\times\widehat{\mathbf{n}}\right]\cdot\nabla_\perp\right\}\left[-\nabla_\perp^2\left(\phi-\phi_s\right)\right] \\
& +\left[\left(-\nabla_\perp\phi_s\times\widehat{\mathbf{n}}\right)\cdot\nabla_\perp\right]\left(-\nabla_\perp^2\phi\right) \\
& +\left[\left(-\nabla_\perp\phi\times\widehat{\mathbf{n}}\right)\cdot\nabla_\perp\right]\left(-\nabla_\perp^2\phi_s\right)
\end{aligned}$$

After adding the equations we obtain

$$\begin{aligned}
& \frac{\partial}{\partial t}\left[\left(\frac{1}{\rho_s^2}-\nabla_\perp^2\right)\left(\phi-\phi_s\right)\right] \\
& +u\frac{\partial}{\partial\eta}\left[\left(-\frac{1}{\rho_s^2}+\nabla_\perp^2\right)\left(\phi-\phi_s\right)\right] \\
& +\frac{\partial}{\partial\eta}\left[-\Omega_i\kappa_n\left(\phi-\phi_s\right)\right] \\
& +\frac{\partial}{\partial\eta}\left[\frac{1}{\rho_s^2}\frac{\kappa_T}{B}\frac{\phi^2}{2}-\frac{\Omega_i\kappa_n'}{uB}\frac{\phi_s^2}{2}\right] \\
& +\frac{1}{B}\left\{\left[-\nabla_\perp\left(\phi-\phi_s\right)\times\widehat{\mathbf{n}}\right]\cdot\nabla_\perp\right\}\left[-\nabla_\perp^2\left(\phi-\phi_s\right)\right] \\
& +\frac{1}{B}\left[\left(-\nabla_\perp\phi_s\times\widehat{\mathbf{n}}\right)\cdot\nabla_\perp\right]\left(-\nabla_\perp^2\phi\right)+\frac{1}{B}\left[\left(-\nabla_\perp\phi\times\widehat{\mathbf{n}}\right)\cdot\nabla_\perp\right]\left(-\nabla_\perp^2\phi_s\right) \\
= \; & 0
\end{aligned}$$

We define the new potential representing the difference between the time dependent solution and the stationary solution

$$\varphi \equiv \phi - \phi_s$$

Then the last two terms of the above equation will become

$$\begin{aligned} & \left[\left(-\nabla_\perp \phi_s \times \widehat{\mathbf{n}} \right) \cdot \nabla_\perp \right] \left(-\nabla_\perp^2 \phi_s - \nabla_\perp^2 \varphi \right) \\ = & \left[\left(-\nabla_\perp \phi_s \times \widehat{\mathbf{n}} \right) \cdot \nabla_\perp \right] \left(-\nabla_\perp^2 \varphi \right) \end{aligned}$$

and

$$\begin{aligned} & \left[\left(-\nabla_\perp \phi_s \times \widehat{\mathbf{n}} \right) \cdot \nabla_\perp + \left(-\nabla_\perp \varphi \times \widehat{\mathbf{n}} \right) \cdot \nabla_\perp \right] \left(-\nabla_\perp^2 \phi_s \right) \\ = & \left[\left(-\nabla_\perp \varphi \times \widehat{\mathbf{n}} \right) \cdot \nabla_\perp \right] \left(-\nabla_\perp^2 \phi_s \right) \end{aligned}$$

We can use in the last term the equation defining the stationary solution, ϕ_s.

$$\begin{aligned} \nabla_\perp^2 \phi_s &= \left(\frac{1}{\rho_s^2} + \frac{\Omega_i \kappa_n}{u} \right) \phi_s - \frac{\Omega_i \kappa_n'}{u^2 B} \frac{\phi_s^2}{2} \\ &\equiv \alpha \phi_s - \beta \phi_s^2 \end{aligned}$$

with notations

$$\begin{aligned} \alpha &\equiv \frac{1}{\rho_s^2} + \frac{\Omega_i \kappa_n}{u} \\ \beta &\equiv \frac{\Omega_i \kappa_n'}{2u^2 B} \end{aligned}$$

Then the equation is

$$\begin{aligned} & \frac{\partial}{\partial t} \left(\frac{1}{\rho_s^2} - \nabla_\perp^2 \right) \varphi - \\ & -u \frac{\partial}{\partial \eta} \left(\frac{1}{\rho_s^2} - \nabla_\perp^2 \right) \varphi \\ & -\Omega_i \kappa_n \frac{\partial}{\partial \eta} \varphi \\ & + \frac{\partial}{\partial \eta} \left[\frac{\kappa_T}{2\rho_s^2 B} \left(\phi_s + \varphi \right) - \frac{\Omega_i \kappa_n'}{2uB} \phi_s^2 \right] \\ & + \frac{1}{B} \left[\left(-\nabla_\perp \varphi \times \widehat{\mathbf{n}} \right) \cdot \nabla_\perp \right] \left(-\nabla_\perp^2 \varphi \right) \\ & + \frac{1}{B} \left[\left(-\nabla_\perp \phi_s \times \widehat{\mathbf{n}} \right) \cdot \nabla_\perp \right] \left(-\nabla_\perp^2 \varphi \right) \\ & + \frac{1}{B} \left[\left(-\nabla_\perp \varphi \times \widehat{\mathbf{n}} \right) \cdot \nabla_\perp \right] \left(-\nabla_\perp^2 \phi_s \right) \\ = & \; 0 \end{aligned} \tag{A.18}$$

In this equation, the quantitities are in *physical units*

(x, η)	m
t	s
φ	V
u	m/s

We change to the units

$$\begin{aligned}\left(\frac{x^{phys}}{\rho_s},\frac{\eta^{phys}}{\rho_s}\right) &\rightarrow \quad (x',\eta')\\ \Omega_i t^{phys} &\rightarrow \quad t'\\ \frac{|e|\varphi^{phys}}{T_e} &\rightarrow \quad \varphi'\end{aligned}$$

Also

$$\frac{|e|\,\phi_s^{phys}}{T_e}\rightarrow\phi_s'$$

After some arrangement, we have

$$\begin{aligned} & \frac{\partial}{\partial t'}\left(1-\nabla_\perp'^2\right)\varphi' \\ & -\frac{u^{phys}}{\Omega_i\rho_s}\frac{\partial}{\partial\eta'}\left(1-\nabla_\perp'^2\right)\varphi' \\ & -\rho_s\kappa_n^{phys}\frac{\partial}{\partial\eta'}\varphi' \\ & +\left(\frac{\kappa_T^{phys}T_e}{B\,|e|\,\rho_s\Omega_i}\right)\frac{1}{2}\frac{\partial}{\partial\eta'}\left(\phi_s'+\varphi'\right)^2 \\ & -\left(\frac{\rho_s T_e}{B\,|e|\,u^{phys}}\kappa_n'^{phys}\right)\frac{1}{2}\frac{\partial}{\partial\eta'}\left(\phi_s'\right)^2 \\ & +\left(\frac{T_e}{B\rho_s^2\,|e|\,\Omega_i}\right)\left[\left(-\nabla_\perp'\varphi'\times\widehat{\mathbf{n}}\right)\cdot\nabla_\perp'\right]\left(-\nabla_\perp'^2\varphi'\right) \\ & +\left(\frac{T_e}{B\rho_s^2\,|e|\,\Omega_i}\right)\left[\left(-\nabla_\perp'\phi_s'\times\widehat{\mathbf{n}}\right)\cdot\nabla_\perp'\right]\left(-\nabla_\perp'^2\varphi'\right) \\ & +\left(\frac{T_e}{B\rho_s^2\,|e|\,\Omega_i}\right)\left[\left(-\nabla_\perp'\varphi'\times\widehat{\mathbf{n}}\right)\cdot\nabla_\perp'\right]\left(-\nabla_\perp'^2\phi_s'\right) \\ = & \; 0 \end{aligned} \tag{A.19}$$

In this form, the *prime* in ϕ_s' means that it will be calculated for (x',η'), *i.e.* with normalised variables.

We introduce new units for the speed of the reference system, u

$$u\equiv\frac{u^{phys}}{\Omega_i\rho_s} \tag{A.20}$$

Also

$$\kappa_n\equiv\rho_s\kappa_n^{phys}$$

$$\begin{aligned}
\frac{\kappa_T^{phys} T_e}{B\,|e|\,\rho_s \Omega_i} &= \frac{\kappa_T^{phys} c_s^2}{\Omega_i^2 \rho_s} \\
&= \rho_s \kappa_T^{phys}
\end{aligned}$$

Then we introduce

$$\kappa_T \equiv \rho_s \kappa_T^{phys}$$

$$\frac{\rho_s T_e}{|e|\,B} \frac{1}{u^{phys}} \kappa_n^{\prime phys} = \frac{1}{u} \rho_s^2 \kappa_n^{\prime phys}$$

where u is the adimensional speed, Eq.(A.20). It is then useful to introduce

$$\kappa_n' \equiv \rho_s^2 \kappa_n^{\prime phys}$$

and the coefficient in the fifth line of Eq.(A.19) is, in terms of adimensional quantities

$$\frac{\rho_s T_e}{|e|\,B} \frac{1}{u^{phys}} \kappa_n^{\prime phys} = \frac{\kappa_n'}{u}$$

Finally we note that

$$\frac{T_e}{B \rho_s^2\,|e|\,\Omega_i} \equiv 1$$

The same adimensionalization is done for α and β. The equation is multiplied by ρ_s^2 and ϕ_s is normalized to ϕ_s'.

$$\begin{aligned}
\rho_s^2 \alpha^{phys} &= \rho_s^2 \left(\frac{1}{\rho_s^2} + \frac{\Omega_i \kappa_n^{phys}}{u^{phys}} \right) \rightarrow \alpha' = 1 + \frac{\rho_s \kappa_n^{phys}}{u^{phys} / \left(\Omega_i \rho_s \right)} \\
&= 1 + \frac{\kappa_n}{u}
\end{aligned}$$

$$\begin{aligned}
\beta^{phys} \frac{T_e}{|e|} \rho_s^2 &= \frac{\Omega_i \kappa_n^{\prime phys}}{2 \left(u^{phys} \right)^2 B} \frac{T_e}{|e|} \rho_s^2 \rightarrow \beta' \\
\beta' &= \frac{1}{2} \left(\rho_s^2 \kappa_n^{\prime phys} \right) c_s^2 \frac{1}{\Omega_i^2} \frac{1}{\left(u^{phys} \right)^2 / \left(\Omega_i^2 \right)} \\
&= \frac{1}{2} \left(\rho_s^2 \kappa_n^{\prime phys} \right) \frac{1}{\left[u^{phys} / \left(\Omega_i \rho_s \right) \right]^2} \\
&= \frac{1}{2} \frac{\kappa_n'}{u^2}
\end{aligned}$$

The equation in adimensional variables (and removing the primes from the variables and operators) becomes

$$\begin{aligned}
& \frac{\partial}{\partial t}\left(1-\nabla_\perp^2\right)\varphi \\
& -u\frac{\partial}{\partial \eta}\left(1-\nabla_\perp^2\right)\varphi \\
& -\kappa_n\frac{\partial}{\partial \eta}\varphi \\
& +\kappa_T\frac{1}{2}\frac{\partial}{\partial \eta}\left(\phi_s+\varphi\right)^2 \\
& -\frac{\kappa_n'}{u}\frac{1}{2}\frac{\partial}{\partial \eta}\phi_s^2 \\
& +\left[\left(-\nabla_\perp\varphi\times\widehat{\mathbf{n}}\right)\cdot\nabla_\perp\right]\left(-\nabla_\perp^2\varphi\right) \\
& +\left[\left(-\nabla_\perp\phi_s\times\widehat{\mathbf{n}}\right)\cdot\nabla_\perp\right]\left(-\nabla_\perp^2\varphi\right) \\
& +\left[\left(-\nabla_\perp\varphi\times\widehat{\mathbf{n}}\right)\cdot\nabla_\perp\right]\left(-\nabla_\perp^2\phi_s\right) \\
= \; & 0
\end{aligned} \tag{A.21}$$

In contrast to Eq.(A.18) this equation is only expressed in terms of adimensional variables.

Let us introduce a new function

$$\psi \equiv \left(1-\nabla_\perp^2\right)\varphi \tag{A.22}$$

Then

$$\begin{aligned}
& \frac{\partial\psi}{\partial t}-u\frac{\partial\psi}{\partial \eta}-\kappa_n\frac{\partial\varphi}{\partial \eta} \\
& +\kappa_T\frac{1}{2}\frac{\partial}{\partial \eta}\left(\phi_s+\varphi\right)^2-\frac{\kappa_n'}{u}\frac{1}{2}\frac{\partial}{\partial \eta}\phi_s^2 \\
& +\left[\left(-\nabla_\perp\varphi\times\widehat{\mathbf{n}}\right)\cdot\nabla_\perp\right]\left(\psi-\varphi\right) \\
& +\left[\left(-\nabla_\perp\phi_s\times\widehat{\mathbf{n}}\right)\cdot\nabla_\perp\right]\left(\psi-\varphi\right) \\
& +\left[\left(-\nabla_\perp\varphi\times\widehat{\mathbf{n}}\right)\cdot\nabla_\perp\right]\left(-\nabla_\perp^2\phi_s\right) \\
= \; & 0
\end{aligned} \tag{A.23}$$

The last three nonlinear terms are

$$\left(-\nabla_\perp\varphi\times\widehat{\mathbf{n}}\right)\cdot\left(\nabla_\perp\psi-\nabla_\perp\varphi\right)=-\frac{\partial\varphi}{\partial\eta}\frac{\partial\psi}{\partial x}+\frac{\partial\varphi}{\partial x}\frac{\partial\psi}{\partial\eta} \tag{A.24}$$

$$\begin{aligned}
\left(-\nabla_\perp\phi_s\times\widehat{\mathbf{n}}\right)\cdot\left(\nabla_\perp\psi-\nabla_\perp\varphi\right) &= -\frac{\partial\phi_s}{\partial\eta}\frac{\partial\psi}{\partial x}+\frac{\partial\phi_s}{\partial x}\frac{\partial\psi}{\partial\eta} \\
&\quad +\frac{\partial\phi_s}{\partial\eta}\frac{\partial\varphi}{\partial x}-\frac{\partial\phi_s}{\partial x}\frac{\partial\varphi}{\partial\eta}
\end{aligned} \tag{A.25}$$

$$
\begin{aligned}
& \left[\left(-\nabla_{\perp}\varphi \times \widehat{\mathbf{n}}\right) \cdot \nabla_{\perp}\right]\left(-\alpha\phi_s + \beta\phi_s^2\right) \\
= & \left(-\alpha + 2\beta\phi_s\right)\left(-\frac{\partial\varphi}{\partial\eta}\frac{\partial\phi_s}{\partial x} + \frac{\partial\varphi}{\partial x}\frac{\partial\phi_s}{\partial\eta}\right)
\end{aligned}
\tag{A.26}
$$

The final form of the equation for ψ is

$$
\begin{aligned}
& \frac{\partial\psi}{\partial t} - u\frac{\partial\psi}{\partial\eta} - \kappa_n\frac{\partial\varphi}{\partial\eta} \\
& -\frac{\partial\varphi}{\partial\eta}\frac{\partial\psi}{\partial x} + \frac{\partial\varphi}{\partial x}\frac{\partial\psi}{\partial\eta} - \frac{\partial\phi_s}{\partial\eta}\frac{\partial\psi}{\partial x} + \frac{\partial\phi_s}{\partial x}\frac{\partial\psi}{\partial\eta} \\
& +\kappa_T\frac{1}{2}\frac{\partial}{\partial\eta}\left(\phi_s + \varphi\right)^2 - \frac{\kappa_n'}{u}\frac{1}{2}\frac{\partial}{\partial\eta}\phi_s^2 + \frac{\partial\phi_s}{\partial\eta}\frac{\partial\varphi}{\partial x} - \frac{\partial\phi_s}{\partial x}\frac{\partial\varphi}{\partial\eta} \\
& + \left(-\alpha + 2\beta\phi_s\right)\left(-\frac{\partial\varphi}{\partial\eta}\frac{\partial\phi_s}{\partial x} + \frac{\partial\varphi}{\partial x}\frac{\partial\phi_s}{\partial\eta}\right) \\
= & \ 0
\end{aligned}
\tag{A.27}
$$

10.4 The time evolution of a perturbation around the stationary poloidal flow

In this case, ϕ_s is the periodic solution expressed in terms of the Weierstrass function, and with parameters chosen such that the pattern of the flow is parallel with the poloidal direction. Then ϕ_s only depends on x and

$$
\frac{\partial\phi_s}{\partial\eta} \equiv 0 \tag{A.28}
$$

which considerably simplifies the equation.

In addition we will take again the fixed (laboratory frame, for this case

$$
\begin{aligned}
t & \rightarrow t' = t \\
\eta & \rightarrow y = \eta + ut
\end{aligned}
$$

which gives

$$
\begin{aligned}
\frac{\partial}{\partial\eta} & = \frac{\partial}{\partial y} \\
\frac{\partial}{\partial t'} & = \frac{\partial}{\partial t} - u\frac{\partial}{\partial y}
\end{aligned}
$$

and the equation becomes

$$\begin{aligned}
&\frac{\partial\psi}{\partial t}-\frac{\partial\varphi}{\partial y}\frac{\partial\psi}{\partial x}+\frac{\partial\varphi}{\partial x}\frac{\partial\psi}{\partial y}+\frac{\partial\phi_s}{\partial x}\frac{\partial\psi}{\partial y}\\
&-\frac{\partial\phi_s}{\partial x}\frac{\partial\varphi}{\partial y}-\kappa_n\frac{\partial\varphi}{\partial y}+\kappa_T\frac{1}{2}\frac{\partial}{\partial y}\left(\phi_s+\varphi\right)^2\\
&+\left(-\alpha+2\beta\phi_s\right)\left(-\frac{\partial\varphi}{\partial y}\frac{\partial\phi_s}{\partial x}\right)\\
=\ & 0
\end{aligned}$$

or

$$\begin{aligned}
&\frac{\partial\psi}{\partial t}+\frac{\partial\psi}{\partial x}\left(-\frac{\partial\varphi}{\partial y}\right)+\frac{\partial\psi}{\partial y}\left(\frac{\partial\varphi}{\partial x}+\frac{\partial\phi_s}{\partial x}\right)\\
&-\left[\left(1-\alpha+2\beta\phi_s\right)\frac{\partial\phi_s}{\partial x}+\kappa_n\right]\frac{\partial\varphi}{\partial y}+\kappa_T\frac{1}{2}\frac{\partial}{\partial y}\left(\phi_s+\varphi\right)^2\\
=\ & 0
\end{aligned} \tag{A.29}$$

This can be written

$$\frac{D\psi}{Dt}+C\left[\varphi\right]=0 \tag{A.30}$$

where

$$\frac{D\psi}{Dt}\equiv\frac{\partial\psi}{\partial t}+v_x\frac{\partial\psi}{\partial x}+v_y\frac{\partial\psi}{\partial y}$$

$$\begin{aligned}
v_x &\equiv -\frac{\partial\varphi}{\partial y}\\
v_y &\equiv \frac{\partial\varphi}{\partial x}+\frac{\partial\phi_s}{\partial x}
\end{aligned} \tag{A.31}$$

$$\begin{aligned}
C\left[\varphi\right] &\equiv -\left[\left(1-\alpha+2\beta\phi_s\right)\frac{\partial\phi_s}{\partial x}+\kappa_n\right]\frac{\partial\varphi}{\partial y}+\kappa_T\frac{1}{2}\frac{\partial}{\partial y}\left(\phi_s+\varphi\right)^2\\
&= \left[-\kappa_n-\left(1-\alpha+2\beta\phi_s\right)\frac{\partial\phi_s}{\partial x}+\kappa_T\left(\phi_s+\varphi\right)\right]\frac{\partial\varphi}{\partial y}
\end{aligned} \tag{A.32}$$

The solution of Eq.(A.30) is

$$\psi\left(x,y\right)=\psi\left(x_0,y_0\right)-\int_{t_0}^{t}d\tau C\left[\varphi\left(x_0+\tau v_x,y_0+\tau v_y\right)\right] \tag{A.33}$$

The Eq.(A.33) gives a possible way to solve numerically the equations. The final system is

$$
\begin{aligned}
-\nabla_\perp^2\varphi+\varphi &= \psi \qquad \text{(A.34)}\\
\psi(x,y) &= \psi(x_0,y_0) \\
&\quad -\int_{t_0}^{t} d\tau \left[-\kappa_n-(1-\alpha+2\beta\phi_s)\frac{\partial\phi_s}{\partial x}+\kappa_T(\phi_s+\varphi)\right]\frac{\partial\varphi}{\partial y}\Big|_{\substack{x=x_0+\tau v_x\\ y=y_0+\tau v_y}} \qquad \text{(A.35)}
\end{aligned}
$$

We note that the vector field of Eq.(A.31) does not have zero divergence

$$
\nabla_\perp\cdot\mathbf{v}=-\frac{\partial^2\phi_s}{\partial x^2} \qquad \text{(A.36)}
$$

10.5 The equation of Petviashvili with time dependence due to the temperature gradient

10.5.1 Physical model of the ion drift instability

This equation has been derived in several papers and used in numerical calculations (Laedke Spatschek 1986). From the dynamical equations of ion density (as above) it is obtained

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(1-\nabla_\perp^2\right)\phi \qquad \text{(A.37)}\\
&-\left(-\nabla_\perp\phi\times\widehat{\mathbf{n}}\right)\cdot\mathbf{v}_d^*+\left(-\nabla_\perp\phi\times\widehat{\mathbf{n}}\right)\cdot\mathbf{v}_{dT}\phi\\
&+\left[\left(-\nabla_\perp\phi\times\widehat{\mathbf{n}}\right)\cdot\nabla_\perp\right]\left(-\nabla_\perp^2\phi\right)\\
=&\ 0
\end{aligned}
$$

where

$$
\mathbf{v}_d^*=-\nabla_\perp\ln n_0-\nabla_\perp\ln T_e
$$

$$
\mathbf{v}_{dT}=-\nabla_\perp\ln T_e
$$

This is Eq.(8) from [**?**]. It differs of what we have obtained before.

Later the equation is expressed as, explained in the article as the limit

$$
L_T\rightarrow\infty
$$

$$
\frac{\partial}{\partial t}\left(1-\nabla_\perp^2\right)\phi+v_*\frac{\partial\phi}{\partial y}-v_{dT}\frac{1}{2}\frac{\partial}{\partial y}\phi^2=0 \qquad \text{(A.38)}
$$

This is the equation (12) from paper of 1986 by Laedke and Spatschek [7]. Here the units are: time is measured in Ω_i^{-1}; space in ρ_s; the potential ϕ is $|e|\,\phi^{phys}/T_e$.

10.5.2 The transformation of the equation

A referential moving with the velocity u is introduced

$$\begin{aligned} y &\rightarrow y' = y - ut \\ t &\rightarrow t' = t \end{aligned}$$

Then

$$\left(\frac{\partial}{\partial t'} - u\frac{\partial}{\partial y'}\right)\left(1-\nabla_\perp^2\right)\phi + v_*\frac{\partial \phi}{\partial y'} - v_{dT}\frac{1}{2}\frac{\partial}{\partial y'}\phi^2 = 0$$

or

$$\frac{\partial}{\partial t'}\left(1-\nabla_\perp^2\right)\phi - u\frac{\partial}{\partial y'}\left(1-\nabla_\perp^2\right)\phi + v_*\frac{\partial \phi}{\partial y'} - v_{dT}\frac{1}{2}\frac{\partial}{\partial y'}\phi^2 = 0$$

We will drop the prime

$$\frac{\partial}{\partial t}\left(1-\nabla_\perp^2\right)\phi = \frac{\partial}{\partial y}\left[-u\nabla_\perp^2 + (u - v_*) + \frac{v_{dT}}{2}\phi^2\right] \tag{A.39}$$

We divide by u

$$\frac{1}{u}\frac{\partial}{\partial t}\left(1-\nabla_\perp^2\right)\phi = \frac{\partial}{\partial y}\left[-\nabla_\perp^2 + \frac{(u - v_*)}{u} + \frac{v_{dT}}{2u}\phi^2\right]$$

and replace the time variable by

$$t \rightarrow t' \equiv ut$$

Here the time in Eq.(A.39) had been already adimensionalized by expressing it in units of Ω_i^{-1}. Since y in the same equation is measured in ρ_s, the velocity is measured in units of

$$\frac{\rho_s}{\Omega_i^{-1}}$$

This is the case for both u and v_*. The equation becomes

$$\frac{\partial}{\partial t}\left(1-\nabla_\perp^2\right)\phi = \frac{\partial}{\partial y}\left[-\nabla_\perp^2 + 4\eta^2 + \frac{v_{dT}}{2u}\phi^2\right]$$

$$\frac{\partial}{\partial t}\left(1-\nabla_\perp^2\right)\phi = \frac{\partial}{\partial y}\left(-\nabla_\perp^2 + 4\eta^2\right)\phi + \frac{v_{dT}}{u}\phi\frac{\partial \phi}{\partial y} \tag{A.40}$$

with

$$4\eta^2 \equiv \frac{u - v_*}{u} > 0 \tag{A.41}$$

We now renormalise the potential ϕ by including in it the factor $-v_{dT}/u$.

$$\phi \rightarrow \phi' \equiv -\frac{v_{dT}}{u}\phi \tag{A.42}$$

For this both sides of the equation are multiplied by $-v_{dT}/u$ and the equation is

$$\frac{\partial}{\partial t}\left(1-\nabla_\perp^2\right)\phi' = \frac{\partial}{\partial y}\left(-\nabla_\perp^2 + 4\eta^2\right)\phi' - \phi'\frac{\partial \phi'}{\partial y} \tag{A.43}$$

The units are:

$$\begin{aligned} \phi|_{phys} &\rightarrow \frac{e\phi|_{phys}}{T_e} \rightarrow -\frac{v_{dT}}{u}\frac{e\phi|_{phys}}{T_e} \quad \text{or} \\ \phi' &= -\frac{v_{dT}}{u}\frac{e\phi|_{phys}}{T_e} \end{aligned} \tag{A.44}$$

The velocities are measured as

$$\begin{aligned} u|_{phys} &\rightarrow \frac{1}{\rho_s\Omega_i^{-1}}u|_{phys} \quad \text{or} \\ u' &= \frac{1}{\rho_s\Omega_i^{-1}}u|_{phys} \end{aligned} \tag{A.45}$$

This quantity disappears from the equation, being absorbed into the new time variable. The time is measured as

$$\begin{aligned} t|_{phys} &\rightarrow \Omega_i^{-1}t|_{phys} \rightarrow u'\Omega_i^{-1}t|_{phys} \quad \text{or} \\ t' &= \frac{1}{\rho_s\Omega_i^{-1}}u|_{phys}\Omega_i^{-1}t|_{phys} \\ &= \frac{1}{\rho_s}u|_{phys}t|_{phys} \end{aligned} \tag{A.46}$$

The distances are measured in ρ_s

$$y' = \frac{1}{\rho_s}y|_{phys} \tag{A.47}$$

Naturally, the physical variables are measured in SI

$$\begin{array}{cc} y|_{phys} & m \\ t|_{phys} & s \\ u, v_* |_{phys} & m/s \\ \phi|_{phys} & V \end{array}$$

and at this point the *primes* are suppressed.

If we have a result about some time duration $\Delta t'$ (*i.e.* expressed in terms of the non-dimensional variables) then in order to recuperate the *physical* duration, we have to do

$$\Delta t|_{phys} = \frac{\rho_s}{u|_{phys}}\Delta t'$$

Only for

$$u > v_*$$

the localised stationary solutions are posible.

The stationary form of the Petviashvili equation is

$$\frac{\partial}{\partial y}\left(-\nabla_{\perp}^{2}+4\eta^{2}\right)\phi-\phi\frac{\partial\phi}{\partial y}=0 \tag{A.48}$$

$$\Delta\phi=4\eta^{2}\phi-\frac{1}{2}\phi^{2} \tag{A.49}$$

which is the same as the general form for

These can be found in [7].

10.6 The equation derived from Ertel 's theorem

The derivation of the equation is done on the same basis as the derivation of Laedke and Spatschek 1988. However, in a series of papers, Su and Horton [8], [4], develop a more physical justification of the scalar nonlinearity model, although their primary aim was to investigate the stability of the dipolar (Larichev-Reznik) solution of the Hasegawa-Mima equation.

The equations for the ion instabilities are

$$\begin{aligned}\left[\frac{\partial}{\partial t}+(\mathbf{v}\cdot\nabla)\right]\mathbf{v} &= -\frac{|e|}{m_i}\nabla_{\perp}\Phi+\Omega_i\mathbf{v}\times\widehat{\mathbf{n}} \\ \frac{\partial n}{\partial t}+\nabla_{\perp}(n\mathbf{v}) &= 0\end{aligned}$$

It is derived the Ertel's theorem

$$\frac{d}{dt}\left[\frac{\Omega_i+\widehat{\mathbf{n}}\cdot(\nabla_{\perp}\times\mathbf{v})}{n(x)}\right]=0 \tag{A.50}$$

This essentially means that: the sum of the cyclotron frequency and the vertical component of the vorticity, divided by the density, is constant along the line of Lagrangean evolution with the velocity $E \times B$.

If one neglects the cyclotron frequency and takes the density constant, this theorem is equivalent with the Euler theorem for ideal fluid

$$\frac{d\omega}{dt}=0$$

The ordering is

$$\varepsilon_t \equiv \frac{1}{\Omega_i}\frac{\partial}{\partial t} \sim \frac{\mathbf{v}\cdot\nabla_{\perp}}{\Omega_i} \ll 1$$

The equation becomes

$$\frac{\partial}{\partial t}\left(\frac{1+\varepsilon_n\nabla_\perp^2\varphi}{n}\right)+\left[\left(-\nabla_\perp\varphi\times\widehat{\mathbf{n}}\right)\cdot\nabla_\perp\right]\left(\frac{1+\varepsilon_n\nabla_\perp^2\varphi}{n}\right)=0 \tag{A.51}$$

It is assumed the Boltzmann density distribution for the ions

$$\begin{aligned} n &= n_0(x)\exp\left(-\frac{|e|\,\Phi}{T_e}\right) \\ &= n_0(x)\exp\left[-\varepsilon_n\frac{\varphi}{T(x)}\right] \end{aligned} \tag{A.52}$$

10.7 Stationary traveling solutions

This means solutions of the type

$$\varphi=\varphi\left(x,y-ut\right)$$

where u is the speed of the system of reference. The *stationary* equation in the moving frame is

$$-u\frac{\partial}{\partial y}\left(\frac{1+\varepsilon_n\nabla_\perp^2\varphi}{n_0(x)\exp\left[\frac{\varepsilon_n\varphi}{T(x)}\right]}\right)+\left[\left(-\nabla_\perp\varphi\times\widehat{\mathbf{n}}\right)\cdot\nabla_\perp\right]\left(\frac{1+\varepsilon_n\nabla_\perp^2\varphi}{n_0(x)\exp\left[\frac{\varepsilon_n\varphi}{T(x)}\right]}\right)=0$$

This equation is identically verified if the expression in the brackets is a general function of

$$\varphi-ux$$

i.e.

$$\frac{1+\varepsilon_n\nabla_\perp^2\varphi}{n_0(x)\exp\left[\frac{\varepsilon_n\varphi}{T(x)}\right]}=F\left(\varphi-ux\right)$$

One can take this function to be

$$F\left(\varphi-ux\right)=n_0\left(x-\frac{\varphi}{u}\right)$$

which gives

$$\varepsilon_n\nabla_\perp^2\varphi=\frac{n_0(x)}{n_0\left(x-\frac{\varphi}{u}\right)}\exp\left[\frac{\varepsilon_n\varphi}{T(x)}\right]-1 \tag{A.53}$$

In the case where the density profile is exponential

$$n_0(x)=\exp\left(-\varepsilon_n x\right)$$

the equation becomes

$$\varepsilon_n \nabla_\perp^2 \varphi = \exp\left[\varepsilon_n \left(\frac{1}{T(x)} - \frac{1}{u}\right)\varphi\right] - 1 \tag{A.54}$$

The dimensional parameters are

$$\kappa_T \equiv \rho_{s0} \frac{d}{dx}\left(\frac{1}{T}\right) \tag{A.55}$$

$$v'_{d0} \equiv \rho_{s0} \frac{dv_d}{dx} \sim \varepsilon_n v_{d0} \tag{A.56}$$

Thaking the logarithm of the Eq.(A.53)

$$\ln\left(1 + \varepsilon_n \nabla_\perp^2 \varphi\right) = \ln n_0(x) - \ln n_0\left(x - \frac{\varphi}{u}\right) + \varepsilon_n \frac{\varphi}{T(x)}$$

The density profile is expanded

$$\ln n_0(x) \approx -\varepsilon_n \left(v_{d0} x + \frac{v'_{d0}}{2} x^2 + \cdots\right)$$

gives

$$\ln\left(1 + \varepsilon_n \nabla_\perp^2 \varphi\right) \approx \varepsilon_n k^2(u, x)\varphi + \varepsilon_n \frac{v'_{d0}}{2u^2}\varphi^2 + \cdots$$

$$k^2(u, x) \equiv \frac{1}{T(x)} - \frac{v_d(x)}{u} \tag{A.57}$$

$$\begin{aligned} v_d(x) &= v_{d0} + v'_{d0} x \\ &= 1 + v'_{d0} x \end{aligned}$$

The ordering is

$$\rho_{s0}^2 \nabla^2 \sim \frac{|e|\,\Phi}{T_e} \sim \kappa_T \sim v'_{d0} \sim \frac{\rho_{s0}}{L_n} \equiv \varepsilon_n \sim \varepsilon \ll 1 \tag{A.58}$$

The velocity of the moving referential u and of v_{d0} are of the same amplitude, normalized at 1.

$$u \sim v_{d0} = 1$$

Keeping only terms of order ε^2

$$\nabla_\perp^2 \varphi = k_0^2 \varphi + \frac{v'_{d0}}{2u^2}\varphi^2 \tag{A.59}$$

where

$$k^2(u,x) = k_0^2 + \alpha x + \cdots$$

with the notations

$$k_0^2 = 1 - \frac{v_{d0}}{u} \sim \varepsilon$$

$$\alpha = \kappa_T - \frac{v_{d0}}{u} \sim \varepsilon^2$$

The gradient of the drift velocity gives rise to a linear damping

$$-\frac{v_{d0}}{u} x\varphi$$

The temperature gradient generates also a linear term that compensates the linear term of the gradient of the drift velocity.

10.8 The energy of the solutions

The density of energy has the expression

$$E(x,y,t) = \frac{1}{2}\left[\frac{\varphi^2(x)}{T(x)} + (\nabla_\perp \varphi)^2\right]$$

The units are

$$\varphi = \frac{L_n}{\rho_{s0}} \frac{|e|\,\Phi}{T_0}$$

and $T(x)$ is adimansional. The distances are measured in ρ_{s0}. This means that

$$E(x,y,t) = \left(\frac{L_n}{\rho_{s0}}\right)^2 \frac{1}{2}\left[\frac{1}{T(x)}\left(\frac{|e|\,\Phi}{T_0}\right)^2 + \left(\nabla_\perp \frac{|e|\,\Phi}{T_0}\right)^2\right]$$

References

[1] E.J. Kim and P.H. Diamond, Phys. Rev. Lett.**90**, 185006 (2003).

[2] P. H. Diamond *et al*, in *Plasma Phys. and Controlled Fusion Research*, IAEA, Vienna 1998.

[3] M. Malkov and P.H.Diamond, Phys. Plasmas **8**, 3996 (2001).

[4] W. Horton and A. Hasegawa, Chaos **4**, 227 (1994).

[5] V. I. Petviashvili, Fiz. Plazmy **3**, 270 (1977) [Sov. J. Plasma Phys. **3**, 150 (1977)].

[6] K. H. Spatschek, E. W. Laedke, Chr. Marquardt, S. Musher and H. Wenk, Phys. Rev. Lett. **64**, 3027 (1990).

[7] E.W. Laedke and K.H. Spatschek, Phys.Fluids **29** (1986) 133.

[8] X. N. Su, W. Horton and P. J. Morrison, Phys. Fluids **B3**, 921 (1991).

[9] V. P. Lakhin, A. B. Mikhailovskii and O. G. Onishcenko, Phys. Lett. **A119**, 348 (1987).

[10] G. R. Flierl, Dyn. Atmos. Oceans **3**, 15 (1979).

[11] J. D. Meiss and W. Horton, Phys. Fluids **25**, 1838 (1982).

[12] B. B. Kadomtsev and V. I. Petviashvili, Sov. Phys. Doklady **6** (1970) 539.

[13] V. I. Petviashvili, O. A. Pokhotelov, Fiz. Plazmy **12** (1986) 651 [Sov. J. Plasma Phys. **12** (1986) 657].

[14] B. Tan and J. P. Boyd, Wave Motion **26** (1997) 239.

[15] V. E. Zakharov and E. A. Kuznetsov, Sov. Phys. JETP **39** (1974) 285.

[16] J. P. Boyd and B. Tan, Chaos, Solitons and Fractals **9**, 2007 (1998).

[17] H. Iwasaki, S. Toh and T. Kawahara, Physica D **43**, 293 (1990).

[18] F. Spineanu and M. Vlad, *arXiv.org/physics/0310027.*

[19] W. R. Thickstun, Journal of Mathematical Analysis and Applications, **55**, 335 (1976).

[20] D. V. Choodnovsky and G. V. Choodnovsky, Il Nuovo Cimento **40 B**, 339 (1977).

[21] B. Deconinck and H. Segur, Preprint 1999.

[22] M. Mulase, in *Perspectives in Mathematical Physics*, Eds. R. Penner and T. S. Yau, 151 (1994).

[23] F. Spineanu, M. Vlad, K. Itoh, H. Sanuki and S.-I.Itoh, *arXiv.org/physics/0305129.*

[24] F. Spineanu and M. Vlad, Phys. Rev. Letter **84,** 4854 (2000).

[25] F. Spineanu and M. Vlad, Phys. Rev. E**65**, 026406 (2002).

[26] F. H. Busse, Chaos **4**, 123 (1994).

[27] S. Coda, M. Porkolab and K.H. Burrell, Phys.Rev.Lett. **86**, 4835 (2001).

[28] S.Brunner, M. Fivaz, T. M. Tran and J. Vaclavik, Phys. Plasmas **5**, 3929 (1998).

[29] E.W. Laedke and K.H. Spatschek, Phys.Fluids **31** (1988) 1493.

[30] S.J. Alfrey, R. Hatzky, A. Bottino, L. Villard, *Role of self-generated and externally applied radial electric fields in the development of ITG driven modes*, in Joint Varenna-Lausanne International Workshop on "Theory of Fusion Plasmas", Varena, Italy, 2002.

[31] L. N. Howard and R. Krishnamurti, J. Fluid Mech. **170**, 385 (1986).

[32] D. Fyfe, D. Montgomery and G. Joyce, J. Plasma Phys. **17**, 369 (1976).

[33] D. Biskamp, Phys.Reports, **237,** 179 **(**1994).

[34] G.C. Craddock, P.H. Diamond and P.W. Terry, Phys. Fluids **B 3**, 304 (1991).

[35] M. Kono and E. Miyashita, Phys. Fluids **31**, 326 (1988).

[36] T.Tajima, W.Horton,P.J.Morrison,S.Shutkeker, T. Kamimura, K.Mima and Y.Abe, Phys.Fluids **B 3** 938 (1991).

[37] R.D. Hazeltine, D.D.Holm and P.J.Morrison, J.Plasma Phys.**34** 103 (1985).

[38] J. Nycander, Phys.Fluids **B 3** 931 (1991).

[39] R. Kinney, J.C. McWilliams and T. Tajima, Phys.Plasmas **2** 3623 (1995).

[40] J.D. Meiss and W. Horton, Phys. Fluids **25**, 1838 (1982).

[41] J.P. Boyd and B. Tan, Chaos, Solitons & Fractals **9**, 2007 (1998).

[42] J.D. Meiss and W. Horton, Phys.Fluids **26** 990 (1983).

[43] W. Horton, Phys.Reports **192** 1 (1990).

[44] P. W. Terry and P.H. Diamond, Phys. Fluids **28**, 1419 (1985).

[45] J. A. Krommes, in *Handbook of Plasma Physics* edited by A. A. Galeev and R. N. Sydan (North-Holland, Amsterdam, 1984), Vol. 2, Chap. 5.5, p. 183.

[46] W. Horton, J. Liu, J.D.Meiss and J.E. Sedlak, Phys.Fluids **29**, 1004, (1986).

[47] T. Schafer and E. V. Shuryak, Rev. Mod. Phys.**70,** 323 (1998).

[48] P. C. Martin, E. D. Siggia and H. A. Rose, Phys. Rev. **A 8**, 423 (1973).

[49] R. V. Jensen, J. Stat. Phys. **25**, 183 (1981).

[50] D. J. Amit, *Field Theory, the Renormalization Group and Critical Phenomena,* Singapore: World Scientific, (1984).

[51] F.Spineanu, M. Vlad and J.H. Misguich, J.Plasma Phys. **51**, 113 (1994).

[52] F. Spineanu and M. Vlad, J.Plasma Phys. **54,** 333 (1995).

[53] G. Eilenberger, *Solitons*, (Mathematical methods for physicists). Springer Series in Solid-State Sciences, Vol. 19, Springer, Berlin, Heidelberg, 1981.

[54] P.G. Drazin and R.S. Johnson, *Solitons: an introduction*, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 1989.

[55] F.Spineanu and M. Vlad, Physics of Plasmas, **4**, 2106 (1997).

[56] J. Todoroki and H. Sanuki, Phys. Letters **48A** (1974) 277.

[57] H. Sanuki and G. Schmidt, J. Phys. Soc. Jpn. **42** (1977) 260.

[58] D. H. Peregrine, J. Fluid Mechanics **27** (1967) 815.

[59] P. F. Byrd and M. D. Friedman, *Handbook of Elliptic Integrals for Engineers and Scientists*, Springer-Verlag, New-York, 1971.

[60] M. Makino, T. Kamimura and T. Taniuti, J. Phys. Soc. Jpn. **50** (1981) 980.

[61] F.Spineanu, M.Vlad, J.-D. Reuss, J.H. Misguich, Plasma Physics and Controlled Fusion **41** (1999) 485.

[62] H. Sanuki and J. Weiland, J. Plasma Physics **23** (1988) 516.

[63] J. Weiland, H. Sanuki and C. S. Liu, Phys. Fluids **24** (1981) 93.

[64] S. Mahajan and J. Weiland, Plasma Physics and Controlled Fusion **42** (2000) 987.